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ON GAUSSIAN KERNEL ESTIMATES ON GROUPS

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Abstract. We give new and simple sufficient conditions for Gaussian upper bounds for a convolution semigroup on a unimodular locally compact group. These conditions involve certain semigroup estimates in $L^2(G)$. We describe an application for estimates of heat kernels of complex subelliptic operators on unimodular Lie groups.

1. Introduction. Many authors have investigated the question of obtaining pointwise upper bounds for the heat kernel of a suitable differential operator, on various classes of manifolds and Lie groups (see, for example, [17, 16, 2] for relevant background). In this paper, we offer a new method of obtaining Gaussian upper bounds for a convolution semigroup on a locally compact group, which in particular applies to heat kernels on Lie groups.

Our main results show that Gaussian upper bounds are a consequence of certain L^2 operator estimates on the semigroup. In comparison, many standard approaches to Gaussian estimates (see the references above) rely on L^1 estimates on the semigroup. Such L^1 estimates may be difficult to obtain, unless one is dealing with the special case of heat kernels for second order operators with real coefficients. There are a number of works dealing with heat kernel estimates on Lie groups outside this special case: see [16, 7, 1, 8, 12, 9, 11, 5, 10] and references therein. But in general, these works seem to rely on specific structural features of the Lie groups considered, for example, local or global scaling properties. Our approach is more general and applies to large classes of convolution semigroups on (unimodular) locally compact groups.

In many cases the assumptions of our main theorems are necessary, as well as sufficient, for Gaussian estimates to hold (for a precise statement, see Remark 2.6 below). Partly for this reason, we suspect that many of the known examples of Gaussian estimates on groups could be derived using our methods.

The present paper can be seen as an extension of [6], where L^∞ kernel bounds were considered. We emphasize that, as in [6], our arguments and results rely crucially on the group invariance of the semigroups considered.

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In fact, for semigroups generated by elliptic operators in $\mathbb{R}^d$ which are not translation invariant, it is well known that the L^2 theory does not always suffice to give Gaussian estimates, or L^p estimates: see, for example, [4, 3] and references therein.

We present our theoretical results in Section 2 below. In Section 3 we give a sample application: a new proof of Gaussian heat kernel estimates of [10] (see also [7]) for second order complex operators on a Lie group of polynomial growth. Compared with earlier proofs, our proof is rather more direct and has the advantage of applying to any unimodular Lie group.

There are other applications of our methods which will not be detailed here. For example, our results would probably yield a new proof of the Gaussian estimates for convolution powers of a probability density obtained in [13], or could be used to extend the theory of heat kernels corresponding to operators which are sums of even powers of vector fields on a Lie group (compare [12, 9, 11]). It might also be possible to alter our method to prove pointwise estimates which do not have the standard Gaussian form. Some of these ideas might be explored in future.

2. The basic theorems. This section contains our main theoretical results. The two essential theorems are Theorem 2.3, which gives large time Gaussian estimates for convolution powers of a fixed function, and Theorem 2.7, which gives small time Gaussian estimates for a semigroup of operators. In applications to heat kernels (see Section 3), one may use these theorems to estimate respectively the large and small time behaviour.

The proofs in this section can be regarded as a non-trivial extension of the proofs of L^∞ estimates in [6].

Let us fix notation. Throughout, G will denote a unimodular, second countable locally compact group, with identity element e. We fix a Haar measure dg and consider the spaces $L^p = L^p(G; dg)$, $1 \le p \le \infty$, with norms $\|\cdot\|_p$. The norm of a bounded linear operator $T\colon L^p \to L^q$ is written as $\|T\|_{p\to q}$, or simply as $\|T\|$ in the case $p = q = 2$. In general, c, c', b and so on denote positive constants whose value may change from line to line when convenient.

Given a locally integrable function $f\colon G \to \mathbb{C}$, we set $(L(g)f)(h) = f(g^{-1}h)$, $g, h \in G$, that is, L is the left regular representation of G, and we define the convolution operator $L(f)$ by

$$(L(f)f_1)(g) = (f * f_1)(g) = \int_G dh\, f(h) f_1(h^{-1}g)$$

for $g \in G$ and suitable functions $f_1\colon G \to \mathbb{C}$. This makes sense at least when f_1 is bounded and compactly supported, and hence the domain of $L(f)$ is dense in L^p for $1 \le p < \infty$.

Let us say that a Borel measurable function $w\colon\ G \to (0,\infty)$ is a *weight function* on G if w and $1/w = w^{-1}$ are both locally bounded functions on G (that is, they are bounded over any compact subset of G), and moreover $w(g) = w(g^{-1})$ for all $g \in G$.

For convenience, we also denote by w the operator of pointwise multiplication $f \mapsto wf$. For locally integrable f, we can then consider the (possibly infinite) norms

$$\|wL(f)w^{-1}\|_{p\to q} = \sup\{\|wL(f)w^{-1}f_1\|_q\colon\ f_1 \in C_{\mathrm{c}}(G),\ \|f_1\|_p \le 1\}$$

when $p < \infty$, where $C_{\mathrm{c}}(G)$ denotes the continuous, compactly supported functions on G.

We will need the following inequalities.

