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THE BOREL STRUCTURE OF SOME NON-LEBESGUE SETS

BY

DON L. HANCOCK (Malibu, CA)

Abstract. For a given function in some classes related to real derivatives, we examine the structure of the set of points which are not Lebesgue points. In particular, we prove that for a summable approximately continuous function, the non-Lebesgue set is a nowhere dense nullset of at most Borel class 4.

1. Introduction. The classes of Darboux Baire 1, approximately continuous, and Lebesgue functions play a significant role in differentiation theory for real functions of a real variable. In [5] it is shown that every Darboux Baire 1 (and thus every approximately continuous) function defined on a closed interval can be transformed into a Lebesgue function by an appropriate homeomorphic change of variables, while in [2] necessary and sufficient conditions are obtained for an approximately continuous function to be transformed into a Lebesgue function by a homeomorphic change of scale. This suggests considering, loosely speaking, how far removed can an approximately continuous function be from a Lebesgue function. More precisely, what properties are possessed by the collection of non-Lebesgue points of an approximately continuous function? In this paper we show that under the additional assumption that the function is summable, the non-Lebesgue set of an approximately continuous function is a nowhere dense nullset of at most Borel class 4.

2. Preliminaries. Let us begin by introducing some notation, terminology, and background results necessary to our discussion. The real line is denoted by $\mathbb{R}$ and $[a,b]$ is an arbitrary nondegenerate closed interval. The letters m and n always denote positive integers. Given a set S, we write $\overline{S}$ and S^{c} for, respectively, the closure of S and the complement of S relative to $[a,b]$. It is assumed that all functions are measurable real-valued functions with domain $[a,b]$, and any integrals are to be interpreted as Lebesgue integrals. For a given function f and a subset E of $[a,b]$, $f|E$ means the restriction of f to the set E. Recall that a number x is a *Lebesgue point* of

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a function f if

$$\lim_{y\to x}\frac{1}{y-x}\int_x^y |f(t)-f(x)|\,dt=0;$$

otherwise x is called a *non-Lebesgue point* of f. The set consisting of all the non-Lebesgue points of f is called the *non-Lebesgue set* and denoted by L_f^c. (If L_f^c is the empty set, then f is a *Lebesgue function*.) Definitions and fundamental properties of both approximately continuous and Darboux Baire 1 functions may be found in [1] and [4], and we will use these properties without restraint. In particular we note that at any of its Lebesgue points a Darboux Baire 1 function f is approximately continuous and is the derivative of its integral. On the other hand, if f is approximately continuous at x, A is a measurable subset of $[a,b]$, and $f|A$ is bounded, then

$$\lim_{y\to x}\frac{1}{y-x}\int_{(x,y)\cap A} |f(t)-f(x)|\,dt=0, \tag{1}$$

and if x is additionally a density point of A, then

$$\lim_{y\to x}\frac{1}{y-x}\int_{(x,y)\cap A} |f(t)|\,dt=|f(x)|. \tag{2}$$

We also assume basic knowledge of Borel sets and their classifications, as found in [3].

3. Results and proofs. Every approximately continuous function f is Darboux Baire 1, and a Baire 1 function has a dense set of points of continuity. About each such point x there exists a sufficiently small open interval I_x on which f must be bounded. Using formula (1), we conclude that each point of I_x is a Lebesgue point of f. Thus L_f contains a dense open set, proving that L_f^c is nowhere dense. In addition, if f is summable, then almost every point is a Lebesgue point, so L_f^c is a nowhere dense nullset.

EXAMPLE 1. The non-Lebesgue set of a summable approximately continuous function is "small" in the sense of measure and Baire category, but not necessarily in the sense of cardinality. To see this, first let $\{I_m\}_{m=1}^\infty = \{(a_m,b_m)\}_{m=1}^\infty$ be an enumeration of the open intervals contiguous to a nowhere dense perfect nullset $P\subset[a,b]$, and for each m let (α_m,β_m) be the subinterval of I_m having length $(b_m-a_m)/m$ and midpoint $(a_m+b_m)/2$. Next, create a "spike function" f such that $f(x)=2m$ if $x=(a_m+b_m)/2$, f is identically 0 on the complement of the union of the intervals (α_m,β_m), and f is defined elsewhere by linear extension. It can be verified by routine computations that f is a summable approximately continuous function for which L_f^c is the uncountable set P.

