COLLOQUIUM MATHEMATICUM

VOL. 102 2005 NO. 2

A NOTE ON SEMISIMPLE DERIVATIONS OF COMMUTATIVE ALGEBRAS

BY

ANDRZEJ TYC (Toruń)

Abstract. A concept of a slice of a semisimple derivation is introduced. Moreover, it is shown that a semisimple derivation d of a finitely generated commutative algebra A over an algebraically closed field of characteristic 0 is nothing other than an algebraic action of a torus on $\mathrm{Max}(A)$, and, using this, that in some cases the derivation d is linearizable or admits a maximal invariant ideal.

Introduction. Let A be a commutative algebra over an algebraically closed field k. Recall that a *derivation* of the algebra A is a k-linear map $d : A \to A$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in A$. If d is a derivation of A and $t \in k$, then we denote by A_t the subspace $\{a \in A;\ d(a) = ta\} \subset A$. It is known that $A_0 = \operatorname{Ker} d$ is a subalgebra of A called the *algebra of constants* of d. A derivation $d : A \to A$ is said to be *semisimple* if it is semisimple as a linear map, that is, if $A = \bigoplus_{t \in k} A_t$. Denote by k^+ the additive group of the field k. It is easily seen that for every semisimple derivation $d : A \to A$ the decomposition $A = \bigoplus_{t\in k} A_t$ is a k^+-*grading* of the algebra A, i.e., $1 \in A_0$ and $A_t A_{t'} \subset A_{t+t'}$ for all $t, t' \in k$. Conversely, if $A = \bigoplus_{t\in k} A_t$ is a k^+-grading of the algebra A, then one easily verifies that the map $d : A \to A$, $d(x = \sum_t x_t) = \sum_t tx_t$, is a semisimple derivation of A with $A_t = \{a \in A;\ d(a) = ta\}$ for all $t \in k$. So, a semisimple derivation of the algebra A is nothing other than a k^+-grading of A. This observation implies (see Lemma 1) that if A is finitely generated and $\mathrm{char}(k) = 0$, then the semisimple derivations of A are in one-to-one correspondence with the rational actions of a torus on the algebraic variety $\mathrm{Max}(A)$ of all maximal ideals in A. The same observation permits introducing a concept of a slice for semisimple derivations which is an analog of the well known concept of a slice for locally nilpotent derivations. This is done in Section 1, where also a corresponding structure theorem is proved. In Section 2 the linearization problem and existence of maximal invariant ideals for semisimple derivations is considered in some special cases. The main theorems of this section

2000 *Mathematics Subject Classification*: 13A02, 13B10, 14L30.

Key words and phrases: slice of semisimple derivation, torus action, linearization.

are basically translations (into the language of derivations) of some results concerning actions of the algebraic tori on the affine spaces $\mathbb{A}^n$ or actions of Hopf algebras on algebras.

1. Slices and a structure theorem. In what follows, k denotes a fixed algebraically closed field, and A denotes a fixed commutative k-algebra with unity.

Given a derivation $d : A \to A$, we denote by $E(d)$ the set of eigenvalues of d, and by $G(d)$ the subgroup of k^+ generated by $E(d)$. Notice that if $\mathrm{char}(k) = 0$, then $G(d)$ is a torsion free abelian group, and if $\mathrm{char}(k) = p > 0$, then $G(d)$ is a vector space over the simple field $\mathbb{F}_p \subset k$. The *rank* of d (we write $\mathrm{rk}(d)$) is meant to be the rank of the abelian group $G(d)$ provided $\mathrm{char}(k) = 0$, and $\dim_{\mathbb{F}_p} G(d)$ provided $\mathrm{char}(k) = p > 0$. It is clear that the group $G(d)$ is important for semisimple derivations, because then $A = \bigoplus_{t\in G(d)} A_t$. Observe that if A is a domain, then $E(d)$ is a submonoid of $G(d)$.

EXAMPLES. 1. If $A = k[X_1, \ldots, X_n]$ and $t_1, \ldots, t_n \in k$, then the derivation $d : A \to A$ given by $d(X_i) = t_iX_i$, $i = 1, \ldots, n$, is semisimple and $G(d) = \mathbb{Z}t_1 + \cdots + \mathbb{Z}t_n \subset k^+$.

