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ON THE DOUBLING OF QUADRATIC ALGEBRAS

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Abstract. The concept of doubling, which was introduced around 1840 by Graves and Hamilton, associates with any quadratic algebra $\mathcal{A}$ over a field k of characteristic not 2 its double $\mathcal{V}(\mathcal{A}) = \mathcal{A} \times \mathcal{A}$ with multiplication $(w,x)(y,z) = (wy - \overline{z}x, x\overline{y} + zw)$. This yields an endofunctor on the category of all quadratic k-algebras which is faithful but not full. We study in which respect the division property of a quadratic k-algebra is preserved under doubling and, provided this is the case, whether the doubles of two non-isomorphic quadratic division algebras are again non-isomorphic.

Generalizing a theorem of Dieterich [9] from $\mathbb{R}$ to arbitrary square-ordered ground fields k we prove that the division property of a quadratic k-algebra of dimension smaller than or equal to 4 is preserved under doubling. Generalizing an aspect of the celebrated (1,2,4,8)-theorem of Bott, Milnor [4] and Kervaire [21] from $\mathbb{R}$ to arbitrary ground fields k of characteristic not 2 we prove that the division property of an 8-dimensional doubled quadratic k-algebra is never preserved under doubling. Finally, we contribute to a solution of the still open problem of classifying all 8-dimensional real quadratic division algebras by extending an approach of Dieterich and Lindberg [12] and proving that, under a mild additional assumption, the doubles of two non-isomorphic 4-dimensional real quadratic division algebras are again non-isomorphic.

1. Introduction. Let k be a field. An *algebra* over k is a vector space endowed with a k-bilinear multiplication $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $(x,y) \mapsto xy$. A *division algebra* over k is an algebra such that $0 < \dim \mathcal{A} < \infty$ and $xy = 0$ only if $x = 0$ or $y = 0$. The latter condition, requiring the absence of zero divisors, is equivalent to the requirement that the k-linear endomorphisms $a? : \mathcal{A} \to \mathcal{A}$, $x \mapsto ax$, and $?a : \mathcal{A} \to \mathcal{A}$, $x \mapsto xa$, are bijective for all $a \in \mathcal{A} \setminus \{0\}$. A morphism between two algebras $\mathcal{A}$ and $\mathcal{B}$ over k is a k-linear map $\sigma : \mathcal{A} \to \mathcal{B}$ satisfying $\sigma(xy) = \sigma(x)\sigma(y)$. When facing the problem of describing—completely and explicitly—the category $\mathcal{D}(k)$ of all division algebras over k, for any given ground field k, one ends up with four subproblems.

(i) Determine $\mathbb{D}(k) = \{n \in \mathbb{N} \mid \mathcal{D}_n(k) \neq \emptyset\}$, where $\mathcal{D}_n(k)$ denotes the full subcategory of $\mathcal{D}_k$ formed by all objects of dimension n.

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(ii) Determine a classifying list $\mathcal{L}_n(k) \subset \mathcal{D}_n(k)$ for each $n \in \mathbb{D}(k)$ (i.e. construct an explicit list for $\mathcal{D}_n(k)/\simeq$).
(iii) Determine $\mathrm{Aut}(\mathcal{A})$ for all $\mathcal{A} \in \mathcal{L}_n(k)$ and all $n \in \mathbb{D}(k)$.
(iv) Determine all inclusions $\mathcal{A} \hookrightarrow \mathcal{B}$ for all $(\mathcal{A}, \mathcal{B}) \in \mathcal{L}_m(k) \times \mathcal{L}_n(k)$ and all $(m, n) \in \mathbb{D}(k) \times \mathbb{D}(k)$ such that $m < n$.

The present article will contribute to the first problem and will improve the existing classifying list for the 8-dimensional real quadratic division algebras.

If k is *algebraically closed*, then the solution to these problems is trivial: $\mathbb{D}(k) = \{1\}$, $\mathcal{L}_1(k) = \{k\}$ and $\mathrm{Aut}(k) = \{\mathbb{I}_k\}$ (see e.g. [13, Proposition 1.1]). For a general ground field k, the solution seems to be highly non-trivial and very heavily dependent on k. So far, none of the problems (i)–(iv) has been solved for general k, beyond the trivial information that $1 \in \mathbb{D}(k)$, $\mathcal{L}_1(k) = \{k\}$ and $\mathrm{Aut}(k) = \{\mathbb{I}_k\}$, and beyond the information on the associative division k-algebras which is stored in the Brauer group $\mathrm{Br}(k)$. Even in prominent cases the solution to the four problems is yet to be found. In the case of $k = \mathbb{R}$, which will be the target of this article, a famous theorem of Bott and Milnor [4] and Kervaire [21] (1958) states that $\mathbb{D}(\mathbb{R}) = \{1, 2, 4, 8\}$. Classifying lists $\mathcal{L}_2(\mathbb{R})$ have been determined by Burdujan (1985), Gottschling [17] (1998), and Hübner and Petersson [20] (2004). On the other hand, determining $\mathcal{L}_4(\mathbb{R})$ and $\mathcal{L}_8(\mathbb{R})$ remains an open problem. Nowadays there seems to be three main strategies to tackle these challenging problems.

(1) An algebra $\mathcal{A}$ is called *flexible* if $(xy)x = x(yx)$ for all $x, y \in \mathcal{A}$. The classification of all real flexible division algebras is almost accomplished (Benkart, Britten and Osborn [1] 1982; Cuenca Mira *et al.* [5] 1999; Darpö [6] 2004).

(2) The real Lie algebras which arise as the derivation algebras of real division algebras are classified by the eight Lie algebras $\mathbb{R}^0$, $\mathbb{R}^1$, $\mathbb{R}^2$ (abelian), $\mathfrak{su}_2$, $\mathfrak{su}_2 \times \mathbb{R}$, $\mathfrak{su}_2 \times \mathfrak{su}_2$, $\mathfrak{su}_3$, $\mathfrak{g}_2$ (Benkart and Osborn 1981 [2]). Moreover, there are partial results towards the classification of all real division algebras having a fixed derivation algebra type (Benkart and Osborn [3] 1981; Rochdi [23] 1995; Doković and Zhao [14] 2003).

(3) An algebra $\mathcal{A}$ is called *quadratic* if $\mathcal{A} \neq \emptyset$, there exists an identity element $1 \in \mathcal{A}$ and $1, x, x^2$ are linearly dependent for all $x \in \mathcal{A}$. A classification of all real quadratic division algebras is accomplished in dimension 4 (Osborn [22] 1962; Hefendehl-Hebecker [18] 1980; Dieterich [7] 1998).

In the present article we will contribute to the latter of these three approaches. For any algebra $\mathcal{A}$ over a field k with identity element $1 \in \mathcal{A}$ we call an element $x \in \mathcal{A}$ *purely imaginary* if $x \notin k1 \setminus \{0\}$ but $x^2 \in k1$.

Proposition 1.1 (Frobenius Lemma [15]). *Let $\mathcal{A}$ be a quadratic algebra over a field k of characteristic not two. Then the set V of all purely imaginary elements in $\mathcal{A}$ is a linear subspace in $\mathcal{A}$ such that $\mathcal{A} = k1 \oplus V$.*

Henceforth the ground field k is assumed to have characteristic not two. For any quadratic k-algebra $\mathcal{A}$, the decomposition $\mathcal{A} = k1 \oplus V$ given by the Frobenius lemma and henceforth referred to as the *Frobenius decomposition* determines a linear form $\lambda : \mathcal{A} \to k$ and a linear map $\iota : \mathcal{A} \to V$ such that $x = \lambda(x)1 + \iota(x)$ for all $x \in \mathcal{A}$. These in turn give rise to a symmetric k-bilinear form $\langle\ ,\ \rangle : \mathcal{A} \times \mathcal{A} \to k$, $\langle x, y\rangle = -\frac{1}{2}\lambda(xy + yx)$, and a quadratic form $q : \mathcal{A} \to k$, $q(x) = \lambda(x)^2 + \langle\iota(x), \iota(x)\rangle$. We call q *anisotropic* if $q^{-1}(0) = \{0\}$, and we say that the quadratic algebra $\mathcal{A}$ is anisotropic if its quadratic form $q : \mathcal{A} \to k$ is anisotropic. We define the *conjugate* of $x \in \mathcal{A}$ to be $\overline{x} = \lambda(x)1_{\mathcal{A}} - \iota(x)$. Finally, we define a *morphism* between two quadratic algebras $\mathcal{A}$ and $\mathcal{B}$ to be a linear map $\sigma : \mathcal{A} \to \mathcal{B}$ such that $\sigma(1_{\mathcal{A}}) = 1_{\mathcal{B}}$ and $\sigma(xy) = \sigma(x)\sigma(y)$ for all $x, y \in \mathcal{A}$. Note that the definition of an algebra morphism, given earlier, does not require that $\sigma(1_{\mathcal{A}}) = 1_{\mathcal{B}}$. It might seem like we lose many morphisms by this additional condition, but this is not the case. Our goal is to classify the *quadratic division algebras* and this additional condition will not affect the isomorphisms between division algebras since if $\sigma : \mathcal{A} \to \mathcal{B}$ is a morphism between two quadratic division algebras over k, then $1_{\mathcal{B}}\sigma(x) = \sigma(x) = \sigma(1_{\mathcal{A}}x) = \sigma(1_{\mathcal{A}})\sigma(x)$ for all $x \in \mathcal{A}$. Hence, since $\mathcal{B}$ has no zero divisors, we see that either $\sigma = 0$ or $\sigma(1_{\mathcal{A}}) = 1_{\mathcal{B}}$. So for quadratic division algebras, the morphisms of quadratic algebras coincide with the non-zero algebra morphisms. Hence, the additional condition will not affect the isoclasses of quadratic division algebras.

Proposition 1.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be quadratic algebras over a field k of characteristic not two*, *with Frobenius decompositions $\mathcal{A} = k1_{\mathcal{A}} \oplus V$ and $\mathcal{B} = k1_{\mathcal{B}} \oplus W$. If $\mathcal{A}$ is anisotropic then every algebra morphism $\sigma : \mathcal{A} \to \mathcal{B}$ is injective*, *orthogonal and decomposes as $\sigma = \mathrm{id} \oplus \widetilde{\sigma}$, where* $\mathrm{id} : k1_{\mathcal{A}} \to k1_{\mathcal{B}}$ *is the identification map and $\widetilde{\sigma} : V \to W$ is induced by σ.*

Proof. We have

$$\begin{array}{c} \mathcal{A} = k1 \oplus V \\ \sigma = \begin{pmatrix} \sigma_1 & \sigma_2 \\ \sigma_3 & \sigma_4 \end{pmatrix} \Big\downarrow \qquad\qquad \Big\downarrow \\ \mathcal{B} = k1 \oplus W \end{array}$$

where $\sigma_1 : k1_{\mathcal{A}} \to k1_{\mathcal{B}}$, $\sigma_2 : V \to k1_{\mathcal{B}}$, $\sigma_3 : k1_{\mathcal{A}} \to W$ and $\sigma_4 : V \to W$ are linear maps. To begin with, $\sigma(1_{\mathcal{A}}) = 1_{\mathcal{B}}$ implies $\sigma_1 = \mathrm{id}$ and $\sigma_3 = 0$. We simplify notation by identifying $k1_{\mathcal{A}}$ and $k1_{\mathcal{B}}$ with k. Then every $x \in \mathcal{A}$ can be written as $x = \alpha + v = (\alpha\ v)^t$, with uniquely determined $\alpha \in k$, $v \in V$.