Lemma 2.1. *If w is a weight function, then*

$$\begin{aligned} \|w(f_1 * f_2)\|_2 &\le \|wL(f_1)w^{-1}\|\,\|wf_2\|_2, \\ \|w(f_2 * f_1)\|_2 &\le \|w^{-1}L(f_1)w\|\,\|wf_2\|_2 \end{aligned} \tag{1}$$

whenever f_1, f_2 are locally integrable functions on G such that the norms on the right sides of the inequalities are finite.

Proof. By density, it is enough to prove inequalities (1) in the case that f_2 is compactly supported and bounded (in fact, only this case will be needed in what follows). The first inequality is immediate from the definitions.

To prove the second inequality, introduce for each locally integrable f the function

$$\widetilde{f}(g) = \overline{f(g^{-1})}, \qquad g \in G. \tag{2}$$

By unimodularity, $\|\widetilde{f}\|_p = \|f\|_p$ for all p, and in particular $\|wf\|_2 = \|w\widetilde{f}\|_2$ because $w(g) = w(g^{-1})$. Observe also that $(f_2 * f_1)^{\sim} = \widetilde{f}_1 * \widetilde{f}_2$. Therefore,

$$\|w(f_2 * f_1)\|_2 = \|w(f_2 * f_1)^{\sim}\|_2 = \|w(\widetilde{f}_1 * \widetilde{f}_2)\|_2 \le \|wL(\widetilde{f}_1)w^{-1}\|\,\|w\widetilde{f}_2\|_2.$$

But it is easily checked that $wL(\widetilde{f}_1)w^{-1}$ is formally adjoint to $w^{-1}L(f_1)w$ with respect to the L^2 inner product, so that $\|wL(\widetilde{f}_1)w^{-1}\| = \|w^{-1}L(f_1)w\|$. The lemma follows. ■

We introduce some convenient (though possibly non-standard) terminology. A *modulus* is a Borel measurable function $\varrho\colon\ G \to [0,\infty)$ such that

$$B_\varrho(r) := \{g \in G : \varrho(g) \le r\}$$

is a relatively compact subset of G for each $r > 0$. The Haar measure of $B_\varrho(r)$ will usually be denoted by $V_\varrho(r) = dg(B_\varrho(r))$.

An *admissible modulus* is a modulus which is also locally bounded and satisfies, for some $c_0 \ge 1$,

$$\varrho(g) = \varrho(g^{-1}), \qquad \varrho(gh) \le c_0(\varrho(g) + \varrho(h)) \tag{3}$$

for all $g, h \in G$. We say that ϱ is *subadditive* if the inequality in (3) holds with $c_0 = 1$. Obviously, if ϱ is an admissible modulus, then $e^{\lambda\varrho}$ is a weight function for all $\lambda \in \mathbb{R}$.

Heuristically, we think of $\varrho(g)$ as the "distance" from g to e, and our Gaussian estimates in this section will be formulated using ϱ.

In passing, we mention that standard examples of admissible modulus functions are (i) on a Lie group G, the Carathéodory modulus associated with a list of generators of the Lie algebra (see [17, 16]), and (ii) on a compactly generated group G, the modulus $\varrho_U(g) = \inf\{n \in \mathbb{N} : g \in U^n\}$ where $U = U^{-1} \subseteq G$ is a fixed compact generating neighbourhood of the identity e. There are many other examples relevant for analysis, for example, "weighted" modulus functions on a Lie group corresponding to filtrations of the Lie algebra (see [15, Section 4] and [9]).

The following lemma will be a crucial tool. Denote by I the identity operator acting on functions over G.

LEMMA 2.2. *Let w be a weight function and let ϱ be a modulus on G. For $r > 0$ write $\|w\|_{\infty,r} = \sup\{w(g) : \varrho(g) \le r\}$. If f is a locally integrable function on G and $wf \in L^2$, then*

$$\|wf\|_2 \le \sup_{g\in G,\, \varrho(g)\le r} \|w(I - L(g))f\|_2 + \|w\|_{\infty,r} V_\varrho(r)^{-1/2} \|w^{-1}L(f)w\|$$

for all $r > 0$ (here, the right side is permitted to be infinite).

REMARK. In the unweighted case $w \equiv 1$, the above inequality was proved by the author in [6], where it is referred to as a convolution Nash inequality. The reason for this name is that it differs essentially from standard Nash inequalities (see [16] for example) by replacing the L^1 norm of f with the convolution norm $\|L(f)\|$. Thus, it seems natural to refer to the inequality of Lemma 2.2 as a *weighted convolution Nash inequality.*

Proof of Lemma 2.2. Let $r > 0$ be given. If $V_\varrho(r) = 0$ then we interpret $V_\varrho(r)^{-1/2} = \infty$, so let us assume that $V_\varrho(r) > 0$.

