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COUNTING OCCURRENCES IN ALMOST SURE LIMIT THEOREMS

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Abstract. Let $X, X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with $X \in L^p$, $0 < p \le 2$. For $n \ge 1$, let $S_n = X_1 + \cdots + X_n$. Developing a preceding work concerning the L^2-case only, we compare, under strictly weaker conditions than those of the central limit theorem, the deviation of the series $\sum_n w_n \mathbf{1}_{\{S_n < s_n\}}$ with respect to $\sum_n w_n \mathbf{P}\{S_n < s_n\}$, for suitable weights (w_n) and arbitrary sequences (s_n) of reals. Extensions to the case $0 < p < 2$, and when the law of X belongs to the domain of attraction of a p-stable law, are also obtained. We deduce strong versions of the a.s. central limit theorem.

1. Setting of the problem and main results. Let $\mathcal{X} = \{X, X_n, n \ge 1\}$ be a sequence of independent, identically distributed (i.i.d.) random variables defined on a probability space $(\Omega, \mathcal{B}, \mathbf{P})$, and let F denote the distribution function of X, and $S_n = X_1 + \cdots + X_n$, $n \ge 1$, the partial sums of $\mathcal{X}$. Assume first, in order to introduce the problem under consideration here, that $\mathbf{E}X^2 < \infty$. Let $s = \{s_k, k \ge 0\}$ be an arbitrary sequence of reals, and consider the events $A_k^{(s)} = A_k = \{S_k < s_k\}$ and a sequence of weights $w = \{w_k, k \ge 1\}$. We study the following natural question: when is the weighted deviation

$$\mathcal{D}_w(A) := \sum_{k=1}^{\infty} w_k(\mathbf{1}_{A_k} - \mathbf{P}(A_k)), \tag{1.1}$$

of the series $\sum_{k=1}^{\infty} w_k \mathbf{1}_{A_k}$ with respect to its mean $\sum_{k=1}^{\infty} w_k \mathbf{P}(A_k)$, finite almost surely?

Some partial results already exist. For any positive integer n, put

$$Y_n^{(s)} = Y_n = \sum_{2^n \le k < 2^{n+1}} \frac{1}{k}\,(\mathbf{1}_{A_k} - \mathbf{P}(A_k)). \tag{1.2}$$

Then the series $\sum_{k \ge 1} c_k Y_k$ converges $\mathbf{P}$-almost surely, for a reasonable choice of the reals $\{c_k, k \ge 1\}$. For instance, one can take $c_k = k^{-1/2}(\log k)^{-b}$ with

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$b > 3/2$; then in view of Kronecker's Lemma, (1.2) implies, with the choice $s_k = x_k \|X\|_2 \sqrt{k} + k\mathbf{E}X$,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \big[\mathbf{1}_{\{(S_k - k\mathbf{E}X)/\|X\|_2\sqrt{k} < x_k\}} - \mathbf{P}\{(S_k - k\mathbf{E}X)/\|X\|_2\sqrt{k} < x_k\}\big] \stackrel{\text{a.s.}}{=} 0. \tag{1.3}$$

By using the CLT, and letting $x_k \equiv x$ in (1.3), one obtains the classical *Almost Sure Central Limit Theorem* (ASCLT) [9]: $\mathbf{P}$-almost surely, for every real number x,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbf{1}_{\{(S_k - k\mathbf{E}X)/\sqrt{k} \le x\}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du. \tag{1.4}$$

When (x_k) are not constant, the stronger property (1.3) does not seem connected to the CLT, although it is established in [5] under the CLT assumptions. In this paper, we show that (1.3) in turn holds true under a strictly weaker assumption.