Then we get

$$\sigma(x) = \begin{pmatrix} \mathrm{id} & \sigma_2 \\ 0 & \sigma_4 \end{pmatrix} \begin{pmatrix} \alpha \\ v \end{pmatrix} = \begin{pmatrix} \alpha + \sigma_2(v) \\ \sigma_4(v) \end{pmatrix}.$$

If $\sigma(x) = 0$ then $\sigma_2(v) = -\alpha$ and $\sigma_4(v) = 0$. Hence $\alpha^2 = \sigma_2(v)^2 = \sigma(v)^2 = \sigma(v^2) = \sigma_1(v^2) = v^2$. Now

$$q(x) = \lambda(x)^2 + \langle \iota(x), \iota(x) \rangle = \alpha^2 + \langle v, v \rangle = \alpha^2 - \lambda(v^2) = \alpha^2 - v^2 = 0$$

and since q is anisotropic it follows that $x = 0$, hence σ is injective.

For all $v \in V$ we get

$$v^2 = \sigma(v^2) = \sigma(v)^2 = (\sigma_2(v) + \sigma_4(v))^2 = (\sigma_2(v)^2 + \sigma_4(v)^2) + 2\sigma_2(v)\sigma_4(v).$$

Hence $2\sigma_2(v)\sigma_4(v) = 0$. Since σ is injective we must have $\sigma_4(v) \neq 0$ for all $v \in V \setminus \{0\}$. Hence $\sigma_2 = 0$. Accordingly $\sigma = \mathrm{id} \oplus \widetilde{\sigma}$, where $\widetilde{\sigma} = \sigma_4$.

Finally, for all $x, y \in \mathcal{A}$,

$$\begin{aligned} \langle \sigma(x), \sigma(y) \rangle &= -\tfrac{1}{2}\lambda(\sigma(x)\sigma(y) + \sigma(y)\sigma(x)) = -\tfrac{1}{2}\lambda(\sigma(xy) + \sigma(yx)) \\ &= -\tfrac{1}{2}(\lambda(xy + yx) + \sigma_2\iota(xy + yx)) = -\tfrac{1}{2}\lambda(xy + yx) = \langle x, y \rangle. \ \blacksquare \end{aligned}$$

Corollary 1.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be quadratic algebras over a field k of characteristic not two, with Frobenius decompositions $\mathcal{A} = k \oplus V$ and $\mathcal{B} = k \oplus W$. If $\mathcal{A}$ is anisotropic and* $\dim \mathcal{A} = \dim \mathcal{B} < \infty$, *then every algebra morphism $\sigma : \mathcal{A} \to \mathcal{B}$ is an orthogonal isomorphism and its induced linear map $\widetilde{\sigma} : V \to W$ is an orthogonal isomorphism of anticommutative k-algebras.*

Proof. The algebra structure on V (W respectively) given by $(v, w) \mapsto \iota(vw)$ is anticommutative, since $\iota(vw) + \iota(wv) = \iota((v + w)^2 - v^2 - w^2) = 0$ for all $v, w \in V$ (W respectively). If $\sigma : \mathcal{A} \to \mathcal{B}$ is an algebra morphism, then $\sigma = \mathrm{id} \oplus \widetilde{\sigma}$ implies that $\widetilde{\sigma}\iota_{\mathcal{A}} = \iota_{\mathcal{B}}\sigma$ and hence that $\widetilde{\sigma} : V \to W$ is an algebra morphism. The remaining asserted properties of σ and $\widetilde{\sigma}$ are immediate consequences of Proposition 1.2. $\blacksquare$

Recall that a field is called *square-ordered* if it is ordered and every positive element is a square [13]. Square-ordered fields are also known as *Hilbert fields of type A* [16].

We will now briefly describe the set-up of this article. In the next section, we will introduce the doubling as an endofunctor on the category of all quadratic algebras over k and state and prove some properties of the functor. In Section 3 we will restrict ourselves to a subcategory to be able to get a more explicit description of the doubling functor. Section 4 will connect this to the theory of division algebras over square-ordered fields, and from Section 5 and on we will apply this in the case of $k = \mathbb{R}$ and hence make a contribution to the classification of the real quadratic division algebras.

2. The doubling functor. Let $\mathcal{Q}(k)$ denote the category of all quadratic k-algebras. The endofunctor $\mathcal{V} : \mathcal{Q}(k) \to \mathcal{Q}(k)$ is defined on objects by $\mathcal{V}(\mathcal{A}) = \mathcal{A} \times \mathcal{A}$ with multiplication $(w,x)(y,z) = (wy - \overline{z}x, x\overline{y} + zw)$, and on morphisms by $\mathcal{V}(\sigma) = \sigma \times \sigma$. (It is an easy verification to see that $\mathcal{V}(\mathcal{A})$ is again a quadratic k-algebra.) The functor $\mathcal{V}$ is called the *doubling functor*. If an algebra $\mathcal{B} \in \mathcal{Q}(k)$ is isomorphic to $\mathcal{V}(\mathcal{A})$ for some $\mathcal{A} \in \mathcal{Q}(k)$, then $\mathcal{B}$ is called *doubled*. We begin with some properties of the doubling functor.

Recall the classical examples $(\mathbb{R}, \mathcal{V}(\mathbb{R}), \mathcal{V}^2(\mathbb{R}), \mathcal{V}^3(\mathbb{R})) = (\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O})$, which are known to have automorphism groups $\mathrm{Aut}(\mathbb{R}) \simeq C_1$, $\mathrm{Aut}(\mathbb{C}) \simeq C_2$, $\mathrm{Aut}(\mathbb{H}) \simeq \mathrm{SO}_3$ (a real Lie group of dimension 3) and $\mathrm{Aut}(\mathbb{O}) \simeq \mathbb{G}_2$ (a real Lie group of dimension 14). These examples give an easy proof of the fact that the doubling functor is not full in general; however, the next proposition will show that the doubling functor fails to be full "everywhere".

PROPOSITION 2.1. *The doubling functor $\mathcal{V} : \mathcal{Q}(k) \to \mathcal{Q}(k)$ is faithful but not full. More precisely, for all $\mathcal{A}, \mathcal{B} \in \mathcal{Q}(k)$ with $\mathrm{Mor}(\mathcal{A}, \mathcal{B}) \neq \emptyset$ the morphism map $\mathcal{V} : \mathrm{Mor}(\mathcal{A}, \mathcal{B}) \to \mathrm{Mor}(\mathcal{V}(\mathcal{A}), \mathcal{V}(\mathcal{B}))$, $\mathcal{V}(\sigma) = \sigma \times \sigma$, is not surjective.*

Proof. The property of being faithful follows directly from the definition of $\mathcal{V}$. The morphism map is not surjective since $\sigma \times -\sigma \in \mathrm{Mor}(\mathcal{V}(\mathcal{A}), \mathcal{V}(\mathcal{B}))$ whenever $\sigma \in \mathrm{Mor}(\mathcal{A}, \mathcal{B})$. ∎

Another interesting property of the doubling functor concerns the way the quadratic form $q^{\mathrm{d}} : \mathcal{V}(\mathcal{A}) \to k$ is induced from $q : \mathcal{A} \to k$, $q(x) = \lambda(x)^2 + \langle \iota(x), \iota(x) \rangle$. In fact, an easy verification shows the following.

PROPOSITION 2.2. *Let k be a field and let $\mathcal{A}$ be a quadratic k-algebra with quadratic form $q : \mathcal{A} \to k$, $q(x) = \lambda(x)^2 + \langle \iota(x), \iota(x) \rangle$. Then the quadratic form $q^{\mathrm{d}} : \mathcal{V}(\mathcal{A}) \to k$ satisfies $q^{\mathrm{d}}(x,y) = q(x) + q(y)$ for all $(x,y) \in \mathcal{V}(\mathcal{A})$.*

It follows that the quadratic form $q^{\mathrm{d}} : \mathcal{V}(\mathcal{A}) \to k$ is not necessarily anisotropic even if $q : \mathcal{A} \to k$ is. Sometimes anisotropy of the quadratic form is preserved and sometimes it is not, as the following examples show.

EXAMPLE 2.3. Let $\mathcal{A} = k = \mathbb{F}_p$, where $p > 2$ is a prime. Then $q : A \to k$ is given by $q(x) = x^2$ and is clearly anisotropic.

(i) If $\mathcal{A} = k = \mathbb{F}_5$ then $q^{\mathrm{d}} : \mathcal{V}(\mathbb{F}_5) \to \mathbb{F}_5$ is isotropic since, by Proposition 2.2, $q^{\mathrm{d}}(1,2) = q(1) + q(2) = 1^2 + 2^2 = 0$.
(ii) If $k = \mathbb{F}_3$ then the quadratic form $q^{\mathrm{d}} : \mathcal{V}(\mathbb{F}_3) \to \mathbb{F}_3$ is anisotropic but $(q^{\mathrm{d}})^{\mathrm{d}} : \mathcal{V}(\mathcal{V}(\mathbb{F}_3)) \to \mathbb{F}_3$ is isotropic.

When will the property of q being anisotropic be preserved under the doubling functor? A sufficient (but, as seen in the example, not necessary) condition is that the ground field k be *square-ordered*. This can be summarized in the following proposition.

Proposition 2.4. *Let $\mathcal{A}$ denote a quadratic algebra over a square-ordered field k. If $\mathcal{A}$ is anisotropic, then so is $\mathcal{V}(\mathcal{A})$.*

Proof. If k is square-ordered then it is easy to see that $q:\mathcal{A}\to k$, $q(x)=\lambda(x)^2+\langle\iota(x),\iota(x)\rangle$, anisotropic implies q positive definite. Now Proposition 2.2 implies that q^{d} is positive definite and hence anisotropic. ∎

We say that $\mathcal{V}$ *separates the isoclasses* of a full subcategory $\mathcal{P}\subset\mathcal{Q}(k)$ if $\mathcal{V}(\mathcal{A})\xrightarrow{\sim}\mathcal{V}(\mathcal{B})$ implies $\mathcal{A}\xrightarrow{\sim}\mathcal{B}$ for all $\mathcal{A},\mathcal{B}\in\mathcal{P}$. Even for prominent full subcategories $\mathcal{P}\subset\mathcal{Q}(k)$, the question whether $\mathcal{V}$ separates the isoclasses of $\mathcal{P}$ is still open. The following result will be used when dealing with this problem.