Following an idea of Robinson [16, p. 267], we consider the function $\chi = V_\varrho(r)^{-1} 1_{B_\varrho(r)}$, where 1_E denotes the characteristic function of a subset $E \subseteq G$. Observing that $\int_G \chi = 1$, we find the identity

$$wf = w(\chi * f) + \int_G dg\, \chi(g)\, w(I - L(g))f$$

and take L^2 norms on both sides. Applying (1) yields

$$\|w(\chi * f)\|_2 \le \|w\chi\|_2 \|w^{-1}L(f)w\| \le \|w\|_{\infty,r} V_\varrho(r)^{-1/2} \|w^{-1}L(f)w\|,$$

and the lemma follows easily. ∎

In the following theorem, we consider a fixed function $K \in L^2$ and obtain Gaussian estimates for $K_{(n)}$, $n \in \mathbb{N} = \{1, 2, \ldots\}$, where $K_{(n)} = K * \cdots * K$ denotes the nth convolution power of K. Observe that, if one assumes that the operator $T = L(K)$ is bounded in L^2, then the $K_{(n)} = T^{n-1}K$ are well defined elements of L^2.

THEOREM 2.3. *Let $K \in L^2$ be such that $T = L(K)$ is bounded in L^2. Suppose $\varrho\colon G \to [0, \infty)$ is an admissible modulus, and denote by U_λ the multiplication operator $f \mapsto e^{\lambda\varrho} f$ for $\lambda \in \mathbb{R}$. Assume there exist an even positive integer $M \geq 2$ and constants $\omega > 0$, $\nu \in (0, 1]$ such that*

$$\|U_\lambda K\|_2 \leq ce^{\omega\lambda^M} \tag{4}$$

for all $\lambda \geq 0$,

$$\|U_\lambda T^n U_{-\lambda}\| \leq ce^{\omega\lambda^M n} \tag{5}$$

for all $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$, and

$$\|U_\lambda(I - L(g))T^n U_{-\lambda}\| \leq c(\varrho(g)n^{-1/M})^\nu e^{\omega\lambda^M n} \tag{6}$$

for all $\lambda \in \mathbb{R}$, $n \in \mathbb{N}$ and $g \in G$ such that $\varrho(g) \leq n^{1/M}$. Suppose there are $a, D > 0$ with $V_\varrho(r) \geq ar^D$ for all $r \geq 1$. Then there exist $c', b > 0$ with

$$|K_{(n)}(g)| \leq c'n^{-D/M} e^{-b(\varrho(g)^M/n)^{1/(M-1)}} \tag{7}$$

for all $n \in \mathbb{N}$ with $n \geq 2$ and $g \in G$, and

$$|K_{(n)}(g) - K_{(n)}(g_1^{-1}g)| \leq c'(\varrho(g_1)n^{-1/M})^\nu n^{-D/M} e^{-b(\varrho(g)^M/n)^{1/(M-1)}} \tag{8}$$

for all $n \in \mathbb{N}$ with $n \geq 3$ and $g, g_1 \in G$ such that $\varrho(g_1) \leq n^{1/M}$.

REMARK. Davies [3, Section 4] establishes L^2 estimates of a type analogous to (5) and (6), for the semigroups generated by a large class of divergence-form elliptic operators in $\mathbb{R}^d$. But for non-translation invariant elliptic operators, such estimates are not in general sufficient to guarantee Gaussian heat kernel bounds. We can therefore say that the group invariance plays an essential role in Theorem 2.3, as well as in Theorem 2.7 below.

Proof of Theorem 2.3. We first observe that $K_{(n+m)} = T^n K_{(m)}$, so that

$$\|U_\lambda K_{(n+m)}\|_2 \leq \|U_\lambda T^n U_{-\lambda}\| \, \|U_\lambda K_{(m)}\|_2 \leq ce^{\omega\lambda^M n}\|U_\lambda K_{(m)}\|_2 \tag{9}$$

for all $n, m \in \mathbb{N}$ and $\lambda \geq 0$. By choosing $m = 1$ and recalling (4), it follows that

$$\|U_\lambda K_{(n)}\|_2 \leq c'e^{\omega\lambda^M n} \tag{10}$$

for all $n \in \mathbb{N}$ and $\lambda \geq 0$. This last bound lacks a desired factor of $n^{-D/(2M)}$, but will be useful for small n.

Let us fix an $\omega' > \omega$, and define, for each $n \in \mathbb{N}$ and $\lambda \geq 0$,

$$\beta_{n,\lambda} = n^{D/(2M)} e^{-\omega' \lambda^M n} \|U_\lambda K_{(n)}\|_2.$$

Note that

$$\begin{aligned} \|U_\lambda (I - L(g)) K_{(n+m)}\|_2 &\leq \|U_\lambda (I - L(g)) T^n U_{-\lambda}\| \, \|U_\lambda K_{(m)}\|_2 \\ &\leq c(\varrho(g) n^{-1/M})^\nu e^{\omega \lambda^M n} \|U_\lambda K_{(m)}\|_2 \end{aligned} \tag{11}$$

for all $n, m \in \mathbb{N}$, $\lambda \geq 0$ and $g \in G$ with $\varrho(g) \leq n^{1/M}$. Now in Lemma 2.2 put $f = K_{(2n)}$ and choose the weight function $w = w_\lambda = e^{\lambda \varrho}$, $\lambda \geq 0$. Then by (11) with $m = n$, and because

$$\|w_\lambda^{-1} L(K_{(2n)}) w_\lambda\| = \|U_{-\lambda} T^{2n} U_\lambda\| \leq c e^{2\omega \lambda^M n},$$