For any summable f that is not a Lebesgue function, L_f^{c} is never open and it need not be closed. In fact, we can create a summable approximately continuous f for which $\overline{L_f^{\mathrm{c}}}$ has positive measure, and therefore is also not σ-porous. To see this, consider a bounded nowhere dense perfect set Q having positive measure, and let $\{J_k\}_{k=1}^{\infty}$ be an enumeration of the open intervals contiguous to Q. Now define a function f so that it is identically 0 on Q, while on each J_k it is constructed just as the function in Example 1 was on the arbitrary interval (a, b), except we now scale down the base widths of each spike by a constant factor of $1/k$. This f is easily verified to be a summable approximately continuous function such that the endpoints of each J_k are non-Lebesgue points, and thus $\overline{L_f^{\mathrm{c}}}$ contains the set Q of positive measure.

We now proceed to show that L_f^{c} must also be a Borel set of type $\mathrm{G}_{\delta\sigma\delta\sigma}$ for any summable approximately continuous function f. Since the proof only requires f to be a Baire 1 summable function, we state and prove our result in this more general setting. The proof will be based on three lemmas. Assume throughout that

$$D = \{(x, y, z) : x \in [a, b],\ y \in [a, b],\ z \in \mathbb{R}\}.$$

LEMMA 1. *If f is summable, then for each m the set T_m defined by*

$$T_m = \left\{(x, y, z) : \frac{1}{y - x} \int_x^y |f(t) - z|\, dt \geq \frac{1}{m}\right\}$$

is of type F_σ *in D.*

Proof. The set-theoretic difference of two closed sets is an F_σ, so it is sufficient to show that $T_m = \overline{T}_m - \{(x, y, z) : x = y\}$. Containment in one direction is obvious. To show containment in the opposite direction we show that if $(x, y, z) \in \overline{T}_m$ and $x \neq y$, then $(x, y, z) \in T_m$. For convenience we suppose that $y > x$, as the proof is analogous if $y < x$. There exists a sequence $\{(x_k, y_k, z_k)\}_{k=1}^{\infty}$ that converges to (x, y, z) in D, with each (x_k, y_k, z_k) in T_m. Thus x_k tends to x, y_k tends to y, z_k tends to z, and $y_k > x_k$ for large enough k.

Choose $\varepsilon > 0$ arbitrarily. Since f is summable, there exists a $\delta > 0$ such that

$$\int_B |f(t) - z|\, dt < \frac{\varepsilon}{3}$$

for any subset B of $[a, b]$ having measure less than δ. Now choose a large enough integer K so that the following four conditions are satisfied whenever $k > K$:

(i) $y_k > x_k$,
(ii) the set $B_k = ([x_k, y_k] \cup [x, y]) - ([x_k, y_k] \cap [x, y])$ has measure less than δ,

(iii) $|z_k - z| \cdot 2(y-x) < \varepsilon/3$,
(iv) $(y-x) - m\varepsilon/3 < y_k - x_k < 2(y-x)$.

Then for all $k > K$ we have

$$\Big|\int_{x_k}^{y_k} |f(t) - z_k|\,dt - \int_x^y |f(t)-z|\,dt\Big|$$
$$\le \Big|\int_{x_k}^{y_k} |f(t) - z_k|\,dt - \int_{x_k}^{y_k} |f(t)-z|\,dt\Big| + \Big|\int_{x_k}^{y_k} |f(t) - z|\,dt - \int_x^y |f(t)-z|\,dt\Big|$$
$$\le \int_{x_k}^{y_k} |z_k - z|\,dt + \int_{B_k} |f(t)-z|\,dt$$
$$< |z_k - z|(y_k - x_k) + \varepsilon/3 \qquad \text{(by (ii))}$$
$$\le |z_k - z| \cdot 2(y-x) + \varepsilon/3 \qquad \text{(by (iv))}$$
$$< \varepsilon/3 + \varepsilon/3 = 2\varepsilon/3 \qquad \text{(by (iii))}.$$