Lemma 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be quadratic algebras over a field k of characteristic not two. Then a k-linear map $\sigma=\begin{pmatrix}\sigma_{11}&\sigma_{12}\\ \sigma_{21}&\sigma_{22}\end{pmatrix}:\mathcal{V}(A)\to\mathcal{V}(\mathcal{B})$ is a morphism of quadratic k-algebras if and only if the k-linear maps $\sigma_{ij}:\mathcal{A}\to\mathcal{B}$ satisfy the following system of equations for all $w,x,y,z\in\mathcal{A}$:*

$$\sigma_{11}(1_{\mathcal{A}})=1_{\mathcal{B}}, \tag{2.1}$$
$$\sigma_{21}(1_{\mathcal{A}})=0, \tag{2.2}$$
$$\sigma_{11}(wy)=\sigma_{11}(w)\sigma_{11}(y)-\overline{\sigma_{21}(y)}\,\sigma_{21}(w), \tag{2.3}$$
$$\sigma_{11}(\overline{z}x)=\overline{\sigma_{22}(z)}\,\sigma_{22}(x)-\sigma_{12}(x)\sigma_{12}(z), \tag{2.4}$$
$$\sigma_{12}(x\overline{y})=\sigma_{12}(x)\sigma_{11}(y)-\overline{\sigma_{21}(y)}\,\sigma_{22}(x), \tag{2.5}$$
$$\sigma_{12}(zw)=\sigma_{11}(w)\sigma_{12}(z)-\overline{\sigma_{22}(z)}\,\sigma_{21}(w), \tag{2.6}$$
$$\sigma_{21}(wy)=\sigma_{21}(w)\,\overline{\sigma_{11}(y)}+\sigma_{21}(y)\sigma_{11}(w), \tag{2.7}$$
$$\sigma_{21}(\overline{z}x)=-\sigma_{22}(x)\,\overline{\sigma_{12}(z)}-\sigma_{22}(z)\sigma_{12}(x), \tag{2.8}$$
$$\sigma_{22}(x\overline{y})=\sigma_{22}(x)\,\overline{\sigma_{11}(y)}+\sigma_{21}(y)\sigma_{12}(x), \tag{2.9}$$
$$\sigma_{22}(zw)=\sigma_{21}(w)\,\overline{\sigma_{12}(z)}+\sigma_{22}(z)\sigma_{11}(w). \tag{2.10}$$

Proof. Conditions (2.1) and (2.2) are equivalent to $\sigma(1_{\mathcal{A}})=1_{\mathcal{B}}$. From the condition $\sigma((w,x)(y,z))=\sigma(w,x)\sigma(y,z)$, setting w,x,y,z pairwise to 0 we get conditions (2.3)–(2.10). On the other hand, if (2.3)–(2.10) are satisfied, then $\sigma((w,x)(y,z))=\sigma(w,x)\sigma(y,z)$. ∎

Proposition 2.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be quadratic algebras over a field k of characteristic not two, such that $\mathcal{A}$ is anisotropic and $\dim\mathcal{A}=\dim\mathcal{B}<\infty$. Then $\mathcal{V}(\mathcal{A})$ and $\mathcal{V}(\mathcal{B})$ are isomorphic if and only if there exists a k-linear map $\sigma:\mathcal{V}(A)\to\mathcal{V}(\mathcal{B})$ such that $\sigma=\begin{pmatrix}\sigma_{11}&\sigma_{12}\\ \sigma_{21}&\sigma_{22}\end{pmatrix}$ and the k-linear maps $\sigma_{ij}:\mathcal{A}\to\mathcal{B}$ satisfy the system* (2.1)–(2.10) *and in addition for all $x\in\mathcal{A}$,*

$$\sigma_{11}(\overline{x})=\overline{\sigma_{11}(x)}, \tag{2.11}$$
$$\overline{\sigma_{12}(x)}=-\sigma_{12}(x). \tag{2.12}$$

Proof. By Lemma 2.5 we see that σ is an algebra morphism if and only if the system (2.1)–(2.10) is satisfied. Now, Corollary 1.3 asserts that, in the given situation, every algebra morphism is an isomorphism and decomposes as $\sigma = \mathrm{id} \oplus \widetilde{\sigma}$. Using this fact we get the additional equations (2.11) and (2.12). ∎

3. Strongly anisotropic quadruples. Let V be a vector space over k. A quadratic form $q : V \to k$ is said to be *strongly anisotropic* if $\widehat{q} : k \times V \to k$, $\widehat{q}(\alpha, v) = \alpha^2 + q(v)$, is anisotropic. A *strongly anisotropic k-quadruple* (V, q, ξ, η) consists of a vector space V over k, a strongly anisotropic quadratic form $q : V \to k$, a linear form $\xi : V \wedge V \to k$ and a linear map $\eta : V \wedge V \to V$. The class $\mathcal{E}(k)$ of all strongly anisotropic k-quadruples is endowed with the structure of a category by declaring morphisms $\sigma : (V, q, \xi, \eta) \to (V', q', \xi', \eta')$ to be orthogonal k-linear maps $\sigma : V \to V'$ satisfying $\xi'(\sigma \wedge \sigma) = \xi$ and $\sigma\eta = \sigma'(\sigma \wedge \sigma)$. Each strongly anisotropic quadruple (V, q, ξ, η) determines, via the equivalence $\mathcal{H}$, an anisotropic quadratic k-algebra $\mathcal{H}(V, q, \xi, \eta) = k \times V$, with multiplication $(\alpha, v)(\beta, w) = (\alpha\beta - \langle v, w\rangle + \xi(v \wedge w), \alpha w + \beta v + \eta(v \wedge w))$, where $\langle v, w\rangle = \frac{1}{2}(q(v+w) - q(v) - q(w))$. We denote by $\mathcal{Q}^{\mathrm{a}}(k)$ the full subcategory of $\mathcal{Q}(k)$ formed by all anisotropic algebras $\mathcal{A} \in \mathcal{Q}(k)$. Then the assignment $(V, q, \xi, \eta) \mapsto \mathcal{H}(V, q, \xi, \eta)$ establishes a functor $\mathcal{H} : \mathcal{E}(k) \to \mathcal{Q}^{\mathrm{a}}(k)$ acting on morphisms by $\mathcal{H}(\widetilde{\sigma}) = \mathbb{I}_k \otimes \widetilde{\sigma}$.

Proposition 3.1 ([10]). *The functor $\mathcal{H} : \mathcal{E}(k) \to \mathcal{Q}^{\mathrm{a}}(k)$ is an equivalence of categories. A quasi-inverse functor $\mathcal{I} : \mathcal{Q}^{\mathrm{a}}(k) \to \mathcal{E}(k)$ to $\mathcal{H}$ is given on objects by $\mathcal{I}(\mathcal{A}) = (V, q, \xi, \eta)$, where $\mathcal{A}$ has Frobenius decomposition $\mathcal{A} = k \oplus V$, $q(v) = -\lambda(v^2)$, $\xi(v \wedge w) = \frac{1}{2}\lambda(vw - wv)$, $\eta(v \wedge w) = \iota(vw)$, and on morphisms by $\mathcal{I}(\sigma) = \widetilde{\sigma}$, where $\sigma = \mathrm{id} \oplus \widetilde{\sigma}$.*

Let $\mathcal{E}^{\mathrm{a}}(k)$ denote the full subcategory of $\mathcal{E}(k)$ formed by all strongly anisotropic quadruples (V, q, ξ, η) such that $\mathcal{V}(\mathcal{H}(V, q, \xi, \eta))$ is an anisotropic quadratic algebra. Then the composed functor of full subcategories $\mathcal{V}\mathcal{H} : \mathcal{E}(k) \to \mathcal{Q}(k)$ induces a functor of full subcategories $\mathcal{V}\mathcal{H} : \mathcal{E}^{\mathrm{a}}(k) \to \mathcal{Q}^{\mathrm{a}}(k)$. Defining $\mathcal{J} = \mathcal{I}\mathcal{V}\mathcal{H}$, we have established the commutative diagram

$$\begin{tikzcd}
\mathcal{E}(k) & \mathcal{Q}^{\mathrm{a}}(k) \arrow[l, "\mathcal{J}"'] \arrow[r, hook] & \mathcal{Q}(k) \\
\mathcal{E}^{\mathrm{a}}(k) \arrow[u, "\mathcal{I}"] \arrow[ur, "\mathcal{V}\mathcal{H}"'] \arrow[r, hook] & \mathcal{E}(k) \arrow[r, "\mathcal{H}"'] & \mathcal{Q}^{\mathrm{a}}(k) \arrow[u, "\mathcal{V}"']
\end{tikzcd}$$

where the hookarrows denote inclusion functors of full subcategories. Calculation shows that $\mathcal{J}$ admits the following explicit description.

Proposition 3.2. *The functor $\mathcal{J} : \mathcal{E}^{\mathrm{a}}(k) \to \mathcal{E}(k)$ is given on objects by $\mathcal{J}(V, q, \xi, \eta) = (V^{\mathrm{d}}, q^{\mathrm{d}}, \xi^{\mathrm{d}}, \eta^{\mathrm{d}})$, where $V^{\mathrm{d}} = V \times k \times V$,*

$$(3.1)\qquad q^{\mathrm{d}}\begin{pmatrix} v\\ \alpha\\ w\end{pmatrix} = \langle v,v\rangle + \alpha^2 + \langle w,w\rangle,$$

$$(3.2)\qquad \xi^{\mathrm{d}}\left(\begin{pmatrix} v\\ \alpha\\ w\end{pmatrix}\wedge\begin{pmatrix} v'\\ \alpha'\\ w'\end{pmatrix}\right) = \xi(v\wedge v') - \xi(w\wedge w'),$$

$$(3.3)\qquad \eta^{\mathrm{d}}\left(\begin{pmatrix} v\\ \alpha\\ w\end{pmatrix}\wedge\begin{pmatrix} v'\\ \alpha'\\ w'\end{pmatrix}\right) = \begin{pmatrix} \alpha w' - \alpha' w + \eta(v\wedge v') - \eta(w\wedge w')\\ \langle w,v'\rangle - \langle v,w'\rangle - \xi(w\wedge v') - \xi(v\wedge w')\\ -\alpha v' + \alpha' v - \eta(w\wedge v') - \eta(v\wedge w')\end{pmatrix},$$

and on morphisms by $\mathcal{J}(\sigma) = \sigma\times\mathbb{I}_k\times\sigma$.

If $\mathcal{E}^{\mathrm{a}}(k) = \mathcal{E}(k)$, we call $\mathcal{J}:\mathcal{E}(k)\to\mathcal{E}(k)$ the *doubling functor for strongly anisotropic quadruples*. Note that if k is a square-ordered field, then due to Proposition 2.4 the doubling functor $\mathcal{V}:\mathcal{Q}(k)\to\mathcal{Q}(k)$ induces an endofunctor $\mathcal{V}:\mathcal{Q}^{\mathrm{a}}(k)\to\mathcal{Q}^{\mathrm{a}}(k)$. Hence $\mathcal{E}^{\mathrm{a}}(k) = \mathcal{E}(k)$.