we obtain an estimate

$$\beta_{2n,\lambda} \leq c_1 (r n^{-1/M})^\nu \beta_{n,\lambda} + c_1 (r n^{-1/M})^{-D/2} e^{\lambda r} e^{-2(\omega' - \omega) \lambda^M n} \tag{12}$$

for all $\lambda \geq 0$, $n \in \mathbb{N}$ and $r \geq 1$ such that $r \leq n^{1/M}$. Fix $\varepsilon \in (0, 1)$ small enough so that

$$c_1 \varepsilon^\nu < 2^{-1}, \qquad \varepsilon < 2(\omega' - \omega),$$

and fix $k_0 \in \mathbb{N}$ with $2^{k_0} > \varepsilon^{-M}$. If $n \geq 2^{k_0}$ then $\varepsilon n^{1/M} \geq 1$. Therefore, we may choose $r = \varepsilon n^{1/M}$ in (12) to obtain

$$\beta_{2n,\lambda} \leq 2^{-1} \beta_{n,\lambda} + c_2 e^{\varepsilon \lambda n^{1/M} - 2(\omega' - \omega) \lambda^M n}$$

for all $n \in \mathbb{N}$ with $n \geq 2^{k_0}$ and all $\lambda \geq 0$. Since

$$\varepsilon \lambda n^{1/M} \leq \varepsilon (1 + \lambda^M n) \leq 2(\omega' - \omega)(1 + \lambda^M n),$$

we have

$$\beta_{2n,\lambda} \leq 2^{-1} \beta_{n,\lambda} + c_3 \leq \max\{\beta_{n,\lambda}, 2c_3\}$$

for all $n \in \mathbb{N}$ with $n \geq 2^{k_0}$ and all $\lambda \geq 0$. This inequality implies, by induction, that

$$\beta_{2^k,\lambda} \leq \max\{\beta_{2^{k_0},\lambda}, 2c_3\}$$

for all $k \in \mathbb{N}$ with $k \geq k_0$. But (10) shows that $\sup_{\lambda \geq 0} \beta_{2^{k_0},\lambda} < \infty$, and consequently there is a $c_4 > 0$ such that

$$\beta_{2^k,\lambda} \leq c_4$$

for all $k \in \mathbb{N}$ with $k \geq k_0$ and all $\lambda \geq 0$. In other words,

$$\|U_\lambda K_{(n)}\|_2 \leq c_4 n^{-D/(2M)} e^{\omega' \lambda^M n} \tag{13}$$

whenever $n = 2^k$ with $k \in \mathbb{N}$, $k \geq k_0$, and $\lambda \geq 0$.

Let us remove the restriction on n in (13). By applying (9) with $m = 2^k$, $k \geq k_0$, and adjusting the value of c_4, we can easily see that (13) holds for all integers $n \geq 2^{k_0}$ and all $\lambda \geq 0$. In case $1 \leq n \leq 2^{k_0}$, a bound of the form (13) follows immediately from (10). Therefore, a bound of the form (13) holds for all $n \in \mathbb{N}$ and $\lambda \geq 0$.

The technique of obtaining Gaussian estimates (7) from the inequality (13) is essentially well known. First use (3) to observe that

$$\begin{aligned} e^{\lambda\varrho(g)}|K_{(n+m)}(g)| &\leq \int_G dh\, e^{c_0\lambda\varrho(h)}|K_{(n)}(h)|e^{c_0\lambda\varrho(h^{-1}g)}|K_{(m)}(h^{-1}g)| \\ &\leq \|U_{c_0\lambda}K_{(n)}\|_2\|U_{c_0\lambda}K_{(m)}\|_2 \end{aligned}$$

for $n, m \in \mathbb{N}$ and $\lambda \geq 0$. Then choose $m = n$ or $m = n+1$, apply (13), and set $\lambda = \tau(\varrho(g)/n)^{1/(M-1)}$ for a suitable small $\tau > 0$.

Similarly, one has

$$e^{\lambda\varrho(g)}|K_{(n+m)}(g) - K_{(n+m)}(g_1^{-1}g)| \leq \|U_{c_0\lambda}(I - L(g_1))K_{(n)}\|_2\|U_{c_0\lambda}K_{(m)}\|_2.$$

Thus, by applying (13) and (11), we can deduce estimates (8) for $n \geq 3$, under the condition that $\varrho(g_1) \leq \kappa n^{1/M}$ for some small constant $\kappa \in (0,1)$.

Finally, in case $\kappa n^{1/M} \leq \varrho(g_1) \leq n^{1/M}$, by (3) there is $c > 0$ with $\varrho(g_1^{-1}g)^M \geq c^{-1}\varrho(g)^M - cn$. Hence, in this case estimates of the form (8) follow directly from (7). The proof of Theorem 2.3 is complete. ∎