Before stating the result, we have to recall the full formulation of (1.3), and a useful notion ([8]) from the theory of orthogonal series. Let $(T, \mathcal{C}, \tau)$ be some probability space and consider a sequence (f_n) of elements of $L^2(\tau)$. Let $a_{j,k} = \int_T f_j f_k \, d\tau$. If the quadratic form defined on ℓ_2 by $(x_n)_n \mapsto \sum_{h,k} a_{h,k} x_h x_k$ is bounded, then the system of functions (f_n) is said to be *quasi-orthogonal.* Say also that a sequence $c = (c_k)_k \in \ell_2$ is *universal* when the series $\sum_n c_n \psi_n$ converges almost everywhere for every orthonormal system $(\psi_n)_n$. According to Schur's Theorem [10, p. 56], if c is universal, then the series $\sum c_n f_n$ converges almost everywhere for any quasi-orthogonal system of functions (f_n). It follows from the Rademacher–Men'shov Theorem that we can choose $c_k = k^{-1/2} (\log k)^{-b}$ with $b > 3/2$. In [5, Theorem 1.1], it is shown that *for any sequence s of reals,*

(1.5) *the system $(Y_n^{(s)}, n \ge 1)$ is quasi-orthogonal.*

The result is originally stated under the conditions: $\mathbf{E}X = 0$, $\mathbf{E}X^2 = 1$, and with $s_k = x_k \sqrt{k}$ where x_k are arbitrary; this does not restrict the validity of (1.5). We refer to [5] for extensions to independent, non-identically distributed random variables, and to more general sequences of sets than $A_k = \{S_k < s_k\}$. Let $0 < p < \infty$, and consider the class $\mathcal{F}_p$ of distribution functions F satisfying

$$\max(F(-x), (1 - F(x)) = \mathcal{O}(x^{-p}), \qquad x \to +\infty. \tag{$\mathcal{F}_p$}$$

When $p \ge 1$, we moreover assume that F is *centered*: $\int_{-\infty}^{\infty} x\, F(dx) = 0$. We prove the following result:

THEOREM 1.1. *Assume that* $F \in \mathcal{F}_2$. *Then property* (1.5) *holds true. Further*, *for any sequence* $\{x_k, k \geq 1\}$ *of reals*,

$$\lim_{n\to\infty} \frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbf{P}\left\{\frac{S_k}{\sqrt{k}} \leq x_k\right\} = c \ \Rightarrow\ \lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbf{1}_{\{S_k/\sqrt{k}\leq x_k\}} \overset{a.s.}{=} c.$$

Theorem 1.1 therefore shows the validity of property (1.5), *independently* of the CLT property. Moreover, in the presence of the CLT property, (1.5) implies the ASCLT in a very strong form.

We also prove results for the case $F \in \mathcal{F}_p$, $p < 2$. In this case, more is required on F. Let $p > 0$ and let $\mathcal{G}_p$ be the class of distribution functions satisfying

$$(\mathcal{G}_p) \qquad x^{-p} = \mathcal{O}(\min(F(-x), (1 - F(x)))), \qquad x \to +\infty.$$

THEOREM 1.2. *Assume that* $F \in \mathcal{F}_p \cap \mathcal{G}_p$ *for some* $0 < p < 2$. *Then property* (1.5) *holds true.*

This result, however, only reflects a part of a more general property. Indeed, let $0 < p_1 \leq p_2 < 2$. There are easy examples of distribution functions with a tail oscillating infinitely often between $|x|^{-p_2}$ and $|x|^{-p_1}$, with $p_1 < p_2$, as $|x|$ tends to infinity. For any real h, let $[h]$ denote the integer part of h. Put $r = p_2/p_1$. Define, for an arbitrary sequence $s = \{s_k, k \geq 0\}$ of reals, the sequence of block sums

$$Z_n^{(s)} = \sum_{v_n \leq k < v_{n+1}} \frac{1}{k\log k}\,(\mathbf{1}_{A_k^{(s)}} - \mathbf{P}(A_k^{(s)})), \tag{1.6}$$

where $v_n = [e^{r^n}]$ for any integer $n \geq 1$.

THEOREM 1.3. *Let* $F \in \mathcal{F}_{p_1} \cap \mathcal{G}_{p_2}$. *Then*, *for any sequence* $s = \{s_k, k \geq 0\}$ *of reals*, *the system* $\{Z_n^{(s)}, n \geq 1\}$ *is quasi-orthogonal.*

We also prove a similar result when F belongs to the domain of attraction of a stable distribution G: there exist constants $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ such that the distribution of $a_n^{-1}S_n - b_n$ tends to G. Apart from the case $\alpha = 1$, it is known [3: p. 315] that the centering constants $\{b_n, n \geq 1\}$ are unnecessary.