4. Dissident quadruples and quadratic division algebras. We begin by explaining the notions appearing in the headline of this section. Let k be any field. A division algebra $\mathcal{A}$ over k is understood to be an algebra $\mathcal{A}$ over k satisfying $0<\dim\mathcal{A}<\infty$ and having no zero divisors (i.e. $xy=0$ only if $x=0$ or $y=0$). We denote by $\mathcal{Q}^{\mathrm{d}}(k)$ the full subcategory of $\mathcal{Q}(k)$ formed by all division algebras $\mathcal{A}\in\mathcal{Q}(k)$. Let V be a finite-dimensional vector space over k. A linear map $\eta: V\wedge V\to V$ is said to be *dissident* if $v,w,\eta(v\wedge w)$ are linearly independent whenever v and w are.

The motivation for this section is the following fundamental result of Osborn.

PROPOSITION 4.1 (Osborn's theorem [22]). *Let k be a field of characteristic not two and let $\mathcal{A}$ be a finite-dimensional quadratic algebra over k. Then $\mathcal{A}$ is a division algebra if and only if $\mathcal{A}$ is anisotropic and $\eta: V\wedge V\to V$ is dissident.*

A famous theorem of Bott, Milnor [4] and Kervaire [21] asserts that every real division algebra has dimension $1,2,4$ or 8. Hence if $\mathcal{A}\in Q_8^{\mathrm{d}}(\mathbb{R})$ then $\mathcal{V}(\mathcal{A})$ is not a division algebra. The question whether this fact generalizes from $\mathbb{R}$ to arbitrary fields seems to be open. We can, however, prove an affirmative result in this direction.

PROPOSITION 4.2. *Let k be a field of characteristic not two. If $\mathcal{A}\in\mathcal{Q}_8^{\mathrm{d}}(k)$ is doubled then $\mathcal{V}(\mathcal{A})$ is not a division algebra.*

Proof. If $\mathcal{V}(\mathcal{A})$ is isotropic, then $\mathcal{V}(\mathcal{A})$ is not a division algebra by Osborn's theorem. Assume that $\mathcal{V}(\mathcal{A})$ is anisotropic. We may assume that

$\mathcal{A} = \mathcal{V}(\mathcal{B})$ for some $\mathcal{B} \in \mathcal{Q}_4^{\mathrm{d}}(k)$. Let $\mathcal{B} = k \oplus V$ be the Frobenius decomposition of $\mathcal{B}$. We may choose vectors $v, w \in V \setminus \{0\}$ such that $\langle v, w \rangle = 0$. Since the bilinear form $\langle\ ,\ \rangle : V \times V \to k$ is anisotropic, v and w are non-proportional. Let $x, y \in V \setminus \{0\}$ be given by $x = v + w$ and $y = v - w$. Then x and y are non-proportional vectors such that $\langle x, x \rangle = \langle y, y \rangle$. Define $X, Y \in (V \times k \times V) \times k \times (V \times k \times V)$ by

$$X = ((x\ 0\ 0)\ 0\ (0\ 0\ y))^t, \qquad Y = ((y\ 0\ 0)\ 0\ (0\ 0\ x))^t.$$

Then X and Y are non-proportional. The dissident map $\eta : V \wedge V \to V$ gives rise to a linear map $(\eta^{\mathrm{d}})^{\mathrm{d}} : (V^{\mathrm{d}})^{\mathrm{d}} \wedge (V^{\mathrm{d}})^{\mathrm{d}} \to (V^{\mathrm{d}})^{\mathrm{d}}$ by applying the doubling functor twice. The linear map $(\eta^{\mathrm{d}})^{\mathrm{d}}$ satisfies, by Proposition 3.2, $(\eta^{\mathrm{d}})^{\mathrm{d}}(X \wedge Y) = 0$. Hence $(\eta^{\mathrm{d}})^{\mathrm{d}}$ is not dissident and by Osborn's theorem, $\mathcal{V}(\mathcal{A}) = \mathcal{V}(\mathcal{V}(\mathcal{B}))$ is not a division algebra. ∎

From now on we will assume that the ground field k is square-ordered. We begin by noticing that every strongly anisotropic quadratic form q on a vector space V over k is positive definite. We denote by $\mathcal{D}(k) = \{(V, q, \xi, \eta) \in \mathcal{E}(k) \mid \dim V < \infty \text{ and } \eta \text{ is dissident}\}$ the full subcategory of $\mathcal{E}(k)$ formed by all *dissident quadruples*. Then, by Osborn's theorem, we have

PROPOSITION 4.3 ([12]). *The equivalence $\mathcal{H} : \mathcal{E}(k) \to \mathcal{Q}^{\mathrm{a}}(k)$ induces an equivalence of full subcategories $\mathcal{H} : \mathcal{D}(k) \to \mathcal{Q}^{\mathrm{d}}(k)$.*

Via the identity $\langle v, w \rangle = \frac{1}{2}(q(v+w) - q(v) - q(w))$ we shall freely identify $(V, q) = (V, \langle\ \rangle)$ and briefly write V for the inner product space $(V, \langle\ \rangle)$. Accordingly, each $(V, q, \xi, \eta) \in \mathcal{D}(k)$ becomes a *dissident triple* (V, ξ, η). Preparatory to the next proposition we need a lemma describing the vector products on a 3-dimensional inner product space over k. Recall that a linear map $\pi : V \wedge V \to V$ on a finite-dimensional inner product space V is called a *vector product* if it satisfies the following conditions:

(a) $\langle \pi(v \wedge w), x \rangle = \langle v, \pi(w \wedge x) \rangle$ for all $v, w, x \in V$.
(b) $\langle \pi(v \wedge w), \pi(v \wedge w) \rangle = 1$ for all orthonormal $v, w \in V$.

Furthermore, each linear endomorphism $\varepsilon : V \to V$ of a finite-dimensional inner product space $V = (V, \langle\ \rangle)$ over k determines a quadratic form $q_\varepsilon(v) = \langle v, \varepsilon(v) \rangle$. The endomorphism ε is called *positive definite* (resp. *negative definite*) if q_ε is positive definite (resp. negative definite). We call ε *definite* in case ε is either positive definite or negative definite.

LEMMA 4.4 ([13]). *Let V be a 3-dimensional inner product space over k. Let $\underline{b} = (b_1, b_2, b_3)$ be an orthonormal basis in V and denote by $\pi : V \wedge V \to V$ the linear map given by $\pi(b_i \wedge b_j) = b_k$, where i, j, k is a cyclic permutation of $1, 2, 3$. Then:*

(i) *π is an isomorphism and π and $-\pi$ are the only vector products on V.*

(ii) *If* $\eta : V \wedge V \to V$ *is a dissident map then there exists a unique definite linear endomorphism* $\varepsilon : V \to V$ *such that* $\varepsilon\pi = \eta$.

(iii) *If* $\varepsilon : V \to V$ *is a definite linear endomorphism, then* $\varepsilon\pi$ *is dissident.*

(iv) *If* $\xi : V \wedge V \to k$ *is a linear form, then* $\xi = \xi'\pi$ *for a unique linear form* $\xi' : V \to k$.

Proof. For proof of (i)–(iii) we refer to [13, Lemmas 2.1 and 2.2]. (iv) is an easy consequence of (i). ∎

The following proposition is a generalization of [9, Corollary 6] from $k = \mathbb{R}$ to the case of an arbitrary square-ordered field k.

Proposition 4.5. *Let* $\mathcal{A}$ *be a quadratic division algebra over a square-ordered field* k. *If* $\dim \mathcal{A} \in \{1, 2, 4\}$ *then* $\mathcal{V}(\mathcal{A})$ *is again a division algebra.*

Proof. Let $\mathcal{A} = k \oplus V$ be the Frobenius decomposition of $\mathcal{A}$. By Osborn's theorem and Proposition 2.4, $\mathcal{V}(\mathcal{A})$ is anisotropic. Hence, again by Osborn's theorem, we only need to check if $\eta^{\mathrm{d}} : V^{\mathrm{d}} \wedge V^{\mathrm{d}} \to V^{\mathrm{d}}$ is dissident for $\dim V \in \{0, 1, 3\}$. By the equivalence $\mathcal{I}(\mathcal{A}) = (V, q, \xi, \eta)$ and from Proposition 3.2 we know that

$$\eta^{\mathrm{d}}\left(\begin{pmatrix} v \\ \alpha \\ w \end{pmatrix} \wedge \begin{pmatrix} v' \\ \alpha' \\ w' \end{pmatrix}\right) = \begin{pmatrix} \alpha w' - \alpha' w + \eta(v \wedge v') - \eta(w \wedge w') \\ \langle w, v'\rangle - \langle v, w'\rangle - \xi(w \wedge v') - \xi(v \wedge w') \\ -\alpha v' + \alpha' v - \eta(w \wedge v') - \eta(v \wedge w') \end{pmatrix}.$$

Let $\pi : k^3 \wedge k^3 \to k^3$ be the linear map described in Lemma 4.4. If $\dim V = 0$, then $\dim V^{\mathrm{d}} = 1$ and hence η^{d} is dissident. If $\dim V = 1$ then $\eta = 0$ and $\xi = 0$, hence $\eta^{\mathrm{d}} = 0^{\mathrm{d}} \simeq \pi$. Therefore, by Lemma 4.4, π is a vector product and hence dissident.

