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ALMOST SURJECTIVE ε-ISOMETRIES OF BANACH SPACES

BY

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Abstract. We investigate Hyers–Ulam stability of non-surjective ε-isometries of Banach spaces. We also pose and discuss an open problem.

1. Introduction. The classical theorem of Mazur and Ulam [MU] asserts that a surjective isometry between real normed spaces is affine. The hypothesis that an isometry is surjective is essential in general, but can be dropped if the target space is strictly convex, or it can be weakened as shown in [FŠV]. Note also that the Mazur–Ulam theorem is not valid for complex normed spaces (just consider complex conjugation on $\mathbb{C}$).

As real-world observations always have some minimal error, one may not be able to deduce from measurements whether a given mapping is indeed isometric or surjective. Thus it is natural to ask if a mapping which only nearly preserves distances and only almost covers the target space can be well approximated by a surjective (affine) isometry.

In this paper we deal with ε-isometries of one normed space into another which almost cover (in some sense) the target space. Throughout, except in Lemma 5, X and Y denote real normed spaces.

Definition. Let $\varepsilon \geq 0$. A map $f\colon X \to Y$ is called an *ε-isometry* if

$$\big|\|f(y) - f(x)\| - \|y - x\|\big| \leq \varepsilon$$

for all $x, y \in X$.

There is an extensive literature on such mappings, starting with the influential paper [HU] of Hyers and Ulam. They proved that every *surjective* ε-isometry between real Hilbert spaces can be uniformly approximated to within 10ε by an affine surjective isometry. Later this result has been extended to all pairs of real Banach spaces (see [G]), and the constant 10

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has been reduced to 2, which is sharp (see [OŠ]). The example of the map $x \mapsto (x, \sqrt[p]{p\varepsilon}\|x\|^{(p-1)/p})$ from l_p^n to l_p^{n+1} (which is ε-isometric for $1 \le p < \infty$, but far from any affine map) shows that the surjectivity assumption is indispensable in this theorem even for Minkowski spaces.

On the other hand, after some recent partial results ([BŠ], [D] and [T]), Šemrl and Väisälä [ŠV] showed that this theorem remains true when the surjectivity condition is replaced by

$$\sup_{y\in Y}\{\operatorname{dist}(y, f(X))\} < \infty.$$

Väisälä [Va] also showed that in the case when X and Y are real Hilbert spaces, the last condition can be replaced by

$$\sup_{y\neq 0\in Y}\ \liminf_{|t|\to\infty} \frac{\operatorname{dist}(ty, f(X))}{\|ty\|} < 1.$$

Our main result (Theorem 3) extends the Väisälä theorem to the case when Y is a uniformly convex Banach space and X is a Banach space. Also, we pose and discuss a problem concerning this result.

Note that when studying approximations of ε-isometries there is no loss of generality in assuming that $f(0) = 0$. Indeed, if f is an ε-isometry then so is $f - f(0)$, and f can be approximated by an isometry U iff $f - f(0)$ is close to the isometry $U - f(0)$.

2. Results. In this section, we investigate ε-isometries into real uniformly convex Banach spaces.

From the proof of Proposition 2.3 of Šemrl and Väisälä [ŠV] one can conclude:

PROPOSITION 1. *Let Y be a uniformly convex Banach space with modulus of convexity $\delta = \delta_Y(\varepsilon)$. Let $f\colon X \to Y$ be an ε-isometry with $f(0) = 0$. Then the limit $U_f(x) = \lim_{|t|\to\infty} f(tx)/t$ exists for each $x \in X$, the map $U_f\colon X \to Y$ is a linear isometry, and*

$$\|f(x) - U_f x\| \le (\|x\| + \varepsilon)\delta^{-1}\left(\frac{3\varepsilon}{\|x\| + \varepsilon}\right) \tag{1}$$

for every $x \in X$ with $\|x\| > 2\varepsilon$ (where δ^{-1} is the inverse function of δ).

We also need the next statement which follows from an inspection of the proof of [ŠV, Theorem 3.2]; we give the details for the convenience of the reader.