Furthermore, since $(x_k, y_k, z_k) \in T_m$,

$$\int_{x_k}^{y_k} |f(t) - z_k|\,dt \ge \frac{1}{m}\,(y_k - x_k).$$

Hence, $k > K$ implies

$$\int_x^y |f(t)-z|\,dt > \int_{x_k}^{y_k} |f(t)-z_k|\,dt - \frac{2\varepsilon}{3} \ge \frac{y_k - x_k}{m} - \frac{2\varepsilon}{3}$$
$$\ge \frac{y - x - m\varepsilon/3}{m} - \frac{2\varepsilon}{3} \qquad \text{(by (iv))}$$
$$= \frac{y-x}{m} - \varepsilon.$$

Since ε was arbitrary, we deduce that $\int_x^y |f(t)-z|\,dt \ge (y-x)/m$ and thus $(x, y, z) \in T_m$.

LEMMA 2. *Suppose that f is a summable function and that C is a subset of $[a,b] \times \mathbb{R}$ defined as follows: $(x,z) \in C$ if and only if for some m and every n, there exists a $y \in [a,b]$ such that $0 < |y-x| \le 1/n$ and*

$$\frac{1}{y-x}\int_x^y |f(t)-z|\,dt \ge \frac{1}{m}.$$

Then C is of type $F_{\sigma\delta\sigma}$ in $[a,b] \times \mathbb{R}$.

Proof. For each m and n, define a set $A_{m,n}$ as follows: $(x,z) \in A_{m,n}$ if and only if there exists a y such that $0 < |y-x| \le 1/n$ and

$$\frac{1}{y-x}\int_x^y |f(t)-z|\,dt \ge \frac{1}{m}.$$

Then $C = \bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty} A_{m,n}$. To complete the proof, it thus suffices to show that each $A_{m,n}$ is of type F_σ. Let $B_{m,n} = T_m \cap \{(x,y,z) : 0 < |y-x| \le 1/n\}$, where T_m is the set defined in the previous lemma. Then $B_{m,n}$, being the intersection of two F_σ sets, is also of type F_σ. Let P_2 denote the projection map of D onto $[a,b]\times\mathbb{R}$, where P_2 is defined by $P_2(x,y,z) = (x,z)$. It is evident that $P_2(B_{m,n}) = A_{m,n}$. Since $B_{m,n}$ is of type F_σ, we can find compact sets F_k so that $B_{m,n} = \bigcup_{k=1}^{\infty} F_k$. The continuity of P_2 implies that $P_2(F_k)$ is closed. Furthermore, $P_2(B_{m,n}) = P_2(\bigcup_{k=1}^{\infty} F_k) = \bigcup_{k=1}^{\infty} P_2(F_k)$, so $A_{m,n}$ is of type F_σ. This completes the proof.

LEMMA 3. *Let f be a summable Baire 1 function, and let G denote the graph of f. If A is any set of type $\mathrm{F}_{\sigma\delta\sigma}$ in $[a,b]\times\mathbb{R}$, then the projection of $G\cap A$ onto $[a,b]$ is a Borel set of type $\mathrm{G}_{\delta\sigma\delta\sigma}$.*

Proof. This is just a special case of a theorem in [3, p. 385].

THEOREM 1. *Suppose that f is a summable function in Baire class 1. Then L_f^{c} is a Borel set of type $\mathrm{G}_{\delta\sigma\delta\sigma}$.*

Proof. By the definition of a Lebesgue point, it is evident that $x \in L_f^{\mathrm{c}}$ if and only if

$$\limsup_{y\to x} \frac{1}{y-x}\int_x^y |f(t)-f(x)|\,dt > 0.$$

Thus $x \in L_f^{\mathrm{c}}$ if and only if there exists some z in $\mathbb{R}$ for which the following two conditions are satisfied:

(i) $f(x) = z$,
(ii) for some m and every n there is a y such that $0 < |y-x| \le 1/n$ and

$$\frac{1}{y-x}\int_x^y |f(t)-z|\,dt \ge \frac{1}{m}.$$

Let P_1 denote the projection map of $[a,b]\times\mathbb{R}$ onto $[a,b]$ defined by $P_1(x,z) = x$. Then it is obvious that $L_f^{\mathrm{c}} = P_1(G\cap C)$, where C is the $\mathrm{F}_{\sigma\delta\sigma}$ set of Lemma 2 and G is the graph of f. Applying Lemma 3 completes the proof.