Let $\dim V = 3$. By Lemma 4.4 we get $\eta = \varepsilon\pi$ and $\xi = \xi'\pi$, where $\varepsilon : V \to V$ is definite, $\pi : V \wedge V \to V$ is a vector product and $\xi' : V \to k$ is such that $\xi' = \xi\pi^{-1}$. Assume that η^{d} is not dissident. Then there exist non-proportional vectors $(v\ \alpha\ w)^t$ and $(v'\ \alpha'\ w')^t$ and $\mu, \nu \in k$ such that

$$\mu\begin{pmatrix} v \\ \alpha \\ w \end{pmatrix} + \nu\begin{pmatrix} v' \\ \alpha' \\ w' \end{pmatrix} = \begin{pmatrix} \alpha w' - \alpha' w + \varepsilon\pi(v \wedge v') - \varepsilon\pi(w \wedge w') \\ \langle w, v'\rangle - \langle v, w'\rangle - \xi'\pi(w \wedge v') - \xi'\pi(v \wedge w') \\ -\alpha v' + \alpha' v - \varepsilon\pi(w \wedge v') - \varepsilon\pi(v \wedge w') \end{pmatrix}.$$

Hence we get the system of equations

$$\mu v + \nu v' - \alpha w' + \alpha' w = \varepsilon(\pi(v \wedge v') - \pi(w \wedge w')), \tag{4.1}$$

$$\mu\alpha + \nu\alpha' - \langle w, v'\rangle + \langle v, w'\rangle = \xi'(-\pi(w \wedge v') - \pi(v \wedge w')), \tag{4.2}$$

$$\mu w + \nu w' - \alpha' v + \alpha v' = \varepsilon(-\pi(w \wedge v') - \pi(v \wedge w')). \tag{4.3}$$

From (4.1) and (4.3) we get the equations

$$\begin{aligned}&\langle \pi(v\wedge v')-\pi(w\wedge w'),\mu v+\nu v'-\alpha w'+\alpha' w\rangle\\ &\qquad\qquad= \langle \pi(v\wedge v')-\pi(w\wedge w'),\varepsilon(\pi(v\wedge v')-\pi(w\wedge w'))\rangle,\\ &\langle -\pi(w\wedge v')-\pi(v\wedge w'),\mu w+\nu w'-\alpha' v+\alpha v'\rangle\\ &\qquad\qquad= \langle \pi(w\wedge v')+\pi(v\wedge w'),\ \varepsilon(\pi(w\wedge v')+\pi(v\wedge w'))\rangle,\end{aligned}$$

which, added up, become

$$\begin{aligned}&\langle \pi(v\wedge v')-\pi(w\wedge w'),\varepsilon(\pi(v\wedge v')-\pi(w\wedge w'))\rangle\\ &\qquad+\langle -\pi(w\wedge v')-\pi(v\wedge w'),\varepsilon(-\pi(w\wedge v')-\pi(v\wedge w'))\rangle=0.\end{aligned}$$

Accordingly ε definite implies

$$\pi(v\wedge v')-\pi(w\wedge w')=0, \tag{4.4}$$

$$\pi(w\wedge v')+\pi(v\wedge w')=0. \tag{4.5}$$

Hence the right-hand sides in equations (4.1)–(4.3) are 0. Since

$$\begin{aligned}&\langle \mu v+\nu v'-\alpha w'+\alpha' w,\alpha' w-\alpha w'\rangle-\langle \mu w+\nu w'+\alpha v'-\alpha' v,\alpha' v-\alpha v'\rangle\\ &\qquad=(\mu\alpha+\nu\alpha')^2+\langle \alpha' w-\alpha w',\alpha' w-\alpha w'\rangle+\langle \alpha' v-\alpha v',\alpha' v-\alpha v'\rangle\end{aligned}$$

we see that $\mu\alpha+\nu\alpha'=0$, $\alpha' w=\alpha w'$ and $\alpha' v=\alpha v'$; but then $\alpha'(v\ \alpha\ w)^t=\alpha(v'\ \alpha'\ w')^t$ and hence $\alpha=\alpha'=0$. Equations (4.1)–(4.3) now reduce to

$$\mu v+\nu v'=0, \tag{4.6}$$

$$\langle v,w'\rangle-\langle w,v'\rangle=0, \tag{4.7}$$

$$\mu w+\nu w'=0. \tag{4.8}$$

If $v=0$ or $w'=0$ we find, by (4.5) and (4.7), that $w=0$ or $v'=0$. Then, by (4.4), $(v\ 0\ w)^t$ and $(v'\ 0\ w')^t$ are proportional. Hence $v\neq 0$ and $w'\neq 0$. By (4.6) and (4.8) there exist $\beta,\gamma\in k$ such that $v'=\beta v$ and $w=\gamma w'$. From (4.7) and (4.5) we infer that $(1-\beta\gamma)\langle v,w'\rangle=0$ and $(1-\beta\gamma)\pi(v\wedge w')=0$, implying $\beta\gamma=1$. But if $\beta\gamma=1$, then $\beta(v\ 0\ w)^t=(v'\ 0\ w')$, contradicting the linear independence of $(v\ \alpha\ w)^t$ and $(v'\ \alpha'\ w')^t$. Hence η^{d} is dissident. ∎

One may wonder why we excluded the case $\dim\mathcal{A}=3$ from the previous proposition. In fact, there is no quadratic division algebra of dimension three (cf. [13]).

Denote by $\mathcal{Q}^{\mathrm{d}}_n(k)$ the full subcategory of $\mathcal{Q}^{\mathrm{d}}(k)$ which is formed by all objects of dimension n. Similarly, $\mathcal{D}_{n-1}(k)$ denotes the full subcategory of $\mathcal{D}(k)$ formed by all objects of dimension $n-1$.

Corollary 4.6. *Let k be a square-ordered field. For all $n\in\{1,2,4\}$ the doubling functor $\mathcal{V}:\mathcal{Q}(k)\to\mathcal{Q}(k)$ induces a functor between full subcategories $\mathcal{V}:\mathcal{Q}^{\mathrm{d}}_n(k)\to\mathcal{Q}^{\mathrm{d}}_{2n}(k)$ and the doubling functor $\mathcal{J}:\mathcal{E}(k)\to\mathcal{E}(k)$ induces a functor between full subcategories $\mathcal{J}:\mathcal{D}_{n-1}(k)\to\mathcal{D}_{2n-1}(k)$.*

5. Doubling of real quadratic division algebras. Henceforth we assume that $k = \mathbb{R}$. We use the shorthands $\mathcal{D} = \mathcal{D}(\mathbb{R})$, $\mathcal{D}_m = \mathcal{D}_m(\mathbb{R})$, $\mathcal{Q} = \mathcal{Q}(\mathbb{R})$, $\mathcal{Q}_n^{\mathrm{d}} = \mathcal{Q}_n^{\mathrm{d}}(\mathbb{R})$ etc. Accordingly, by a *division algebra* we mean a real division algebra.

By Lemma 4.4(ii), every dissident map η on a 3-dimensional Euclidean space V admits a factorization $\eta = \varepsilon\pi$ into a vector product π on V and a definite linear endomorphism ε of V. This fact not only leads to a complete and irredundant classification of all dissident maps on $\mathbb{R}^3$, but also to a complete and irredundant classification of all 3-dimensional dissident triples. In view of Proposition 4.3 it also leads to a complete and irredundant classification of $\mathcal{Q}_4^{\mathrm{d}}$ [7], [13]. To be able to state this assertion more precisely, we need to recall the category $\mathcal{K}$ of configurations in $\mathbb{R}^3$, which can be found e.g. in [8], [10], [13]. We set $\mathcal{T} = \{d \in \mathbb{R}^3 \mid 0 < d_1 \leq d_2 \leq d_3\}$ and denote, for any $d \in \mathcal{T}$, by D_d the diagonal matrix in $\mathbb{R}^{3\times 3}$ with diagonal sequence d. The category $\mathcal{K}$ is defined by $\mathrm{Ob}(\mathcal{K}) = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{T}$ and

$$\mathrm{Mor}_{\mathcal{K}}((x,y,d),(x',y',d')) = \{S \in \mathrm{SO}_3(\mathbb{R}) \mid (Sx, Sy, SD_dS^t) = (x',y',D_{d'})\}.$$

We can interpret $\mathcal{K}$ geometrically by identifying the objects $(x,y,d) \in \mathcal{K}$ with those configurations in $\mathbb{R}^3$ which consist of a pair of points (x,y) and an ellipsoid $E_d = \{z \in \mathbb{R}^3 \mid zD_dz^t = 1\}$. By identifying $\mathrm{SO}_3(\mathbb{R})$ with $\mathrm{SO}(\mathbb{R}^3)$, the morphisms $S : (x,y,d) \mapsto (x',y',d')$ in $\mathcal{K}$ are identified with those rotations of $E_d = E_{d'}$ which simultaneously send x to x' and y to y'. This explains why the category $\mathcal{K}$ is called the category of configurations.

With any given configuration $\kappa = (x,y,d) \in \mathcal{K}$ we associate the dissident triple $\mathcal{G}(\kappa) = (\mathbb{R}^3, \xi_x, \eta_{yd})$ defined by $\xi_x(v \wedge w) = v^t M_x w = \langle v, \pi(x \wedge w)\rangle$ and $\eta_{yd} = (M_y + D_d)\pi(v \wedge w)$ for all $v, w \in \mathbb{R}^3$, where $\pi : \mathbb{R}^3 \wedge \mathbb{R}^3 \to \mathbb{R}^3$ is the vector product described in Lemma 4.4 with respect to the standard basis in $\mathbb{R}^3$ and

$$M_z = \begin{pmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{pmatrix}$$

for all $z \in \mathbb{R}^3$. By Lemma 4.4(iii), η_{yd} is a dissident map on $\mathbb{R}^3$, and hence $\mathcal{G}(\kappa)$ is a dissident triple. Moreover, the construction $\mathcal{G} : \mathcal{K} \to \mathcal{D}$ is functorial, acting on morphisms identically.

PROPOSITION 5.1 ([12, Proposition 1.5]). *The functor $\mathcal{G} : \mathcal{K} \to \mathcal{D}$ induces an equivalence of categories $\mathcal{G} : \mathcal{K} \to \mathcal{D}_3$.*

A cross-section $\mathcal{C}$ for $\mathcal{K}/\simeq$ was first given in [7, pp. 17–18], thus solving the problem of classifying $\mathcal{D}_3$, hence classifying $\mathcal{Q}_4^{\mathrm{d}}$. Hence, by applying the doubling functor, we get all doubled 8-dimensional quadratic division algebras as the following commutative diagram shows.

$$\begin{tikzcd}
 & \mathcal{D}_7 & \mathcal{Q}_8^{\mathrm{d}} \arrow[l, "\mathcal{I}"'] \\
\mathcal{K} \arrow[r, "\mathcal{G}"'] & \mathcal{D}_3 \arrow[u, "\mathcal{J}"] \arrow[r, "\mathcal{H}"'] & \mathcal{Q}_4^{\mathrm{d}} \arrow[u, "\mathcal{V}"']
\end{tikzcd}$$

The doubling functor $\mathcal{V}$ is faithful but not full, according to Proposition 2.1. Hence, the cross-section $\mathcal{C}$ for $\mathcal{K}/\simeq$ gives rise to a complete, but maybe redundant, list of doubled 8-dimensional quadratic division algebras. Despite this, we have strong indications that the list is in fact irredundant and in the last section we will prove partial results supporting the following conjecture.