REMARK 2.4. It is a useful technical remark that, if ϱ is subadditive, then the hypothesis (6) in Theorem 2.3 can be replaced by the alternative form

$$\|(I - L(g))U_\lambda T^n U_{-\lambda}\| \leq c(\varrho(g)n^{-1/M})^\nu e^{\omega\lambda^M n} \tag{14}$$

for $n \in \mathbb{N}$, $g \in G$ with $\varrho(g) \leq n^{1/M}$. To see this, note the general identity

$$(e^\psi(I - L(g))e^{-\psi}f)(h) = f(h) - f(g^{-1}h) + [1 - e^{\psi(h)-\psi(g^{-1}h)}]f(g^{-1}h)$$

for $g, h \in G$ and functions f and ψ on G. This leads, via the bound $|1-e^s| \leq |s|e^{|s|}$, $s \in \mathbb{R}$, to an inequality

$$\begin{aligned} &\|e^\psi(I - L(g))e^{-\psi}f\|_2 \\ &\qquad \leq \|(I - L(g))f\|_2 + \|(I - L(g))\psi\|_\infty e^{\|(I-L(g))\psi\|_\infty}\|f\|_2. \end{aligned} \tag{15}$$

Now if (14) holds, then it is not hard to deduce (6) by setting $\psi = \lambda\varrho$ in (15) and adjusting the constant ω. Note that $\|(I - L(g))\varrho\|_\infty \leq \varrho(g)$ since ϱ is assumed subadditive. We leave further details to the reader.

The next result is a well known consequence of the Gaussian estimates (7).

COROLLARY 2.5. *Assume the hypotheses of Theorem* 2.3, *and suppose that* $c^{-1}r^D \leq V_\varrho(r) \leq cr^D$ *for all* $r \geq 1$. *Then there exists* $c' > 0$ *with*

$$\|T^n\|_{p\to p} \leq c'$$

uniformly for all $1 \leq p \leq \infty$ *and* $n \in \mathbb{N}$, $n \geq 2$.

Proof. A standard integration of the Gaussian estimates (7), using a volume doubling bound $V_\varrho(2r) \le c' V_\varrho(r)$, $r \ge 1$, yields $\|K_{(n)}\|_1 \le c'' < \infty$ for all $n \ge 2$ (compare, for example, [17, p. 111]). The result follows since $\|T^n\|_{p\to p} = \|L(K_{(n)})\|_{p\to p} \le \|K_{(n)}\|_1$. ∎

REMARK 2.6. It should be noted that Theorem 2.3 often has a converse, so that in many cases the hypotheses of Theorem 2.3 are necessary conditions for Gaussian estimates.

To be more precise, suppose that the admissible modulus ϱ is subadditive, and that $c^{-1} r^D \le V_\varrho(r) \le c r^D$, $r \ge 1$. Then we claim that the Gaussian estimates (7) and (8) imply estimates of the form

$$\|U_\lambda T^n U_{-\lambda}\|_{p\to p} \le c' e^{\omega \lambda^M n},$$
$$\|U_\lambda (I - L(g)) T^n U_{-\lambda}\|_{p\to p} \le c' (\varrho(g) n^{-1/M})^\nu e^{\omega\lambda^M n}$$

whenever $\lambda \in \mathbb{R}$, $p \in [1,\infty]$ and $\varrho(g) \le n^{1/M}$. In particular, when $p = 2$ we recover (5) and (6).

To prove this claim, one observes that $U_\lambda T^n U_{-\lambda}$ has an integral kernel

$$K_{n,\lambda}(g,h) = e^{\lambda \varrho(g)} K_{(n)}(gh^{-1}) e^{-\lambda\varrho(h)}$$

for $g, h \in G$, so that

$$|K_{n,\lambda}(g,h)| \le e^{|\lambda| \varrho(gh^{-1})} |K_{(n)}(gh^{-1})|$$

by the assumed subadditivity of ϱ. (In fact, here subadditivity could be replaced by a weaker condition $|\varrho(g) - \varrho(h)| \le c_1 \varrho(gh^{-1})$ for all $g, h \in G$.) Then the estimates on $\|U_\lambda T^n U_{-\lambda}\|_{p\to p}$ follow by a standard integration argument from the Gaussian estimates (7). Similarly, one derives estimates on $U_\lambda(I - L(g_1)) T^n U_{-\lambda}$ using the estimates (8). We omit further details. ∎

We will now give an analogue of Theorem 2.3 for small time Gaussian estimates. The following result differs from Theorem 2.3 in that one does not assume the existence of a convolution kernel, but only of a (right invariant) semigroup of bounded operators in L^2. The existence of a kernel K_t is deduced in the conclusion.

THEOREM 2.7. *Let $t_0 \in (0,\infty)$ and suppose $\{T_t\}_{0<t<t_0}$ is a family of right invariant, bounded operators in L^2, with $T_{s+t} = T_s T_t$ whenever s, t, $s + t \in (0, t_0)$. Let $\varrho\colon G \to [0,\infty)$ be an admissible modulus, which is continuous, subadditive and satisfies $\varrho(e) = 0$. Set $U_\lambda f = e^{\lambda\varrho} f$ for $\lambda \in \mathbb{R}$. Assume that M is an even positive integer, $\omega > 0$, and $\nu \in (0,1]$, such that*

$$\|U_\lambda T_t U_{-\lambda}\| \le c e^{\omega\lambda^M t}, \qquad \|U_\lambda (I - L(g)) T_t U_{-\lambda}\| \le c(\varrho(g) t^{-1/M})^\nu e^{\omega\lambda^M t}$$

for all $t \in (0, t_0)$, $\lambda \in \mathbb{R}$ and $g \in G$ with $\varrho(g) \le t^{1/M}$. Suppose there are $a, D > 0$ with $V_\varrho(r) \ge a r^D$ for all $r \in (0,1)$. Then there exist functions