THEOREM 1.4. *Assume that* F *belongs to the domain of attraction of a stable distribution* G *with exponent* $p \in \,]0, 2]$, *and* $\mathbf{E}X = 0$ *when* $1 < p \leq 2$. *Then property* (1.5) *holds true. In particular*, $\mathbf{P}$*-almost surely*, *for every continuity point* x *of* G, *we have*

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\,\mathbf{1}_{\{S_k/a_k\leq x\}} = G(x).$$

The last part of the statement expresses the form of the ASCLT for i.i.d. random variables lying in the domain of attraction of a p-stable law. It is already known, and we refer to [2] and [7] (see also references therein).

2. Proofs. The proofs rely upon two different sorts of estimates: first, an estimate of $\mathbf{E}|S_n|^\alpha$, where $0 < \alpha \le 1$; next, an upper bound for the concentration function (see (2.5)) of S_n, for which Esseen's inequality is used. Concerning the moments of S_n, some results already exist, and we refer the interested reader to the works of von Bahr and Esseen [1], and Hall [6] (see also references therein). In [6], the knowledge of the precise asymptotics of the distribution function F is needed, and thus the result of [6] cannot be used if we only know that $F \in \mathcal{F}_p$. It applies, however, when F belongs to the domain of attraction of a stable law, but the computations of $\mathbf{E}|S_n - \mathrm{med}(S_n)|$ are only sketched in the discussion following the main result (Theorem 1 of [6]). The proofs of our results require precise estimates of $\mathbf{E}|S_n|^\alpha$, and that is why we have displayed them in Lemma 2.2 below. In [1], useful estimates are given, in the independent case notably, under the assumption that $\mathbf{E}|X_i|^\alpha$ are finite. However, these estimates do not apply here. Indeed, we need a bound for $\mathbf{E}|S_n|^\alpha$ knowing that $F \in \mathcal{F}_p$ for $0 < \alpha < p$; and the bound should be expressed in terms of α of course, but also p (see Lemma 2.2(b) below). This fact, which seems inherent to the problem considered, thus also precludes the use of the estimates of [1] in our study, since the bounds there are only expressed in terms of α.

We use a notational convention: let C (resp. $C_{\alpha,\beta,\ldots}$) denote a constant depending on F (resp. $F, \alpha, \beta, \ldots$) only, which may change its value at each occurrence. We begin with some general lemmas. Let $\mathcal{X} = \{X, X_n, n \ge 1\}$ be a sequence of i.i.d. random variables on the basic probability space $(\Omega, \mathcal{B}, \mathbf{P})$.

LEMMA 2.1. *Let $0 < \alpha \le 1$. Assume that $\mathbf{E}|X|^\alpha < \infty$. Let $b = \{b_n, n \ge 1\}$ be a sequence of positive reals. For any integer n,*

$$\mathbf{E}|S_n|^\alpha \le n\Big[b_n^\alpha \mathbf{P}\{|X| > b_n\} + \alpha \int_{b_n}^{\infty} \mathbf{P}\{|X| > v\} v^{\alpha-1}\, dv\Big] + n^\alpha |\mathbf{E} X \mathbf{1}_{\{|X| \le b_n\}}|^\alpha + n^{\alpha/2} \{\mathbf{E} X^2 \mathbf{1}_{\{|X| \le b_n\}}\}^{\alpha/2}.$$

Proof. Write

$$\mathbf{E}|S_n|^\alpha \le \mathbf{E}\Big|\sum_{k=1}^n X_k \mathbf{1}_{\{|X_k| \le b_n\}}\Big|^\alpha + \mathbf{E}\Big|\sum_{k=1}^n X_k \mathbf{1}_{\{|X_k| > b_n\}}\Big|^\alpha.$$

Then

$$\mathbf{E}\Big|\sum_{k=1}^{n} X_k \mathbf{1}_{\{|X_k|>b_n\}}\Big|^{\alpha} \le n\mathbf{E}|X|^{\alpha}\mathbf{1}_{\{|X|>b_n\}}$$

$$= n\Big[b_n^{\alpha}\mathbf{P}\{|X|>b_n\} + \alpha\int_{b_n}^{\infty}\mathbf{P}\{|X|>v\}v^{\alpha-1}\,dv\Big],$$

$$\mathbf{E}\Big|\sum_{k=1}^{n} X_k \mathbf{1}_{\{|X_k|\le b_n\}}\Big|^{\alpha} \le \mathbf{E}\Big|\sum_{k=1}^{n}(X_k\mathbf{1}_{\{|X_k|\le b_n\}} - \mathbf{E}X\mathbf{1}_{\{|X|\le b_n\}})\Big|^{\alpha} + |n\mathbf{E}X\mathbf{1}_{\{|X|\le b_n\}}|^{\alpha}.$$