CONJECTURE 5.2. *If κ and κ' are configurations in $\mathcal{K}$ such that $\mathcal{VHG}(\kappa)$ and $\mathcal{VHG}(\kappa')$ are isomorphic then κ and κ' are isomorphic.*

6. Separating isomorphism classes of doubled 8-dimensional real quadratic division algebras. From the standard basis $\underline{e}=(e_1,e_2,e_3\,|\,e_4\mid e_5,e_6,e_7)$ in $\mathbb{R}^7$ we construct the subset $\{\pm e_i\wedge e_j \mid 1\le i<j\le 7\}$ of $\mathbb{R}^7\wedge\mathbb{R}^7$, which after any choice of signs and total order becomes a basis in $\mathbb{R}^7\wedge\mathbb{R}^7$, denoted by $\underline{e}\wedge\underline{e}$. Using the shorthands $e_{ij}=e_i\wedge e_j$, we fix our choice so that $\underline{e}\wedge\underline{e}=(e_{23},e_{31},e_{12}\mid e_{72},e_{17},e_{61}\mid e_{14},e_{24},e_{34}\mid e_{15},e_{26},e_{37}\mid e_{45},e_{46},e_{47}\mid e_{36},e_{53},e_{25}\mid e_{76},e_{57},e_{65})$. Given a matrix $\mathcal{Y}\in\mathbb{R}^{7\times 21}$, we denote by $\underline{\mathcal{Y}}:\mathbb{R}^7\wedge\mathbb{R}^7\to\mathbb{R}^7$ the linear map represented by the matrix $\mathcal{Y}$ in the bases $\underline{e}\wedge\underline{e}$ and $\underline{e}$. Similarly, the notation $\underline{\mathcal{X}}:\mathbb{R}^7\wedge\mathbb{R}^7\to\mathbb{R}$ stands for the linear map defined by $\underline{\mathcal{X}}(v\wedge w)=v^t\mathcal{X}w$, represented by the matrix $\mathcal{X}\in\mathbb{R}^{7\times 7}$. By a *doubled* dissident triple we mean a dissident triple (V,ξ,η) such that $\mathcal{H}(V,\eta,\xi)$ is a doubled division algebra. Furthermore, a dissident map $\eta: V\wedge V\to V$ is called *doubled* if it occurs as the third component in a doubled dissident triple (V,ξ,η).

We now proceed to build up an exhaustive family of doubled division algebras on $\mathbb{R}^7$ by constructing an exhaustive family of doubled dissident triples. We set

$$E_{yd}=M_y+D_d=\begin{pmatrix} d_1 & -y_3 & y_2\\ y_3 & d_2 & -y_1\\ -y_2 & y_1 & d_3\end{pmatrix},\qquad \mathcal{X}(\kappa)=\left(\begin{array}{c|c|c} M_x & 0 & 0\\ \hline 0 & 0 & 0\\ \hline 0 & 0 & -M_x\end{array}\right),$$

$$\mathcal{Y}(\kappa)=\left(\begin{array}{c|c|c|c|c|c|c} E_{yd} & 0 & 0 & 0 & \mathbb{I}_3 & 0 & E_{yd}\\ \hline 0 & -x^t & 0 & -1_3 & 0 & -x^t & 0\\ \hline 0 & E_{yd} & \mathbb{I}_3 & 0 & 0 & E_{yd} & 0\end{array}\right)$$

and using the chain of functors

$$\mathcal{K} \xrightarrow{\mathcal{G}} \mathcal{D}_3 \xrightarrow{\mathcal{J}} \mathcal{D}_7$$

we deduce from $(\mathbb{R}^3, \xi_x, \eta_{yd}) = \mathcal{G}(\kappa)$ that $\mathcal{J}\mathcal{G}(\kappa) = (\mathbb{R}^7, \underline{\mathcal{X}(\kappa)}, \underline{\mathcal{Y}(\kappa)})$. Here, $\underline{\mathcal{X}(\kappa)}$ and $\underline{\mathcal{Y}(\kappa)}$ can be viewed as functions of κ and henceforth we will look at them as maps from $\mathcal{K}$ to $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^7, \mathbb{R}^7)$ and $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^7 \wedge \mathbb{R}^7, \mathbb{R}^7)$ respectively. Accordingly, we will write $\underline{\mathcal{X}}(\kappa) = \underline{\mathcal{X}(\kappa)}$ and $\underline{\mathcal{Y}}(\kappa) = \underline{\mathcal{Y}(\kappa)}$.

LEMMA 6.1. *Each configuration $\kappa \in \mathcal{K}$ determines a 4-dimensional real quadratic division algebra $\mathcal{A}(\kappa) = \mathcal{H}\mathcal{G}(\kappa)$ and an 8-dimensional real quadratic division algebra $\mathcal{B}(\kappa) = \mathcal{V}(\mathcal{A}(\kappa))$. The latter has the following properties.*

(i) *Denote the standard basis in $\mathcal{A}(\kappa)$ by (e_0, e_1, e_2, e_3). Then the sequence $\underline{b} = ((e_1, 0), (e_2, 0), (e_3, 0) \mid (0, e_0) \mid (0, e_1), (0, e_2), (0, e_3))$ of vectors in $\mathcal{B}(\kappa)$ is an orthonormal basis for the purely imaginary hyperplane V in $\mathcal{B}(\kappa)$.*

(ii) *The linear form $\xi(\kappa) : V \wedge V \to \mathbb{R}$ depends on x only and is represented in $\underline{b}$ by the matrix $\mathcal{X}(\kappa)$.*

(iii) *The dissident map $\eta(\kappa) : V \wedge V \to V$ is represented in $\underline{b}$ and $\underline{b} \wedge \underline{b}$ by the matrix $\mathcal{Y}(\kappa)$.*

(iv) *Let $\sigma : V \xrightarrow{\sim} \mathbb{R}^7$ be the orthogonal isomorphism identifying $\underline{b}$ with the standard basis $\underline{e}$ in $\mathbb{R}^7$. Then σ is also an isomorphism of dissident triples $\sigma : \mathcal{I}(\mathcal{B}(\kappa)) = (V, \xi(\kappa), \eta(\kappa)) \xrightarrow{\sim} (\mathbb{R}^7, \underline{\mathcal{X}}(\kappa), \underline{\mathcal{Y}}(\kappa))$.*

PROPOSITION 6.2 ([12]). (i) *For each configuration $\kappa \in \mathcal{K}$ the dissident triple $(\mathbb{R}^7, \underline{\mathcal{X}}(\kappa), \underline{\mathcal{Y}}(\kappa))$ is a doubled dissident triple.*

(ii) *Each doubled dissident triple $(\mathbb{R}^7, \xi, \eta)$ is isomorphic to the triple $(\mathbb{R}^7, \underline{\mathcal{X}}(\kappa), \underline{\mathcal{Y}}(\kappa))$ for some $\kappa \in \mathcal{K}$.*

(iii) *If $\kappa, \kappa' \in \mathcal{K}$ are isomorphic configurations, then the dissident triples $(\mathbb{R}^7, \underline{\mathcal{X}}(\kappa), \underline{\mathcal{Y}}(\kappa))$ and $(\mathbb{R}^7, \underline{\mathcal{X}}(\kappa'), \underline{\mathcal{Y}}(\kappa'))$ are isomorphic.*

With any dissident map on a Euclidean space V, we associate the subspace $V_\eta = \{v \in V \mid \langle u, \eta(v \wedge w)\rangle = \langle \eta(u \wedge v), w\rangle \text{ for all } u, w \in V\}$ of V. Dissident maps $\eta : V \wedge V \to V$ with $V_\eta = V$ are called *flexible* [11]. Note that by Proposition 3.2 for all $\kappa \in \mathcal{K}$ we have

$$\eta^{\mathrm{d}}_{yd}(\kappa)\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} v \\ \alpha \\ w \end{pmatrix}\right) = \begin{pmatrix} w \\ 0 \\ -v \end{pmatrix} \quad \text{for all } \begin{pmatrix} v \\ \alpha \\ w \end{pmatrix} \in \mathbb{R}^7.$$

Accordingly, the linear endomorphism $\underline{\mathcal{Y}}(\kappa)(e_4 \wedge ?) : \mathbb{R}^7 \wedge \mathbb{R}^7$ to $\mathbb{R}^7$, given by $\underline{\mathcal{Y}}(\kappa)(e_4 \wedge ?) := \eta^{\mathrm{d}}_{yd}(\kappa)(e_4 \wedge ?)$ is antisymmetric, so $e_4 \in \mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)}$ for all $\kappa \in \mathcal{K}$.

The subspace $\mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)}$ of $\mathbb{R}^7$ proves to be useful in our attempt to prove Conjecture 5.2. We proceed by constructing a complete and explicit description of the subspaces $\mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)}$ of $\mathbb{R}^7$, for all $\kappa \in \mathcal{K}$. Before stating this description we need to partition the category $\mathcal{K}$ into the pairwise disjoint subsets

$$\begin{aligned}
\mathcal{K}_7 &= \{(x,y,d) \in \mathcal{K} \mid x = y = 0 \wedge d_1 = d_2 = d_3\},\\
\mathcal{K}_{31} &= \{(x,y,d) \in \mathcal{K} \mid x \neq 0 \wedge y = 0 \wedge d_1 = d_2 = d_3\},\\
\mathcal{K}_{32} &= \{(x,y,d) \in \mathcal{K} \mid x_1 = x_2 = 0 \wedge y = 0 \wedge d_1 = d_2 < d_3\},\\
\mathcal{K}_{33} &= \{(x,y,d) \in \mathcal{K} \mid x_2 = x_3 = 0 \wedge y = 0 \wedge d_1 < d_2 = d_3\},\\
\mathcal{K}_{34} &= \{(x,y,d) \in \mathcal{K} \mid x \in [x_{yd}] \wedge y = \pm\varrho_d e_2 \wedge d_1 < d_2 < d_3\},\\
\mathcal{K}_1 &= \mathcal{K} \setminus (\mathcal{K}_7 \cup \mathcal{K}_{31} \cup \mathcal{K}_{32} \cup \mathcal{K}_{33} \cup \mathcal{K}_{34}),
\end{aligned}$$

where $[x_{yd}] = (-y_2\ 0\ d_3 - d_2)^t$ and $\varrho_d = \sqrt{(d_3-d_2)(d_2-d_1)}$. Moreover, we set $\mathcal{K}_3 = \mathcal{K}_{31} \cup \mathcal{K}_{32} \cup \mathcal{K}_{33} \cup \mathcal{K}_{34}$. We also introduce the linear injections $\iota_{<4} : \mathbb{R}^3 \to \mathbb{R}^7$ given by $\iota_{<4}(x) = x_1e_1 + x_2e_2 + x_3e_3$ and $\iota_{>4} : \mathbb{R}^3 \to \mathbb{R}^7$ given by $\iota_{>4}(x) = x_1e_5 + x_2e_6 + x_3e_7$, identifying $\mathbb{R}^3$ with the first respectively the last factor of $\mathbb{R}^7 = (\mathbb{R}^3)^{\mathrm{d}} = \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3$. We can now recall the following two lemmas from [12].