As an immediate application of Theorem 1, we see that if L_f^{c} is uncountable then it contains a nonempty nowhere dense perfect set. This is true because every uncountable Borel set has that property. We also mention that there is an analogue to Lemma 3 which is valid for any function f of

Borel class α. Using this result, the proof of Theorem 1 shows that L_f^c is a Borel set of additive class $\alpha+3$, provided that f is also summable.

For any x in $[a,b]$ and a given approximately continuous function f, let c_x be the extended real number defined by

$$c_x = \limsup_{y\to x} \frac{1}{y-x}\int_x^y |f(t)-f(x)|\,dt.$$

If c_x is 0 then x is a Lebesgue point of f, while if c_x is a positive number or ∞, then $x \in L_f^c$. A point $x \in L_f^c$ is called *weak* if $0 < c_x < \infty$ or *firm* if $c_x = \infty$. (This terminology is appropriate because for any homeomorphism h of $\mathbb{R}$ onto $\mathbb{R}$ satisfying a mild growth condition, each weak non-Lebesgue point of f is a Lebesgue point of $h\circ f$, for every approximately continuous function f. See [2].) It is a straightforward argument to prove from formulas (1) and (2) that x is a firm non-Lebesgue point of an approximately continuous function f if and only if

$$\limsup_{y\to x} \frac{1}{y-x}\int_x^y |f(t)|\,dt = \infty. \tag{3}$$

Every non-Lebesgue point x of the "spike" function in Example 1 above is weak, since crude estimates show that $c_x < 2$. However, suppose that we modify Example 1 by increasing the height of each spike by defining $f((a_m+b_m)/2) = 2ms_m$, where $\{s_m\}_{m=1}^{\infty}$ is an increasing, unbounded sequence of positive numbers such that $\sum_{m=1}^{\infty} s_m(b_m-a_m)$ converges. This produces a summable approximately continuous function with an uncountable set of firm non-Lebesgue points. Although such sets need not be closed, the next theorem shows they must be Borel of type $\mathrm{F}_{\sigma\delta}$.

THEOREM 2. *If f is a summable approximately continuous function, then the set of firm non-Lebesgue points is of type* $\mathrm{F}_{\sigma\delta}$.

Proof. For each m and n, define

$$E_{m,n} = \left\{ x : \text{for some } y,\ 0 < |y-x| \le \frac{1}{n} \text{ and } \frac{1}{y-x}\int_x^y |f(t)|\,dt \ge m \right\}.$$

We can deduce that $E_{m,n}$ is of type F_σ by a slight modification of the argument that was used in the proof of Lemma 2 to show the set $A_{m,n}$ was of type F_σ. (Just take $z=0$, using the notation in that proof.) By formula (3), it is clear that the set of firm non-Lebesgue points equals $\bigcap_{m=1}^{\infty}\bigcap_{n=1}^{\infty} E_{m,n}$, and is therefore of type $\mathrm{F}_{\sigma\delta}$.

4. A question. We have shown that for any summable approximately continuous function, L_f^c is a nowhere dense Borel nullset of type $\mathrm{G}_{\sigma\delta\sigma\delta}$.

However, it is extremely unlikely that for an arbitrary nowhere dense Borel nullset S of type $G_{\sigma\delta\sigma\delta}$ there is some summable approximately continuous function for which $S = L_f^c$. We thus pose the following question: what are necessary and sufficient conditions for a set S to be the non-Lebesgue set of some summable approximately continuous function?

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Department of Mathematics
Pepperdine University
Malibu, CA 90263, U.S.A.
E-mail: don.hancock@pepperdine.edu

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