PROPOSITION 2. *Let $f\colon X \to Y$ be an ε-isometry with $f(0) = 0$. If there is a surjective linear isometry $U\colon X \to Y$ such that $\|f(x) - Ux\| = o(\|x\|)$ as $\|x\| \to \infty$, then $\|f(x) - Ux\| \le 2\varepsilon$ for all $x \in X$.*

Proof. Replacing f by $U^{-1}f$, we may assume that $X = Y$ and that $U = \mathrm{id}$. Let $x \in X$ and set $\lambda = \|f(x) - x\|$. We have to show that

$$\lambda \leq 2\varepsilon. \tag{2}$$

If $\lambda = 0$, we have to do nothing. So, suppose that $\lambda > 0$ and put $u = (x - f(x))/\lambda$ (so $\|u\| = 1$). For $s > 0$, set $x_s = x + su$ and $y_s = f(x_s) - f(x)$, and

$$\alpha = \limsup_{s\to\infty}(\|f(x_s)\| - \|y_s\|).$$

Observe that $|\alpha| \leq \|f(x)\| < \infty$. Since $\|f(x_s) - x_s\| = o(s)$ as $s \to \infty$ by the assumption,

$$\lim_{s\to\infty} y_s/s = \lim_{s\to\infty} f(x_s)/s = \lim_{s\to\infty} x_s/s = u.$$

Let $0 < t < s$. Then

$$\|f(x_s)\| - \left\|f(x) + \frac{t}{s}\,y_s\right\| \leq \left\|f(x_s) - f(x) - \frac{t}{s}\,y_s\right\| = \left(1 - \frac{t}{s}\right)\|y_s\|.$$

Consequently,

$$\|f(x_s)\| - \|y_s\| \leq \left\|f(x) + t\,\frac{y_s}{s}\right\| - t\left\|\frac{y_s}{s}\right\|.$$

As $s \to \infty$, this yields

$$\alpha \leq \|f(x) + tu\| - t\|u\| = \|x + (t - \lambda)u\| - (t - \lambda)\|u\| - \lambda.$$

Since f is an ε-isometry, this implies that

$$\alpha \leq \|f(x_{t-\lambda})\| - \|f(x_{t-\lambda}) - f(x)\| + 2\varepsilon - \lambda = \|f(x_{t-\lambda})\| - \|y_{t-\lambda}\| + 2\varepsilon - \lambda$$

for every $t > \lambda$. Letting $t \to \infty$ gives (2). ∎

The next statement extends [Va, 5.4].

THEOREM 3. *Let X and Y be real Banach spaces, and let Y be uniformly convex. Let $f\colon X \to Y$ be an ε-isometry with $f(0) = 0$ and*

$$\sup_{y\neq 0\in Y}\ \liminf_{|t|\to\infty} \frac{\mathrm{dist}(ty, f(X))}{\|ty\|} = q < 1. \tag{3}$$

Then there is a (unique) surjective linear isometry $U\colon X \to Y$ such that

$$\|f(x) - Ux\| \leq 2\varepsilon.$$

Proof. The desired isometry is U_f (see Proposition 1), and we only have to show that U_f is surjective. The theorem then follows from Proposition 2, because of (1).

Assume U_f is not surjective. Let $q < q' < 1$. Since $U_f X$ is a closed linear subspace of Y, there is a unit vector $y \in Y$ such that $\mathrm{dist}(y, U_f X) > q'$. Then by (3), there are two sequences $\{x_n\} \subset X$ and $\{t_n\} \subset \mathbb{R}$ with $|t_n| \to \infty$ such that $\lim_{n\to\infty}\|y - f(x_n)/t_n\| < q'$. Hence $2 > \|f(x_n)\|/|t_n| > 1 - q'$ for large n's. Since $\big|\|f(x)\| - \|x\|\big| \le \varepsilon$, we have $\|x_n\|/|t_n| < 2 + \varepsilon/|t_n|$ and $\|x_n\| \to \infty$. Therefore by (1),

$$\lim_{n\to\infty} \frac{\|f(x_n) - U_f x_n\|}{|t_n|} = \lim_{n\to\infty} \frac{\|f(x_n) - U_f x_n\|}{\|x_n\|} \frac{\|x_n\|}{|t_n|} = 0$$

and hence

$$\limsup_{n\to\infty} \|y - U_f x_n/t_n\| \le \limsup_{n\to\infty}(\|y - f(x_n)/t_n\| + \|f(x_n) - U_f x_n\|/|t_n|) < q',$$

a contradiction. ∎

Remark 4. Theorem 3 does not hold for $Y = l^2_\infty$, for instance. This follows from the next simple example: Consider the map $f(t) = (t, |t|)$ from $\mathbb{R}$ to l^2_∞. This is a non-linear isometry, and it is easy to check that

$$\sup_{y\neq 0\in Y} \liminf_{|t|\to\infty} \frac{\operatorname{dist}(ty, f(X))}{\|ty\|} = \frac{1}{2}.$$

3. Open problem. The results above raise the following geometric problem.

Problem 1. *Describe all Banach spaces Y with the following property: If X is a Banach space and $f\colon X \to Y$ is an ε-isometry such that*

$$\operatorname{dist}(y, f(X)) = o(\|y\|), \tag{4}$$

then it necessarily follows that

$$\sup_{y\in Y}\{\operatorname{dist}(y, f(X))\} < \infty \tag{5}$$

(and so, by the result of Šemrl and Väisälä [ŠV] mentioned in the Introduction, there exists a surjective linear isometry $U\colon X \to Y$ such that $\|f(x) - f(0) - Ux\| \le 2\varepsilon$).