LEMMA 6.3 ([12, Lemma 5.2]). *The subspace $\mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} \subset \mathbb{R}^7$ determined by any configuration $\kappa = (x,y,d) \in \mathcal{K}$ admits the following description.*

(i) *If $\kappa \in \mathcal{K}_7$ then $\mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} = \mathbb{R}^7$.*
(ii) *If $\kappa \in \mathcal{K}_{31}$ then $\mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} = [\iota_{<4}(x), e_4, \iota_{>4}(x)]$.*
(iii) *If $\kappa \in \mathcal{K}_{32}$ then $\mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} = [e_3, e_4, e_7]$.*
(iv) *If $\kappa \in \mathcal{K}_{33}$ then $\mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} = [e_1, e_4, e_5]$.*
(v) *If $\kappa \in \mathcal{K}_{34}$ then $\mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} = [\iota_{<4}(x_{yd}), e_4, \iota_{>4}(x_{yd})]$.*
(vi) *If $\kappa \in \mathcal{K}_1$ then $\mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} = [e_4]$.*

LEMMA 6.4 ([12, Lemma 5.1]). *Let $\sigma : (V,\xi,\eta) \xrightarrow{\sim} (V',\xi',\eta')$ be an isomorphism of dissident triples. Then σ induces an isomorphism of Euclidean spaces $\sigma : V_\eta \xrightarrow{\sim} V_{\eta'}$.*

We introduce the map $\delta : \mathcal{K} \to \{0,1,\dots,7\}$, defined by $\delta(\kappa) = \dim \mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)}$. Using the previous two lemmas we get the following proposition.

PROPOSITION 6.5 ([12, Proposition 5.3]). (i) *The image and non-empty fibres of δ are given by* $\mathrm{im}(\delta) = \{1,3,7\}$ *and $\delta^{-1}(m) = \mathcal{K}_m$ for all $m \in \{1,3,7\}$.*
(ii) *Let $\kappa, \kappa' \in \mathcal{K}$ be configurations such that $\underline{\mathcal{Y}}(\kappa)$ and $\underline{\mathcal{Y}}(\kappa')$ are isomorphic. Then $\delta(\kappa) = \delta(\kappa')$.*

By restricting $\mathcal{K}$ to $\mathcal{K}_1$, $\mathcal{K}_3$ and $\mathcal{K}_7$ respectively, the last proposition decomposes the problem of proving Conjecture 5.2 into three separate subproblems. By using results from [11], we will now decompose these subproblems even further.

As in the previous decomposition of the problem we will use the properties of the dissident map $\eta : \mathbb{R}^7 \wedge \mathbb{R}^7 \to \mathbb{R}^7$. This refined decomposition builds upon the theory of liftings of dissident maps [11]. Note that [11] does not restrict the dimension of the vector space, V, where $\eta : V \wedge V \to V$, to be 7, nor does it require that η is a doubled dissident map. In the present article we will restrict ourselves to the single case of $\dim V = 7$. For proofs and further details we refer the reader to [11]. We begin by recalling what a lifting of a dissident map on $\mathbb{R}^7$ is.

Let $(\mathbb{R}^7, \xi, \eta)$ be a doubled dissident triple. Then η induces a selfmap $\eta_{\mathbb{P}} : \mathbb{P}(\mathbb{R}^7) \to \mathbb{P}(\mathbb{R}^7)$, $\eta_{\mathbb{P}}[v] = (\eta(v \wedge \mathbb{R}^7))^{\perp}$, of the 6-dimensional real projective space. From [12, Proposition 2.2] we know that $\eta_{\mathbb{P}}$ is always bijective, and [12, Proposition 4.5] shows that it may or may not be collinear. In view of the fundamental theorem of projective geometry, this can be equivalently expressed by saying that $\eta_{\mathbb{P}}$ may or may not be lifted to a linear bijection $\Phi : \mathbb{R}^7 \to \mathbb{R}^7$.

We denote by R the polynomial ring $\mathbb{R}[X_1, \ldots, X_7]$ and by R_d the dth homogeneous summand of R. The *greatest common divisor* of a sequence $p_1, \ldots, p_r$ in R is the unique minimal principal ideal (d) in R containing the ideal $(p_1, \ldots, p_r)$ generated by $p_1, \ldots, p_r$. The greatest common divisor will be denoted by $\gcd(p_1, \ldots, p_r)$.

We define a *lifting* of a dissident map on the seven-dimensional Euclidean space to be a map $\Phi : \mathbb{R}^7 \to \mathbb{R}^7$, $\Phi(v) = (\phi_1(v), \ldots, \phi_7(v))$, such that

(a) $\{\phi_1, \ldots, \phi_7\} \subset R_d$ for some $d \in \mathbb{N}$.
(b) $\eta_{\mathbb{P}}[v] = [\Phi(v)]$ for all $v \in \mathbb{R}^7 \setminus \{0\}$.
(c) $\gcd(\phi_1, \ldots, \phi_7) = (1)$.

The *support* of a lifting Φ is defined by $I(\Phi) = \{i \in \underline{7} \mid \phi_i \neq 0\}$ and the *degree* of Φ is $\deg \Phi = \deg \phi_i$ for all $i \in I(\Phi)$. According to (a) we have $\deg \Phi \geq 0$.

Proposition 6.6 ([11, Theorem 2.4 and Proposition 3.1]). *Let* $(\mathbb{R}^7, \xi, \eta)$ *be a dissident triple. Then*:

(i) *There exists a lifting of* $\eta_{\mathbb{P}}$.
(ii) *The degree of a lifting* Φ *of* $\eta_{\mathbb{P}}$ *is uniquely determined by* η. *It satisfies* $1 \leq \deg \Phi \leq 5$. *Moreover*, $\deg \Phi = 1$ *if and only if* $\eta_{\mathbb{P}}$ *is collinear.*
(iii) *Let* $(\mathbb{R}^7, \xi, \eta)$ *and* $(\mathbb{R}^7, \xi', \eta')$ *be isomorphic dissident triples and let* Φ *and* Φ' *be liftings of* $\eta_{\mathbb{P}}$ *respectively* $\eta'_{\mathbb{P}}$. *Then* $\deg \Phi = \deg \Phi'$.

This proposition states all we need for a refined decomposition of $\mathcal{K}$; by (i) every dissident triple $(\mathbb{R}^7, \xi, \eta)$ gives rise to a lifting Φ of $\eta_{\mathbb{P}}$, and by (iii), $\deg \Phi$ is preserved under isomorphisms. Without delving into details on how to calculate the liftings, and hence the degree, we state the following result.

PROPOSITION 6.7 ([11, Proposition 6.3]). *Let* $\kappa = (x, y, d) \in \mathcal{K}$.

(i) *If* $y = 0$ *and* $(x, d) = (0, d_1 1_3)$, *then* $\deg \mathcal{VHG}(\kappa) = 1$.
(ii) *If* $y = 0$ *and* $(x, d) \neq (0, d_1 1_3)$, *then* $\deg \mathcal{VHG}(\kappa) = 3$.
(iii) *If* $y \neq 0$, *then* $\deg \mathcal{VHG}(\kappa) = 5$.

Using this proposition and Proposition 6.5 we can decompose the problem of proving Conjecture 5.2 into separate subproblems by decomposing $\mathcal{K}$ further. Let $\mathcal{K}_i^0 = \{\kappa \in \mathcal{K}_i \mid y = 0\}$ and $\mathcal{K}_i^y = \{\kappa \in \mathcal{K}_i \mid y \neq 0\}$ for each $i \in \{1, 3, 7\}$. Noting that $\mathcal{K}_7^0 = \mathcal{K}_7$, $\mathcal{K}_7^y = \emptyset$, $\mathcal{K}_3^0 = \mathcal{K}_3 \setminus \mathcal{K}_{34}$ and $\mathcal{K}_3^y = \mathcal{K}_{34}$ we get the decomposition $\mathcal{K} = \mathcal{K}_7 \dot\cup (\mathcal{K}_{31} \cup \mathcal{K}_{32} \cup \mathcal{K}_{33}) \dot\cup \mathcal{K}_{34} \dot\cup \mathcal{K}_1^0 \dot\cup \mathcal{K}_1^y$, where $A \dot\cup B$ denotes the union of the pairwise disjoint sets A and B.

COROLLARY 6.8. *Let* $\kappa, \kappa' \in \mathcal{K}$ *be configurations such that* $\mathcal{B}(\kappa)$ *and* $\mathcal{B}(\kappa')$ *are isomorphic. Then* κ *and* κ' *belong to the same set of disjoint subsets of the partition* $\mathcal{K} = \mathcal{K}_7 \dot\cup \mathcal{K}_3^0 \dot\cup \mathcal{K}_{34} \dot\cup \mathcal{K}_1^0 \dot\cup \mathcal{K}_1^y$.

We are now ready to partially prove Conjecture 5.2.

7. On the classification of doubled real division algebras. In this section we will prove that if $\kappa, \kappa' \in \mathcal{K}_7 \cup \mathcal{K}_3^0$ are such that $\mathcal{VHG}(\kappa)$ and $\mathcal{VHG}(\kappa')$ are isomorphic, then κ and κ' are isomorphic.

We set $\mathcal{R} = \{r \in \mathbb{R}^7 \mid r_0 = 0,\, r_1 \leq \cdots \leq r_6\}$. Recall that $\mathcal{T} = \{t \in \mathbb{R}^3 \mid 0 < t_1 \leq t_2 \leq t_3\}$. Let $V = (V, \langle\ ,\ \rangle)$ be a Euclidean space. Recall that each $L \in \mathrm{End}(V)$ can be uniquely written as $L = L^{\mathrm{s}} + L^{\mathrm{a}}$, where $L^{\mathrm{s}}, L^{\mathrm{a}} \in \mathrm{End}(V)$ with L^{s} symmetric and L^{a} antisymmetric. If $L, M \in \mathrm{End}(V)$ and $S \in \mathcal{O}(V)$ is such that $SL = MS$, then $SL^{\mathrm{s}} = M^{\mathrm{s}}S$ and $SL^{\mathrm{a}} = M^{\mathrm{a}}S$.

Given any configuration $\kappa \in \mathcal{K}$ and any vector $v \in \mathbb{R}^7 \setminus \{0\}$ the linear endomorphism $L_v = \underline{\mathcal{Y}}(\kappa)(v \wedge ?)$ of $\mathbb{R}^7$ has kernel $[v]$ and can be written uniquely as $L_v = L_v^{\mathrm{s}} + L_v^{\mathrm{a}}$. Accordingly, there exist an orthonormal basis $\underline{b}$ and a sequence $r \in \mathcal{R}$ such that L_v^{s} is represented in $\underline{b}$ by the diagonal matrix D_r with diagonal sequence r. Furthermore, there exist an orthonormal basis $\underline{c}$ and a triple t of positive real numbers $0 < t_1 \leq t_2 \leq t_3$ such that L_v^{a} is represented in $\underline{c}$ by the matrix

$$N_t = \begin{pmatrix} 0 & & & & & & \\ & 0 & -t_1 & & & & \\ & t_1 & 0 & & & & \\ & & & 0 & -t_2 & & \\ & & & t_2 & 0 & & \\ & & & & & 0 & -t_3 \\ & & & & & t_3 & 0 \end{pmatrix}.$$

Here $r \in \mathcal{R}$ and $t \in \mathcal{T}$ are uniquely determined by the given data $\kappa \in \mathcal{K}$ and

$v \in \mathbb{R}^7 \setminus \{0\}$. We express this by introducing the maps $\varrho_\kappa : \mathbb{R}^7 \setminus \{0\} \to \mathcal{S}$, $\varrho_\kappa(v) = r$, and $\tau_\kappa : \mathbb{R}^7 \setminus \{0\} \to \mathcal{T}$, $\tau_\kappa(v) = t$. Note that if $v \in \mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} \setminus \{0\}$, then $L_v^{\mathrm{s}} = 0$, hence $\varrho_\kappa(v) = 0$.