Now, by a routine symmetrization argument, letting $\varepsilon = \{\varepsilon_n, n \ge 1\}$ be a Rademacher sequence independent of the sequence $\mathcal{X}$, with corresponding expectation symbol $\mathbf{E}_{\varepsilon}$, we have

$$\mathbf{E}\Big|\sum_{k=1}^{n}(X_k\mathbf{1}_{\{|X_k|\le b_n\}} - \mathbf{E}X\mathbf{1}_{\{|X|\le b_n\}})\Big|^{\alpha} \le \mathbf{E}\mathbf{E}_{\varepsilon}\Big|\sum_{k=1}^{n}\varepsilon_k X_k\mathbf{1}_{\{|X_k|\le b_n\}})\Big|^{\alpha}$$

$$\le \mathbf{E}\Big\{\sum_{k=1}^{n} X_k^2\mathbf{1}_{\{|X_k|\le b_n\}}\Big\}^{\alpha/2} \le \Big\{\mathbf{E}\sum_{k=1}^{n} X_k^2\mathbf{1}_{\{|X_k|\le b_n\}}\Big\}^{\alpha/2}$$

$$= n^{\alpha/2}\{\mathbf{E}X^2\mathbf{1}_{\{|X|\le b_n\}}\}^{\alpha/2}.$$

Combining both inequalities gives the claimed estimate. ∎

Put

$$A_1(n,\alpha) = nb_n^{\alpha}\mathbf{P}\{|X|>b_n\}, \qquad A_2(n,\alpha) = n\int_{b_n}^{\infty}\mathbf{P}\{|X|>v\}v^{\alpha-1}\,dv,$$

$$A_3(n,\alpha) = n^{\alpha}|\mathbf{E}X\mathbf{1}_{\{|X|\le b_n\}}|^{\alpha}, \quad A_4(n,\alpha) = (n\mathbf{E}X^2\mathbf{1}_{\{|X|\le b_n\}})^{\alpha/2}.$$

LEMMA 2.2. (a) *Assume that $F \in \mathcal{F}_p$ and $b_n \ge n^{1/p}$. For any $0 < \alpha < p \le 2$, there exists a constant $C_{\alpha,p}$, depending on α, p only, such that for any integer n large enough,*

$$\max(A_1(n,\alpha), A_2(n,\alpha), A_3(n,\alpha)) \le C_{\alpha,p} b_n^{\alpha}$$

and

$$A_4(n,\alpha) \le C_{\alpha,p} b_n^{\alpha} \quad \textit{if } 0 < p < 2.$$

(b) *In particular,*

$$\mathbf{E}|S_n|^{\alpha} \le C_{\alpha,p} n^{\alpha/p} \qquad \textit{if } 0 < \alpha < p \le 1,$$

$$\mathbf{E}|S_n| \le C_p n^{1/p} \qquad \textit{if } 1 < p < 2,$$

$$\mathbf{E}|S_n| \le C(n\mathbf{E}X^2\mathbf{1}_{\{|X|\le n^{1/2}\}})^{1/2} \qquad \textit{if } p = 2 \textit{ and } \mathbf{E}X^2 = \infty.$$

We omitted the trivial case $p = 2$ with $\mathbf{E}X^2 < \infty$, for which $\mathbf{E}|S_n| \le Cn^{1/2}$.

Proof. (a) We have $A_1(n,\alpha) \le Cnb_n^{\alpha-p} \le Cb_n^{\alpha}$, and for n large enough,

$$A_2(n,\alpha) \le Cn \int_{b_n}^{\infty} v^{\alpha-1-p}\,dv \le C_{\alpha,p} n b_n^{\alpha-p} \le C_{\alpha,p} b_n^{\alpha}.$$

If $0 < p < 1$, we observe that for $a > 0$ sufficiently large,

$$\begin{aligned} n\mathbf{E}|X|\mathbf{1}_{\{|X|\le b_n\}} &= n\int_0^{b_n} \mathbf{P}\{|X|>t\}\,dt \le na + Cn\int_a^{b_n} t^{-p}\,dt \\ &\le na + Cnb_n^{1-p} \le Cb_n. \end{aligned}$$

Thus $A_3(n,\alpha) \le C_{\alpha,p} b_n^{\alpha}$.