We do not know whether (4) *implies that X and Y are linearly isomorphic.*

We do not know whether every ε-isometry $f\colon Y \to Y$ (i.e., the special case $X = Y$) which satisfies (4) *satisfies also* (5).

In addition to uniformly convex spaces, all finite-dimensional spaces also have the property from Problem 1.

Indeed, if $\dim X = \dim Y < \infty$, the assertion follows from [D, Theorem 1].

If $\dim X < \dim Y$, then there is no ε-isometry $f\colon X \to Y$ satisfying (4). This follows, for instance, from the next lemma (as well as from other simple topological considerations).

LEMMA 5. *Let X and Y be normed spaces (not necessarily real). Let $\dim X < \infty$ and Y be separable. If there exists a map $f : X \to Y$ satisfying* (4) *and*

$$\limsup_{\substack{x,y\in X\\ \|f(x)\|+\|f(y)\|\to\infty}} \left|\frac{\|f(x)-f(y)\|}{\|x-y\|}-1\right| = q < 1,$$

then Y is $(1-q)^{-1}$-bi-Lipschitz embeddable into X.

Proof. Assume $f(0) = 0$. Let $y \neq 0 \in Y$. By (4), there is a sequence $\{x_n\}$ in X such that $\|ny - f(x_n)\| = o(n)$. Hence $\lim_{n\to\infty} f(x_n)/n = y$, and it follows that $\{x_n/n\}$ is a bounded sequence. Indeed, otherwise we could choose a subsequence $\{x_{j_n}\}$ such that $\|x_{j_n}\| > 4j_n\|y\|/(1-q)$ and $\|f(x_{j_n})\| < 2j_n\|y\|$, which gives

$$1 - \frac{\|f(x_{j_n})\|}{\|x_{j_n}\|} > \frac{1+q}{2} > q,$$

a contradiction. It follows that there is an increasing sequence $\{k_n(y)\} \subset \mathbb{N}$ so that the limit $\varphi(y) = \lim_{n\to\infty} x_{k_n(y)}/k_n(y)$ exists.

Choosing a dense subset $S = \{y_1, y_2, \ldots\}$ in $Y \setminus \{0\}$, applying this procedure to find sequences $\{x_{i,n}\}$ in X such that $\lim_{n\to\infty} f(x_{i,n})/n = y_i$ and using the Cantor diagonal procedure we can find an increasing sequence $\{k_n\} \subset \mathbb{N}$ so that the limit

$$\varphi(y_i) = \lim_{n\to\infty} \frac{x_{i,k_n}}{k_n}$$

exists for all i. Then it follows from the assumption that

$$\left|\frac{\|y_i - y_j\|}{\|\varphi(y_i)-\varphi(y_j)\|} - 1\right| = \lim_{n\to\infty}\left|\frac{\|f(x_{i,k_n}) - f(x_{j,k_n})\|}{\|x_{i,k_n} - x_{j,k_n}\|} - 1\right| \le q.$$

Thus φ is $(1-q)^{-1}$-bi-Lipschitz on S, and extends by continuity to all of Y. ∎

Note that Lemma 5 fails if X is infinite-dimensional. Indeed, by an example of Lindenstrauss and Szankowski [LS, Theorem 2], there exist a Hilbert space X, a non-Hilbert space Y and a *surjective* map $f\colon X \to Y$ such that $\big|\|f(y) - f(x)\| - \|y - x\|\big| = o(\|y - x\|)$ as $\|y - x\| \to \infty$ (thus $q = 0$, but Y is not isometrically embeddable into X).

Note also that if we replace "an ε-isometry" by "a true isometry" in the formulation of Problem 1, then such an isometry is necessarily surjective. More precisely, it follows from a result of Figiel, Šemrl and Väisälä [FŠV] that if f is an *into* isometry with $f(0) = 0$ such that for every unit vector $y \in Y$ there exist $x \in X$ and a real number s so that

$$\|y - sf(x)\| < 1/2,$$

then f is surjective.

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