Lemma 7.1. *Let $\kappa, \kappa' \in \mathcal{K}$. If $\sigma : (\mathbb{R}^7, \underline{\mathcal{X}}(\kappa), \underline{\mathcal{Y}}(\kappa)) \to (\mathbb{R}^7, \underline{\mathcal{X}}(\kappa'), \underline{\mathcal{Y}}(\kappa'))$ is an isomorphism of dissident triples, then $\varrho_\kappa(v) = \varrho_{\kappa'}\sigma(v)$ and $\tau_\kappa(v) = \tau_{\kappa'}\sigma(v)$ for all $v \in \mathbb{R}^7 \setminus \{0\}$.*

Proof. Given κ, κ', σ and v as in the statement, set $L = \underline{\mathcal{Y}}(\kappa)(v \wedge ?)$, $M = \underline{\mathcal{Y}}(\kappa')(\sigma(v) \wedge ?)$ and write $L = L^{\mathrm{s}} + L^{\mathrm{a}}$, $M = M^{\mathrm{s}} + M^{\mathrm{a}}$. Set $\varrho_\kappa(v) = s$ and $\tau_\kappa(v) = t$. This means that L^{s} and L^{a} are represented in suitable orthonormal bases $\underline{b}$ and $\underline{c}$ respectively in $\mathbb{R}^7$ by D_s and N_t respectively. Accordingly, M^{s} and M^{a} are represented in the orthonormal bases $\sigma(\underline{b}) = (\sigma(b_1), \ldots, \sigma(b_7))$ and $\sigma(\underline{c}) = (\sigma(c_1), \ldots, \sigma(c_7))$ respectively again by D_s and N_t respectively. Hence $\varrho_\kappa(v) = \varrho_{\kappa'}\sigma(v)$ and $\tau_\kappa(v) = \tau_{\kappa'}\sigma(v)$. ∎

The study of $\varrho_\kappa(v)$ and $\tau_\kappa(v)$ turns out to be useful in our attempt to prove Conjecture 5.2. Due to Lemma 6.4 we can split this problem into two cases, $v \in \mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} \setminus \{0\}$ and $v \in (\mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)})^\perp \setminus \{0\}$. Given $v = (v_1, \ldots, v_7) \in \mathbb{R}^7$, let $v_{<4} = (v_1, v_2, v_3)$ and $v_{>4} = (v_5, v_6, v_7)$.

Lemma 7.2. *Let $\kappa \in \mathcal{K}$ be any configuration and let $v \in \mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)}$. Then*

$$\tau_\kappa(v) = \begin{cases} (\varepsilon, \varepsilon, |v|) & \text{if } 0 < d_2 \le 1, \\ (|v|, \varepsilon, \varepsilon) & \text{if } 1 \le d_2 < \infty, \end{cases}$$

where $\varepsilon = \sqrt{d_2(|v_{<4}|^2 + |v_{>4}|^2) + v_4^2}$.

Proof. This is a generalization of [12, Lemma 5.5] and the proof goes along the same lines, using the fact that the matrix representing L_v in the basis $\underline{e}$ has the same structure for all $\kappa \in \mathcal{K}$. ∎

Example 7.3. If $\kappa \in \mathcal{K}_1$ then $\mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} = [e_4]$ and $\tau_\kappa(v) = (|v|, |v|, |v|)$, which is consistent with the previous lemma, since $|v_{<4}| = |v_{>4}| = 0$.

If $v \in (\mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)})^\perp \setminus \{0\}$ the calculations are much more difficult, but using Maple to calculate $\varrho_\kappa(v)$ and $\tau_\kappa(v)$ we get the following partial result.

Lemma 7.4. *If $\kappa \in \mathcal{K}_3^0$ and $v \in (\mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)})^\perp \setminus \{0\}$ then*

$$\varrho_\kappa(v) = \big(0, -\tfrac{1}{2}\gamma|v|, -\tfrac{1}{2}(d_3 - d_1)|v|, 0, 0, \tfrac{1}{2}(d_3 - d_1)|v|, \tfrac{1}{2}\gamma|v|\big),$$
$$\tau_\kappa(v) = \big(\tfrac{1}{2}\sqrt{(\mu - \nu)}\,|v|, \tfrac{1}{2}(d_1 + d_3)|v|, \tfrac{1}{2}\sqrt{(\mu + \nu)}\,|v|\big),$$

where $\gamma = \sqrt{(d_3 - d_1)^2 + |x|^2}$ and

$$\mu = \tfrac{1}{2}(|x|^2 + (d_1 + d_3)^2 + 4), \qquad \nu = \tfrac{1}{2}\sqrt{(|x|^2 + (d_1 + d_3)^2 - 4)^2 + 16|x|^2}.$$

Proposition 7.5. *Let $\kappa = (x, y, d)$ and $\kappa' = (x', y', d')$ be configurations in $\mathcal{K}$ such that $(d_2, d_2') \neq (1, 1)$ or $\kappa, \kappa' \in \mathcal{K}_1$ and let σ be an isomorphism* *of dissident triples* $\sigma : (\mathbb{R}^7, \underline{\mathcal{X}}(\kappa), \underline{\mathcal{Y}}(\kappa)) \xrightarrow{\sim} (\mathbb{R}^7, \underline{\mathcal{X}}(\kappa'), \underline{\mathcal{Y}}(\kappa'))$. *Then* $\sigma(e_4) = \pm e_4$.

Proof. Let κ, κ' and σ be as in the statement. If $\kappa, \kappa' \in \mathcal{K}_1$, then we can apply Lemma 6.4 to get $\sigma(e_4) = \pm e_4$.

We may assume $d_2' \neq 1$. Applying Lemmas 7.1 and 7.2 to the vector $v = e_4$ we obtain $\tau_{\kappa'}\sigma(e_4) = \tau_\kappa(e_4) = (1,1,1)$, implying that $\sigma(e_4) = \pm e_4$. ■

PROPOSITION 7.6. *Let* $\kappa, \kappa' \in \mathcal{K}_7 \cup \mathcal{K}_3^0$ *be configurations such that the associated dissident triples* $(\mathbb{R}^7, \underline{\mathcal{X}}(\kappa), \underline{\mathcal{Y}}(\kappa))$ *and* $(\mathbb{R}^7, \underline{\mathcal{X}}(\kappa'), \underline{\mathcal{Y}}(\kappa'))$ *are isomorphic. Then* κ *and* κ' *are isomorphic.*

Proof. Let $\mathcal{F} : \mathcal{K} \to \mathcal{D}_7$ be the composed functor $\mathcal{F} = \mathcal{J}\mathcal{G}$. Then Lemma 6.1(iv) asserts that $\mathcal{F}(\kappa) \xrightarrow{\sim} (\mathbb{R}^7, \underline{\mathcal{X}}(\kappa), \underline{\mathcal{Y}}(\kappa))$ for all $\kappa \in \mathcal{K}$. For each $\kappa \in \mathcal{K}$ we choose

$$\kappa_n = \begin{cases} (|x|e_1, 0, d) & \text{if } \kappa \in \mathcal{K}_{31} \cup \mathcal{K}_{33}, \\ (|x|e_3, 0, d) & \text{if } \kappa \in \mathcal{K}_{32}, \\ \kappa & \text{if } \kappa \in \mathcal{K}_7, \end{cases}$$

to be the normal form of κ. Indeed, if, for example, $\kappa \in \mathcal{K}_{31}$ then any $R \in \mathrm{SO}_3(\mathbb{R})$ with $Rx = |x|e_1$ is an isomorphism $R : \kappa \xrightarrow{\sim} \kappa_n$ in $\mathcal{K}$, determining an isomorphism $\mathcal{F}(R) : \mathcal{F}(\kappa) \xrightarrow{\sim} \mathcal{F}(\kappa_n)$ in $\mathcal{D}_7$. This reduces the proof to the special case where both κ and κ' are in normal form. Furthermore, by Lemmas 6.4 and 6.3 we may assume that either $\kappa, \kappa' \in \mathcal{K}_7$ or $\kappa, \kappa' \in \mathcal{K}_3^0$.

Let $\kappa = (0,0,d)$ and $\kappa' = (0,0,d')$, where $d_1 = d_2 = d_3$ and $d_1' = d_2' = d_3'$. By Lemmas 7.1, 7.2 and Proposition 7.5 we deduce that $d_2 = d_2'$. This proves the statement for $\kappa, \kappa' \in \mathcal{K}_7$.

Let $\kappa = (x, 0, d)$ and $\kappa' = (x', 0, d')$ be configurations in $\mathcal{K}_3^0$. Letting $v \in \mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} \setminus \{0\}$, we see by Lemma 7.1 that the sixth element in $\varrho_\kappa(v)$ is $\frac{1}{2}(d_3 - d_1)|v|$ and the middle element in $\tau_\kappa(v)$ equals $\frac{1}{2}(d_1 + d_3)|v|$. Arguing along the same lines for κ' we show that the sixth element in $\varrho_{\kappa'}(\sigma(v))$ equals $\frac{1}{2}(d_3' - d_1')|v|$ and the middle element in $\tau_{\kappa'}(\sigma(v))$ is $\frac{1}{2}(d_1' + d_3')|v|$. Hence $d_1 = d_1'$ and $d_3 = d_3'$, which implies that κ and κ' belong to the same subset $\mathcal{K}_{31}$, $\mathcal{K}_{32}$ or $\mathcal{K}_{33}$ of $\mathcal{K}_3^0$. Now, by the same arguments and studying the last elements of $\varrho_\kappa(v)$ and $\varrho_{\kappa'}(\sigma(v))$, we find that $|x| = |x'|$ and hence $\kappa \xrightarrow{\sim} \kappa'$. ■

We restate this proposition in terms of doubled real quadratic division algebras.

COROLLARY 7.7. *Let* κ, κ' *be configurations in* $\mathcal{K}_7 \cup \mathcal{K}_3^0$ *such that* $\mathcal{B}(\kappa)$ *and* $\mathcal{B}(\kappa')$ *are isomorphic. Then* κ *and* κ' *are isomorphic.*

Using this corollary we see that what remains for a complete proof of Conjecture 5.2 is to prove the statement for $\kappa, \kappa' \in \mathcal{K}_3^y \cup \mathcal{K}_1$. Apparently the computations become much too complicated when trying to prove it along the same lines as in the present article. However, Propositions 2.6 and 7.5 may turn out useful for another approach to the problem.

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