$K_t \in L^2 \cap L^\infty$ with $T_t = L(K_t)$ and $K_{t+s} = K_t * K_s$ when $s, t, s+t \in (0, t_0)$. Moreover, there are $c', b > 0$ such that

$$|K_t(g)| \le c' t^{-D/M} e^{-b(\varrho(g)^M/t)^{1/(M-1)}},$$

$$|K_t(g_1^{-1}g) - K_t(g)| \le c'(\varrho(g_1)t^{-1/M})^\nu t^{-D/M} e^{-b(\varrho(g)^M/t)^{1/(M-1)}}$$

for all $t \in (0, t_0)$ and $g, g_1 \in G$ such that $\varrho(g_1) \le t^{1/M}$.

Proof. The proof is a variation of the proof of Theorem 2.3, and we only outline the main ideas. The key point is to establish an estimate

$$\|U_\lambda T_t U_{-\lambda}\|_{1\to 2} \le c t^{-D/(2M)} e^{\omega' \lambda^M t} \tag{16}$$

for some $\omega' > \omega$ and all $t \in (0, t_0)$ and $\lambda \ge 0$. To do this, suppose that $\lambda \ge 0$ and $f \in C_c(G)$ with $\|U_\lambda f\|_1 \le 1$, and put

$$\beta_t = t^{D/(2M)} e^{-\omega' \lambda^M t} \|U_\lambda T_t f\|_2.$$

To establish (16), we will bound β_t by a constant independent of t, λ and f (subject to the above conditions on λ and f). One checks, using the subadditivity of ϱ, that $\|U_{-\lambda} L(f) U_\lambda\| \le \|U_\lambda f\|_1 \le 1$. The right invariance of T_t implies that $L(T_t f) f_2 = T_t L(f) f_2$ for arbitrary $f_2 \in C_c(G)$. We thus have

$$\|U_{-\lambda} L(T_t f) U_\lambda\| \le \|U_{-\lambda} T_t U_\lambda\| \, \|U_{-\lambda} L(f) U_\lambda\| \le c e^{\omega \lambda^M t}.$$

Now apply Lemma 2.2 to the function $T_t f$, with $w = e^{\lambda\varrho}$ and $r = \varepsilon t^{1/M} \in (0, 1)$ for a fixed $\varepsilon > 0$, and note that

$$\|U_\lambda (I - L(g)) T_t f\| \le c'(\varrho(g) t^{-1/M})^\nu e^{\omega \lambda^M t/2} \|U_\lambda T_{t/2} f\|_2$$

whenever $\varrho(g) \le (t/2)^{1/M}$. Provided that ε is chosen sufficiently small, this yields an estimate

$$\beta_t \le 2^{-1} \beta_{t/2} + c_1 \le \max\{\beta_{t/2}, 2c_1\}$$

for all $t \in (0, t_0)$, where c_1 is independent of λ and f. Because $U_\lambda f \in L^2$, one has

$$\lim_{t\to 0} \beta_t \le \lim_{t \to 0} t^{D/(2M)} \|U_\lambda T_t U_{-\lambda}\| \, \|U_\lambda f\|_2 = 0.$$

It is then easy to see that $\beta_t \le 2c_1$ for all $t \in (0, t_0)$, and (16) follows.

It follows from the case $\lambda = 0$ of (16), and the right invariance of T_t, that there is a kernel $K_t \in L^2$ with $T_t = L(K_t)$ and $\|K_t\|_2 = \|T_t\|_{1\to 2}$. Then the operator $U_\lambda T_t U_{-\lambda}$ has an integral kernel given by

$$K_t^{(\lambda)}(g; h) = e^{\lambda \varrho(g)} K_t(gh^{-1}) e^{-\lambda \varrho(h)}$$

for $g, h \in G$, and

$$\sup_{h \in G} \Big(\int_G dg \, |K_t^{(\lambda)}(g; h)|^2 \Big)^{1/2} = \|U_\lambda T_t U_{-\lambda}\|_{1\to 2}.$$

Taking $h = e$ in the last expression, we infer from (16) that

$$\|U_\lambda K_t\|_2 \le ct^{-D/(2M)} e^{\omega' \lambda^M t}$$

for all $t \in (0, t_0)$ and $\lambda \ge 0$. The remainder of the proof follows the proof of Theorem 2.3. ∎

Theorem 2.7 has the following corollary which is analogous to Corollary 2.5. The proof is standard and is omitted.

COROLLARY 2.8. *Assume the hypotheses of Theorem* 2.7, *and suppose there exist* $c_1, \mu > 0$ *with*

$$V_\varrho(r) \le c_1 r^D e^{\mu r}$$

for all $r > 0$. *Then there is* $c' > 0$ *with*

$$\|T_t\|_{p\to p} \le \|K_t\|_1 \le c'$$

for all $t \in (0, t_0)$ *and* $p \in [1, \infty]$.

REMARK 2.9. One can extend the estimates of Theorem 2.7 to large values of t, by using the semigroup property.