If $1 < p \le 2$, then by centering,

$$\begin{aligned} |\mathbf{E}X\mathbf{1}_{\{|X|\le b_n\}}| = |\mathbf{E}X\mathbf{1}_{\{|X|>b_n\}}| &\le \mathbf{E}|X|\mathbf{1}_{\{|X|>b_n\}} \\ &= b_n\mathbf{P}\{|X|>b_n\} + \int_{b_n}^{\infty} \mathbf{P}\{|X|>t\}\,dt. \end{aligned}$$

But $nb_n\mathbf{P}\{|X|>b_n\} \le Cnb_n^{1-p} \le Cb_n$, and $\int_{b_n}^{\infty} \mathbf{P}\{|X|>t\}\,dt \le C\int_{b_n}^{\infty} t^{-p}\,dt \le C_p n b_n^{1-p} \le C_p b_n$. Therefore $A_3(n,\alpha) \le C_{\alpha,p} b_n^{\alpha}$.

If $p = 1$, since we assumed centering, we have $\mathbf{E}X\mathbf{1}_{\{|X|\le b_n\}} = o(1)$. Thus $A_3(n,\alpha) \le Cn^{\alpha} \le Cb_n^{\alpha}$, since $b_n \ge n^{1/p} = n$.

If $0 < p < 2$, for $a > 0$ sufficiently large we have

$$\begin{aligned} n\mathbf{E}X^2\mathbf{1}_{\{|X|\le b_n\}} &= 2n\int_0^{b_n} u\mathbf{P}\{|X|>u\}\,du \le 2na + nC\int_a^{b_n} u^{1-p}\,du \\ &\le 2na + Cnb_n^{2-p} \le Cb_n^2. \end{aligned}$$

Hence, $A_4(n,\alpha) \le C_\alpha b_n^{\alpha}$. This proves the first part of the lemma.

(b) Consider first the case $0 < \alpha < p \le 1$. Then $\mathbf{E}|X|^{\alpha} < \infty$, and we can apply Lemma 2.1. Thus $\mathbf{E}|S_n|^{\alpha} \le \sum_{i=1}^{4} A_i(n,\alpha)$, and the result follows from (a) by choosing $b_n = n^{1/p}$. If $1 < p < 2$, we proceed similarly to bound $\mathbf{E}|S_n|$. Finally, if $p = 2$ and $\mathbf{E}X^2 = \infty$, we apply again Lemma 2.1 and the estimates of (a) with $b_n = n^{1/2}$. In this case, the largest contribution is given by $A_4(n,\alpha)$, since $n\mathbf{E}X^2\mathbf{1}_{\{|X|\le b_n\}} \ge n$ for n large enough. We find $\mathbf{E}|S_n| \le C(n\mathbf{E}X^2\mathbf{1}_{\{|X|\le n^{1/2}\}})^{1/2}$. ■

LEMMA 2.3. *Assume that* $F \in \mathrm{DA}(G)$, *where* G *is a stable distribution with index* $0 < p \le 2$, *and* $\mathbf{E}X = 0$ *when* $1 < p \le 2$. *Let* $0 < \alpha < p \le 2$. *Then*

$$\mathbf{E}|S_n|^{\alpha} \le C_\alpha a_n^{\alpha} \quad \text{if } 0 < \alpha < p \le 1, \qquad \mathbf{E}|S_n| \le Ca_n \quad \text{if } 1 < p \le 2.$$

Proof. Since $\mathbf{E}|X|^\alpha < \infty$, by applying Lemma 2.1 with $a_n = b_n$, we get

$$\mathbf{E}|S_n|^\alpha \le n\Big[a_n^\alpha \mathbf{P}\{|X| > a_n\} + \alpha \int_{a_n}^{\infty} \mathbf{P}\{|X| > v\} v^{\alpha-1}\,dv\Big] + n^\alpha |\mathbf{E}X\mathbf{1}_{\{|X|\le a_n\}}|^\alpha + n^{\alpha/2}\{\mathbf{E}X^2\mathbf{1}_{\{|X|\le a_n\}}\}^{\alpha/2}.$$