2. Let $A = k[X, Y]/(X^2 - Y^3)$, and let d be the derivation of A determined by $d(X) = 3X$, $d(Y) = 2Y$. Then d is semisimple and $G(d) = 2\mathbb{Z}1_k + 3\mathbb{Z}1_k = \mathbb{Z}1_k$.

LEMMA 1. *Assume that the algebra A is finitely generated and d is a semisimple derivation of A. Then the group $G(d)$ is finitely generated, and* $\mathrm{rk}(d) \le n(d)$, *where $n(d)$ is the minimal number of eigenvectors of d which generate the algebra A. In particular, if $n = \mathrm{rk}(d)$, then $G(d) \simeq \mathbb{Z}^n$ when* $\mathrm{char}(k) = 0$, *and $G(d) \simeq \mathbb{F}_p^n$ when* $\mathrm{char}(k) = p > 0$.

Proof. Let $E = E(d)$. As $A = \bigoplus_{t\in E} A_t$ and A is finitely generated, there exist $t_1, \ldots, t_n \in E$ and eigenvectors $a_i \in A_{t_i}$, $i = 1, \ldots, n$, such that $A = k[a_1, \ldots, a_n]$. It is sufficient to show that $E \subset \mathbb{N}t_1 + \cdots + \mathbb{N}t_n$. Let J be the subset of $\mathbb{N}^n$ such that $\{a^\alpha; \alpha \in J\}$ is a basis of A as a vector space over k, where $a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. Now let $t \in E$. This means that $d(a) = ta$ for some nonzero $a \in A$. But $a = \sum_{\alpha\in J} l_\alpha a^\alpha$ for some $l_\alpha \in k$. It follows that $\sum_{\alpha\in J} tl_\alpha a^\alpha = ta = d(a) = \sum_{\alpha\in J}(\alpha_1t_1 + \cdots + \alpha_nt_n)l_\alpha a^\alpha$, whence $tl_\alpha = (\alpha_1t_1 + \cdots + \alpha_nt_n)l_\alpha$ for all $\alpha \in J$. Consequently, $t = \alpha_1t_1 + \cdots + \alpha_nt_n$ for some $\alpha = (\alpha_1, \ldots, \alpha_n) \in J$, because $a \neq 0$. The lemma is proved.

COROLLARY (of the proof). *In the situation of the lemma, if the algebra A is a domain and the eigenvectors $a_1 \in A_{t_1}, \ldots, a_n \in A_{t_n}$ generate A, then the monoid $E(d)$ is generated by $t_1, \ldots, t_n$.*

In view of the above, the lemma implies that if the algebra A is finitely generated, then a semisimple derivation of A is simply a G-grading of A, where G is a finitely generated subgroup of k^+.

Below, $U(A)$ stands for the group of units of the algebra A.

DEFINITION. Let $d : A \to A$ be a semisimple derivation. A *slice* of d is a homomorphism of groups $\sigma : G(d) \to U(A)$ such that $\sigma(t) \in A_t$ for all $t \in G(d)$.

It is easy to see that if d admits a slice, then $G(d) = E(d)$.

EXAMPLES. 3. Let $A = k[X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1}]$. Then for any $t_1, \ldots, t_n \in k$ the derivation $d : A \to A$ determined by $d(X_i) = t_i X_i, i = 1, \ldots, n$, is semisimple and $G(d) = \mathbb{Z}t_1 + \cdots + \mathbb{Z}t_n$. If $\operatorname{char}(k) = 0$ and $t_1, \ldots, t_n$ are linearly independent over $\mathbb{Z}$, then $\sigma : G(d) \to U(A)$, $\sigma(t_i) = X_i$, $i = 1, \ldots, n$, is a slice of d.

4. If $A = k[X]$ and $d(X) = X$, then $G(d) = \mathbb{Z}1_k$, but d does not admit any slice: if $\sigma : G(d) \to U(A)$ were a slice, then $\sigma(1_k) \in U(A) \cap A_1 = \emptyset$, because $U(A) = k^*$ $(= k - \{0\})$ and $A_1 = kX$.

Given an algebra B and a group G, BG denotes the group algebra of G over B.