To see this, let us assume the hypotheses of the theorem, and set $t_1 = 2^{-1} t_0$. If $t \ge t_0$, we write $t = nt_1 + s$ with $n \in \mathbb{N}$, $0 < s \le t_1$, and define T_t by

$$T_t = (T_{t_1})^n T_s.$$

Then $T_t = L(K_t)$ where, by definition, $K_t = (K_{t_1})_{(n)} * K_s$. One has $T_t T_{t'} = T_{t+t'}$ for all $t, t' > 0$. Moreover, there is a $\sigma > 0$ such that an estimate of the form

$$|K_t(g)| \le ct^{-D/M} e^{\sigma t} e^{-b(\varrho(g)^M/t)^{1/(M-1)}}$$

holds for all $t > 0$. Of course, since $\sigma > 0$ this bound does not give very precise control of $\|K_t\|_\infty$ for large t, but it is nevertheless of interest when $\varrho(g)^M/t$ is large.

To prove this bound on K_t, it suffices to get suitable estimates on $\|U_\lambda K_t\|_2$, $t \ge t_0$, $\lambda \ge 0$, and these may be obtained by writing

$$\|U_\lambda K_t\|_2 \le \|U_\lambda T_s U_{-\lambda}\| (\|U_\lambda T_{t_1} U_{-\lambda}\|)^{n-1} \|U_\lambda K_{t_1}\|_2,$$

where $t = nt_1 + s$ with $n \in \mathbb{N}$, $0 < s \le t_1$. We leave further details to the reader.

3. Complex second order operators. In this section, we present a new proof of the Gaussian estimates for second order subelliptic operators with complex coefficients, on a Lie group of polynomial growth (see [10, 7]). In fact, our proof will apply to arbitrary unimodular Lie groups.

Thus, in this section let G denote a connected unimodular Lie group with Lie algebra $\mathfrak{g}$. To each $x \in \mathfrak{g}$ we associate a right invariant vector field

$X = dL_G(x)$: as an operator, $dL_G(x) = \lim_{t\to 0} t^{-1}(L(\exp(tx)) - I)$ where $\exp\colon \mathfrak{g} \to G$ is the exponential map.

Let $a_1, \ldots, a_{d'} \in \mathfrak{g}$ be a list of elements which algebraically generate $\mathfrak{g}$, and set $A_i = dL_G(a_i)$. We consider, as in [10] or [7], a subelliptic differential operator of the form

$$H = -\sum_{i,j=1}^{d'} c_{ij} A_i A_j$$

on G, where the c_{ij} are complex constants which are assumed to satisfy $\mathrm{Re} \sum_{i,j} c_{ij} \xi_i \overline{\xi}_j \geq \mu |\xi|^2$ for some $\mu > 0$ and all $\xi \in \mathbb{C}^{d'}$. The most studied case (see [17, 16]) is where $c_{ij} = \delta_{ij}$ for all i, j, in which case $H = -\sum_i A_i^2$ is called a *sublaplacian.*

Note that H can be precisely defined using the theory of sectorial forms (see [14]), as the sectorial operator in L^2 associated with the quadratic form $Q(f) = \sum_{i,j=1}^{d'} \int_G c_{ij} A_j f A_i \overline{f}$ for $f \in L^2$ with $A_i f \in L^2$, $i \in \{1, \ldots, d'\}$. Then standard reasoning shows that H generates a holomorphic contraction semigroup $T_t = e^{-tH}$ in L^2, and $\|HT_t\| \leq ct^{-1}$ for all $t > 0$ (see, for example, [7, Section II.2]).

If H is a sublaplacian, then the general theory of Dirichlet forms shows that T_t is a contraction semigroup in L^p for all $1 \leq p \leq \infty$. But for general complex coefficients c_{ij}, this theory is not applicable and the study of H seems more difficult.

Let $\varrho = \varrho_{\mathbf{A}}$ denote the standard Carathéodory modulus on G associated with $A_1, \ldots, A_{d'}$ (see [17, 16]). We will see that it is not difficult to verify the hypotheses of Theorems 2.3 and 2.7.

Consider the set $\mathcal{D}$ consisting of all smooth bounded functions $\psi\colon G \to \mathbb{R}$ with $\|A_i \psi\|_\infty \leq 1$ for all $i \in \{1, \ldots, d'\}$. Let U_λ and U_λ^ψ denote respectively the operators of multiplication by $e^{\lambda\varrho}$ and $e^{\lambda\psi}$, for $\psi \in \mathcal{D}$, $\lambda \in \mathbb{R}$. From the formula

$$e^{\psi} A_i (e^{-\psi} f) = A_i f - (A_i \psi) f, \tag{17}$$

one sees that $U_\lambda^\psi H U_{-\lambda}^\psi$ is a differential operator which is a perturbation of H by lower order terms. Moreover, some standard arguments then give estimates of the form

$$\|U_\lambda^\psi T_t U_{-\lambda}^\psi\| \leq c e^{\omega\lambda^2 t}, \qquad \|A_i U_\lambda^\psi T_t U_{-\lambda}^\psi\| \leq c t^{-1/2} e^{\omega\lambda^2 t}$$