• First we treat the case $0 < p < 2$. Since $F \in \mathrm{DA}(G)$, by [3, Theorem 1, p. 312 and relation (8.6), p. 313], one has $\mathbf{E}X^2\mathbf{1}_{\{|X|\le x\}} \sim x^{2-p}L(x)$ as $x \to \infty$, where $L : \overline{\mathbb{R}}_+ \to \mathbb{R}$ is a slowly varying function and

$$1 - F(x) + F(-x) \sim \frac{2-p}{p}\, x^{-p} L(x), \qquad x \to \infty.$$

From [3, p. 579], it also follows (for $0 < p \le 2$) that

$$\frac{nL(a_n)}{a_n^p} \ \to\ c > 0. \tag{2.1}$$

Thus, we have immediately $n\mathbf{E}X^2\mathbf{1}_{\{|X|\le a_n\}} = \mathcal{O}(a_n^2)$, and $na_n\mathbf{P}\{|X| > a_n\} = \mathcal{O}(a_n)$. Moreover, for any $0 < \alpha < p$,

$$\begin{aligned} n\int_{a_n}^{\infty} \mathbf{P}\{|X| > v\} v^{\alpha-1}\,dv &= n\sum_{k=0}^{\infty} \int_{a_n 2^k}^{a_n 2^{k+1}} \mathbf{P}\{|X| > v\} v^{\alpha-1}\,dv \\ &\le \frac{2^\alpha - 1}{\alpha}\, n \sum_{k=0}^{\infty} \mathbf{P}\{|X| > a_n 2^k\}(a_n 2^k)^\alpha \\ &\le C_{\alpha,p} \frac{n a_n^\alpha L(a_n)}{a_n^p} \sum_{k=0}^{\infty} 2^{k(\alpha-p)} \frac{L(a_n 2^k)}{L(a_n)} \\ &\le C_{\alpha,p} a_n^\alpha \sum_{k=0}^{\infty} 2^{k(\alpha-p)} \frac{L(a_n 2^k)}{L(a_n)}, \end{aligned}$$

and $C_{\alpha,p}$ depends on α and p only. Since $L(\cdot)$ is slowly varying, it can be represented ([12, Theorem 1.2]) as

$$L(x) = C(1 + o(1)) \exp\left\{\int_1^x \frac{\varepsilon(u)}{u}\,du\right\} \qquad \text{as } x \to \infty,$$

where $C > 0$ and $\lim_{u\to\infty} \varepsilon(u) = 0$. Let $0 < \varepsilon < p - \alpha$. Then, for any n large enough and every k,

$$\frac{L(a_n 2^k)}{L(a_n)} \le C \exp\left\{\int_{a_n}^{a_n 2^k} \frac{\varepsilon(u)}{u}\,du\right\} \le C \exp\{\varepsilon k \log 2\} = C 2^{\varepsilon k},$$

and
$$\sum_{k=0}^{\infty} 2^{k(\alpha-p)} \frac{L(a_n 2^k)}{L(a_n)} \le C \sum_{k=0}^{\infty} 2^{k(\alpha-p+\varepsilon)} < \infty.$$
This implies that
$$n \int_{a_n}^{\infty} \mathbf{P}\{|X| > v\} v^{\alpha-1}\, dv = \mathcal{O}(a_n^{\alpha}). \tag{2.2}$$

Assume first that $1 < p < 2$, and apply this estimate with $\alpha = 1$ to get $n \int_{a_n}^{\infty} \mathbf{P}\{|X| > u\}\, du = \mathcal{O}(a_n)$, which proves the claim in this case. We have used the fact that $\mathbf{E}X\mathbf{1}_{\{|X|\le b_n\}} = -\mathbf{E}X\mathbf{1}_{\{|X|>b_n\}}$, since $\mathbf{E}X = 0$.