THEOREM 1. *Let $d : A \to A$ be a semisimple derivation with $G = G(d)$. If d admits a slice $\sigma : G \to U(A)$, then $f : A_0 \otimes kG \to A$, $f(a \otimes t) = a\sigma(t)$, $a \in A_0$, $t \in G$, is an A_0-linear isomorphism of algebras. The inverse isomorphism $g : A \to A_0 \otimes kG$ is given by*

$$g\Big(a = \sum_{t\in G} a_t\Big) = \sum_{t\in G} a_t \sigma(-t) \otimes t.$$

In particular, A is isomorphic to the group algebra A_0G.

Proof. The proof is an easy exercise and we omit it.

From now on, we assume that $\operatorname{char}(k) = 0$. By $\operatorname{Dim} A$ we denote the Krull dimension of A. If A is a finitely generated domain, then it is known that $\operatorname{Dim} A = \operatorname{tr.deg}_k Q(A)$, where $Q(A)$ is the quotient field of A. Given a multiplicative system S in A, A_S denotes the localization of A with respect to S.

THEOREM 2. *Assume that A is a domain and d is a semisimple derivation of A with $G = G(d)$. Let $S = \bigcup_t A_t - \{0\}$. Then S is a multiplicative system in A, the induced derivation $\widetilde{d} : A_S \to A_S$ is semisimple with $G(\widetilde{d}) = G$, and $K = (A_S)_0$ $(= \operatorname{Ker} \widetilde{d})$ is a field containing A_0. Moreover, if A is finitely generated, then the derivation $\widetilde{d}$ admits a slice and $A_S \simeq K[X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1}]$, where $n = \operatorname{rk}(d)$. In particular,*

$$\operatorname{tr.deg}_k K + \operatorname{rk}(d) = \operatorname{Dim} A.$$

Proof. The first part of the theorem is a simple calculation. Notice only that $\mathrm{Ker}(\widetilde{d} - t\,\mathrm{Id}) = \{a/b \in A_S;\ \exists_{t\in G}\ a \in A_{s+t},\ b \in A_s\}$. If A is finitely generated, then G is a free group of finite rank, by Lemma 1. Let $g_1, \ldots, g_n$ be free generators of G. As G is generated by the set $E(d)$ of eigenvalues of d and $E(d)$ is a submonoid of G, $g_i = t_i - t_i'$ for some $t_i, t_i' \in E(d)$, $i = 1, \ldots, n$. Now for each i choose a nonzero $a_i \in A_{t_i}$, a nonzero $s_i \in A_{t_i'}$, and set $y_i = a_i/s_i$. Then $\widetilde{d}(y_i) = g_i y_i$, which implies that the mapping $\sigma : G(\widetilde{d}) = G \to U(A_S)$ determined by $\sigma(g_i) = y_i$, $i = 1, \ldots, n$, is a slice of the derivation $\widetilde{d} : A_S \to A_S$. Hence, by Theorem 1, A_S is isomorphic to the group algebra KG, where $K = (A_S)_0$. The conclusion is that $A_S \simeq K[X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1}]$, because $G \simeq \mathbb{Z}^n$. This completes the proof of the theorem.

Remark. Theorems 1 and 2 were motivated by [7, Sections I, III]. Besides, they can be deduced from [7, Section I].

2. Linearization and existence of maximal invariant ideals. As above, the field k is assumed to be of characteristic 0. If the algebra A is finitely generated, we denote by $\mu(A)$ the minimal number of generators of A. A derivation $d : A \to A$ is called *linearizable* if there exist eigenvectors $a_1, \ldots, a_{\mu(A)}$ of d which generate the algebra A. Notice that if $A = k[X_1, \ldots, X_n]$, then a derivation $d : A \to A$ is linearizable if there is a change of variables $\{X_i\} \to \{Y_i\}$ such that $d(Y_i) = \alpha_i Y_i$ for some $\alpha_i \in k$, $i = 1, \ldots, n$.

If (A, m) is a local (noetherian) algebra, then a derivation d of A is called *linearizable* if there are eigenvectors $x_1, \ldots, x_n$ of d which form a minimal system of generators of the maximal ideal m. Recall that for a given derivation $d : A \to A$ an ideal $J \subset A$ is said to be *invariant* if $d(J) \subset J$.

Let d be a derivation of the algebra A. If A is finitely generated or local, then obviously the following two problems are of interest.

The linearization problem: When is d linearizable?

Existence of maximal invariant ideals: When does d admit a maximal invariant ideal m (i.e., m is maximal in A and invariant)?