uniformly for all $t > 0$, $\lambda \in \mathbb{R}$ and $\psi \in \mathcal{D}$ (compare, for example, [7, pp. 276–277] or [3]). Now recall an elementary inequality (see [16, p. 268])

$$\|(I - L(g))f\|_2 \leq \varrho(g) \Big(\sum_{i=1}^{d'} \|A_i f\|_2^2 \Big)^{1/2} \tag{18}$$

for $g \in G$ and $f \in L^2$ such that $A_i f \in L^2$, $i \in \{1, \ldots, d'\}$. Using this estimate, and then arguing as in Remark 2.4, one easily deduces a bound

$$\|U_\lambda^\psi (I - L(g)) T_t U_{-\lambda}^\psi\| \le c(\varrho(g) t^{-1/2}) e^{\omega\lambda^2 t}$$

for all $t > 0$, $\lambda \in \mathbb{R}$, $\psi \in \mathcal{D}$ and $g \in G$ satisfying $\varrho(g) \le t^{1/2}$.

But ϱ can be approximated by elements of $\mathcal{D}$, by setting

$$\psi_n = (\min\{\varrho, n\}) * \varphi_n,$$

where $\{\varphi_n\}_{n=1}^\infty \subseteq C_c^\infty(G)$ is a suitable smooth approximation of the identity for G, with $\varphi_n \ge 0$ and $\int_G \varphi_n = 1$. It is not hard to show that $\psi_n \in \mathcal{D}$ and $\lim_{n\to\infty} \psi_n(g) = \varrho(g)$, where the limit is uniform over any compact subset of G. If W is any bounded operator in L^2, and $f_1, f_2 \in C_c(G)$, then

$$\int_G dg\, e^{\lambda\varrho(g)} (W U_{-\lambda} f_1)(g)\, \overline{f_2(g)} = \lim_{n\to\infty} \int_G dg\, (U_\lambda^{\psi_n} W U_{-\lambda}^{\psi_n} f_1)(g)\, \overline{f_2(g)}.$$

Therefore, a simple limiting argument gives the estimates

$$\|U_\lambda T_t U_{-\lambda}\| \le c e^{\omega\lambda^2 t}, \qquad \|U_\lambda (I - L(g)) T_t U_{-\lambda}\| \le c(\varrho(g) t^{-1/2}) e^{\omega\lambda^2 t}$$

for all $t > 0$ and $\varrho(g) \le t^{1/2}$.

It is well known (see for example [17]) that there is an integer $D' \ge 1$ such that $c^{-1} r^{D'} \le V_\varrho(r) \le c r^{D'}$ for $0 < r < 1$. We may apply Theorem 2.7 with $M = 2$ over any finite interval $(0, t_0)$, to deduce that $T_t = L(K_t)$, $t > 0$, where the kernel K_t satisfies

$$|K_t(g)| \le c t^{-D'/2} e^{-b\varrho(g)^2/t}$$

for $0 < t \le 1$. Note that the proof of Theorem 2.7 also gives an estimate of the form $\|U_\lambda K_t\|_2 \le c t^{-D'/4} e^{\omega\lambda^2 t}$, $\lambda \ge 0$, $0 < t \le 1$.

Suppose now that $D > 0$ with $V_\varrho(r) \ge a r^D$ for all $r \ge 1$. We can apply Theorem 2.3, with $K = K_1$, to get

$$|K_t(g)| \le c t^{-D/2} e^{-b\varrho(g)^2/t} \tag{19}$$

for $t \in \mathbb{N} = \{1, 2, \ldots\}$. The latter estimate is then easily extended to all $t \ge 1$ by using the semigroup property $K_t = K_{t-s} * K_s$ with $s \in \mathbb{N}$ (or alternatively, by arguing through the L^2 estimate $\|U_\lambda K_t\|_2 \le \|U_\lambda T_{t-s} U_{-\lambda}\|\, \|U_\lambda K_s\|_2$).

Note that the Gaussian estimate for small t was given in, for example, [9]. The estimate (19) for $t \ge 1$ is proved in [10] or [7] under the stronger assumption that G has polynomial growth of order precisely D, that is, $c^{-1} r^D \le V_\varrho(r) \le c r^D$ for $r \ge 1$. The present proof is more direct than these earlier proofs, since we do not need to exploit detailed structural properties of G. Moreover, our proof displays a unified approach to the small and large time estimates.

Finally, let us briefly consider the case that G has exponential growth, which means that for some $a > 0$ one has

$$V_\varrho(r) \geq a e^{ar}$$

for all $r \geq 1$. Then given any $D > 0$ one has an estimate of the form (19). We can improve this by using the fact that, for G of exponential growth, there are $c, \sigma > 0$ with

$$\|K_t\|_\infty \leq c e^{-\sigma t^{1/3}}$$

for all $t \geq 1$; this bound is well known for a sublaplacian ([17, 16]) and is proved in general in [6]. By writing $|K_t(g)| \leq |K_t(g)|^\varepsilon \|K_t\|_\infty^{1-\varepsilon}$, we find for some $b, c > 0$ that

$$|K_t(g)| \leq c e^{-bt^{1/3}} e^{-b\varrho(g)^2/t}$$

for $t \geq 1$. This estimate is apparently new for complex coefficients c_{ij}.

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