In general, a given derivation $d : A \to A$ is neither linearizable nor admits a maximal invariant ideal. For example, this is the case for $A = k[X]$ and $d = \partial/\partial X$. Observe that this d is not semisimple. So, some positive results can be expected for semisimple derivations. Let us start with the local case.

Theorem 3. *Let d be a derivation of the algebra A, and let m be a maximal invariant ideal in A.*

(1) *If (A, m) is a complete local ring, $k = A/m$, and for each $s \geq 2$ the induced derivation $d_s : A/m^s \to A/m^s$ is semisimple, then the derivation d is linearizable.*

(2) *If the derivation d is semisimple and the algebra A is finitely generated, then the induced derivation $d : A_m \to A_m$ is linearizable, where A_m is the localization of A at the maximal ideal m.*

Proof. (1) Let $n = \dim_k(m/m^2)$. Since the induced derivation $d_2 : A/m^2 \to A/m^2$ is semisimple and $k = A/m$, we can find a minimal system $x_1^{(1)}, \dots, x_n^{(1)}$ of generators of the ideal m with $d(x_i^{(1)}) = t_i x_i^{(1)} \bmod m^2$ for some $t_1, \dots, t_n \in k$ and $i = 1, \dots, n$. Now, proceeding by induction on $j \geq 1$, we construct sequences $x_1^{(j)}, \dots, x_n^{(j)}$, $j \geq 1$, such that $d(x_i^{(j)}) = t_i x_i^{(j)} \bmod m^j$ and $x_i^{(j+1)} = x_i^{(j)} \bmod m^j$ for all j. Suppose that $j \geq 2$, and that the components $x_1^{(i)}, \dots, x_n^{(i)}$ have already been constructed for $i = 1, \dots, j-1$. Denote by p the linear map $A/m^j \to A/m^{j-1}$, $a + m^j \mapsto a + m^{j-1}$. As $d_{j-1}p = pd_j$, and the induced derivations $d_{j-1} : A/m^{j-1} \to A/m^{j-1}$ and $d_j : A/m^j \to A/m^j$ are semisimple, it is easy to see that there exist $x_1^{(j)}, \dots, x_n^{(j)} \in m$ such that $p(x_i^{(j)} + m^j) = x_i^{(j-1)} + m^{j-1}$ and $d(x_i^{(j)}) = t_i x_i^{(j)} \bmod m^j$ for $i = 1, \dots, n$. This means that the inductive procedure gives us sequences $x_1^{(j)}, \dots, x_n^{(j)}$, $j \geq 1$, with the required properties. Now, since the local ring (A, m) is complete, we can consider the limits $x_1, \dots, x_n$ of the respective sequences. It is obvious that $d(x_i) = t_i x_i$ for each i. Moreover, $x_1, \dots, x_n$ form a minimal system of generators of the maximal ideal m, because so do $x_1^{(1)}, \dots, x_n^{(1)}$ and $x_i = x_i^{(1)} \bmod m^2$ for all i. Thus, part (1) of the theorem is proved.

(2) Assume that the derivation d is semisimple and A is finitely generated. As $d(m) \subset m$, we have $m = \bigoplus_{t \in k} m_t$, where $m_t = \{a \in m;\ d(a) = ta\}$. It follows that there exist eigenvectors $x_1, \dots, x_n$ of d such that $x_1 + m^2, \dots, x_n + m^2$ is a basis of the $k = A/m$-vector space m/m^2, because $\bigcup_{t \in t} m_t$ generates the vector space m. This in turn implies that $x_1/1, \dots, x_n/1 \in A_m$ is a minimal system of generators of the maximal ideal $M = mA_m$ of the local ring A_m, because $M/M^2 = mA_m/m^2A_m \simeq m/m^2$. Obviously $x_1/1, \dots, x_n/1$ are eigenvectors of the induced derivation $d : A_m \to A_m$. This proves part (2), and thus the proof of the theorem is complete.

Remark. The above theorem can be deduced from [4, Theorem 4].

Below, the algebra A is supposed to be finitely generated. Moreover, we assume that A is a domain.

Theorem 4. *Let $d : A \to A$ be a semisimple derivation, and let $m = \bigoplus_{t \neq 0} A_t$. If $A_0 = k$ and $U(A) = k^*$, then the following conditions hold.*

(1) *m is the unique maximal invariant ideal in A.*

(2) *There are eigenvectors* $a_1, \dots, a_r$ *of* d *such that each* a_i *belongs to* m *and* $A = k[a_1, \dots, a_r]$, *where* $r = \dim_k(m/m^2)$. *In particular*, d *is linearizable whenever* $\dim_k(m/m^2) = \mu(A)$.

(3) *If* A *is regular* (*as a ring*) *of Krull dimension* n, *then* $A \simeq k[X_1, \dots \dots, X_n]$, *and* d *is linearizable.*

Proof. Part (1) follows from [8, Theorem 4.1]. For completeness we give the proof. Let, as above, $E(d)$ be the set of eigenvalues of d. If $m = \bigoplus_{t \in k^*} A_t$ is an ideal, then clearly m is the unique maximal and invariant ideal in A, because $A/m \simeq k = A_0$ and $m = d(A)$. Therefore, we need only verify that m is an ideal. To this end, it is enough to show that given a nonzero $t \in E(d)$, we have $t + t' \neq 0$ for all $t' \in E(d)$. Suppose that, on the contrary, $t + t' = 0$ for some $t' \in E(d)$. Then $t' \neq 0$, whence there are nonzero $a \in A_t$ and $b \in A_{t'}$ with $ab \in A_{t+t'} = A_0$. As A is a domain and $A_0 = k$, it follows that $ab \in k^*$, which implies that $a \in U(A) = k^* \subset A_0$. This is impossible, because $a \in A_t$ with $t \neq 0$. Thus, part (1) is proved.

For (2), it is clear that there exist eigenvectors $a_1, \dots, a_r$ of d such that $a_1 + m^2, \dots, a_r + m^2$ is a basis of the $k = A/m$-vector space m/m^2. Now in view of (1) and [5, Corollary 1.4 and statement 1.7], $A = k[a_1, \dots, a_r]$.

It remains to prove (3). By regularity of A, the Krull dimension of A equals $\dim_k(m/m^2)$. Therefore, from (2) we infer that there are eigenvectors $a_1, \dots, a_n$ of d such that $A = k[a_1, \dots, a_n]$, where $n = \operatorname{Dim} A$. Let, as above, $Q(A)$ denote the quotient field of A. Then $\operatorname{Dim} A = \operatorname{tr.deg}_k Q(A)$, because A is a finitely generated domain. This implies that the elements $a_1, \dots, a_n$ are algebraically independent over k, which proves (3).

COROLLARY. *Let* $A = k[X, Y]/(X^i - Y^j)$, *where* $(i, j) = 1$, $i, j \geq 2$, *and let* d *be a semisimple derivation of* A. *Then* d *is linearizable* (*and admits an invariant maximal ideal*).

Proof. Obviously one can assume that $d \neq 0$, whence $\operatorname{rk}(d) \geq 1$. It is easy to see that $U(A) = k^*$, $\mu(A) = 2$, and $\dim_k(m/m^2) = 2$ for any maximal ideal in A. Furthermore, by Theorem 2, $\operatorname{rk}(d) + \operatorname{tr.deg}_k K = \operatorname{Dim} A = 1$, where K is a subfield of $Q(A)$ containing A_0. Hence $\operatorname{tr.deg}_k K = 0$, which implies that $A_0 \subset K = k$, because the field k is algebraically closed. The conclusion now follows from part (2) of the theorem.

EXAMPLE 5. If $A = k[X, X^{-1}]$ and $d : A \to A$ is the derivation defined by $d(X) = X$, then d is semisimple, $A_0 = k$, but d has no maximal invariant ideals (in view of the above theorem, the reason is that $U(A) \neq k$). Notice that d is linearizable.

We mentioned above that a semisimple derivation of the algebra A is nothing other than a G-grading of A, where G is a finitely generated subgroup of the group k^+. But we have assumed that $\operatorname{char}(k) = 0$. Therefore, a semisimple derivation of A is simply a $\mathbb{Z}^s$-grading of A, where $s = \mathrm{rk}(d)$. This means that a semisimple derivation of A of rank s induces an action of the algebraic torus $T^s = k^* \times \ldots \times k^*$ (s times) on the algebraic variety $\mathrm{Max}(A)$ of all maximal ideals in A. More precisely, if $d : A \to A$ is a semisimple derivation of rank s and $t_1, \ldots, t_s$ are free generators of the group $G(d)$, then the corresponding action of T^s on $\mathrm{Max}(A)$ is defined as follows. For $\alpha = (\alpha_1, \ldots, \alpha_s) \in T^s$ and $m \in \mathrm{Max}(A)$,

$$\alpha.m = \mathrm{Ker}(p_m \phi_\alpha),$$

where $\phi_\alpha : A \to A$ is the homomorphism of algebras given by $\phi_\alpha(a) = \sum_{(u_1,\ldots,u_s)\in\mathbb{Z}^s} a_u \alpha_1^{u_1} \cdots \alpha_s^{u_s}$ with $a_u = a_{u_1 t_1 + \cdots + u_s t_s}$ in the decomposition $a = \sum_{t\in G(d)} a_t \in \bigoplus_t A_t$, and $p_m : A \to A/m = k$ is defined by $p_m(y) = y + m$. It is easy to see that each action of T^s on the variety $\mathrm{Max}(A)$ comes from a semisimple derivation $d : A \to A$ in the above way. Also it is not difficult to prove that, given a semisimple derivation d of A, the maximal invariant ideals for d are precisely the fixed points of the corresponding action of the torus T^s on the variety $\mathrm{Max}(A)$.

This translation of semisimple derivations into the language of algebraic geometry gives us the following.

THEOREM 5. *Let $A = k[X_1, \ldots, X_n]$ and let d be a semisimple derivation of A.*

(1) *The derivation d admits a maximal invariant ideal.*
(2) *If $\mathrm{rk}(d) = n - 1$ or n, then d is linearizable. In particular, d is linearizable when $n \leq 2$.*
(3) *If $I = \bigoplus_{t\neq 0} A_t$ is an ideal in A and $r = \mathrm{Dim}\, A_0 \leq 2$, then there exists a change of variables $\{X_i\} \to \{Y_i\}$ such that $A_0 = k[Y_1, \ldots, Y_r]$, $A = A_0[Y_{r+1}, \ldots, Y_n]$, and all Y_i's are eigenvectors of d. In particular, d is linearizable.*
(4) *If $n = 3$, then d is linearizable.*

Proof. Parts (1) and (2) are due to Białynicki-Birula (see [2] and [3]). Part (3) was proved by Kambayashi and Russell in [5, proof of Theorem 3.4], and (4) is a joint result by Kaliman, Koras, Makar-Limanov, and Russell [6].

REMARK. If $n = 4$, then it is not known if every semisimple derivation of $A = k[X_1, \ldots, X_n]$ is linearizable. By [1], for each $n > 4$ there is a semisimple derivation of the $\mathbb{R}$-algebra $A = \mathbb{R}[X_1, \ldots, X_n]$ (i.e., $d \in \mathrm{Der}(A)$ such that $A = \sum_{t\in\mathbb{R}^+} A_t$, where $A_t = \mathrm{Ker}(d - t\,\mathrm{Id})$) which is not linearizable.

Acknowledgments. The author thanks the referee for valuable remarks which allowed him to improve the text of the paper.

REFERENCES

[1] T. Asanuma, *Non-linearizable algebraic k^*-actions on affine spaces*, Invent. Math. 138 (1999), 281–306.

[2] A. Białynicki-Birula, *Remarks on the action of an algebraic torus on k^n*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 14 (1966), 177–181.

[3] —, *Remarks on the action of an algebraic torus on k^n II*, ibid. 15 (1967), 123–125.

[4] M. Boratyński and A. Tyc, *A note on linearization of actions of finitely semisimple Hopf algebras on local algebras*, J. Pure Appl. Algebra 130 (1998), 217–222.

[5] T. Kambayashi and P. Russell, *On linearizing algebraic torus actions*, ibid. 23 (1982), 243–250.

[6] S. Kaliman, M. Koras, L. Makar-Limanov and P. Russell, *$\mathbb{C}^*$-actions on $\mathbb{C}^3$ are linearizable*, Electron. Res. Announc. Amer. Math. Soc. 3 (1997), 63–71.

[7] L. Le Bruyn, M. Van den Bergh and F. Van Oystaeyen, *Graded Orders*, Birkhäuser, 1988.

[8] A. Nowicki and A. Tyc, *On semisimple derivations in characteristic zero*, Comm. Algebra 29 (2001), 5115–5130.

Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: atyc@mat.uni.torun.pl

Received 11 October 2004;
revised 6 January 2005 (4515)