Assume now that $0 < p \le 1$ and let $0 < \alpha < p$. By (2.1) and remarks made at the beginning of the proof, $n^{\alpha/2}\{\mathbf{E}X^2\mathbf{1}_{\{|X|\le a_n\}}\}^{\alpha/2} = \mathcal{O}(a_n^{\alpha})$, and $n a_n^{\alpha}\mathbf{P}\{|X| > a_n\} = \mathcal{O}(a_n^{\alpha})$. Further, $n \int_{a_n}^{\infty} \mathbf{P}\{|X| > v\} v^{\alpha-1}\, dv$ is estimated by (2.2). It remains to treat the third term in Lemma 2.1, namely $n^{\alpha}|\mathbf{E}X\mathbf{1}_{\{|X|\le a_n\}}|^{\alpha}$. But $\mathbf{E}|X|\mathbf{1}_{\{|X|\le x\}} = \int_0^x \mathbf{P}\{|X|>t\}\, dt$, and $\mathbf{E}X^2\mathbf{1}_{\{|X|\le x\}} = 2\int_0^x t\mathbf{P}\{|X| > t\}\, dt$. By L'Hospital's rule,
$$\lim_{x\to\infty} \frac{x\mathbf{E}|X|\mathbf{1}_{\{|X|\le x\}}}{\mathbf{E}X^2\mathbf{1}_{\{|X|\le x\}}} = \lim_{x\to\infty} \frac{\int_0^x \mathbf{P}\{|X| > t\}\, dt}{2x^{-1}\int_0^x t\mathbf{P}\{|X| > t\}\, dt}$$
$$= \lim_{x\to\infty} \frac{\mathbf{P}\{|X| > x\}}{-x^{-2}\mathbf{E}X^2\mathbf{1}_{\{|X|\le x\}} + 2\mathbf{P}\{|X| > x\}}.$$
Now recall that
$$\mathbf{P}\{|X| > x\} \sim \frac{2-p}{p}\, x^{-p}L(x) \quad \text{and} \quad \mathbf{E}X^2\mathbf{1}_{\{|X|\le x\}} \sim x^{2-p}L(x)$$
as $x \to \infty$. We thus find that
$$\lim_{x\to\infty} \frac{x\mathbf{E}|X|\mathbf{1}_{\{|X|\le x\}}}{\mathbf{E}X^2\mathbf{1}_{\{|X|\le x\}}} = \frac{\frac{2-p}{p}}{-1 + 2\cdot\frac{2-p}{p}} = \frac{2-p}{4-3p}.$$
Consequently,
$$\frac{2-p}{4-3p} = \lim_{n\to\infty} \frac{a_n\mathbf{E}|X|\mathbf{1}_{\{|X|\le a_n\}}}{\mathbf{E}X^2\mathbf{1}_{\{|X|\le a_n\}}} = \lim_{n\to\infty} \frac{a_n\mathbf{E}|X|\mathbf{1}_{\{|X|\le a_n\}}}{a_n^{2-p}L(a_n)}$$
$$= \lim_{n\to\infty} \frac{n\mathbf{E}|X|\mathbf{1}_{\{|X|\le a_n\}}}{n a_n^{1-p}L(a_n)}$$
$$= \frac{1}{c}\lim_{n\to\infty} \frac{n\mathbf{E}|X|\mathbf{1}_{\{|X|\le a_n\}}}{a_n} \quad \text{(by (2.1))}.$$
The last two lines show that $n\mathbf{E}|X|\mathbf{1}_{\{|X|\le a_n\}} = \mathcal{O}(a_n)$, which finishes the estimate of $\mathbf{E}|S_n|^{\alpha}$ in this case.

• There are only minor changes for the case $p = 2$. Here $U(x) = \mathbf{E}X^2\mathbf{1}_{\{|X|\le x\}} \sim L(x)$ as $x \to \infty$, where L is a slowly varying function, and

$x^2\mathbf{P}\{|X| > x\}/U(x) \to 0$ as $x \to \infty$. Plainly $n\mathbf{E}X^2\mathbf{1}_{\{|X|<a_n\}} = \mathcal{O}(a_n^2)$, and $na_n\mathbf{P}\{|X| > a_n\} = \mathcal{O}(a_n)$. Let $0 < \varepsilon < 1$. By using again Karamata's representation of slowly varying functions, we find that $L(a_n 2^j)/L(a_n) \le 2^{\varepsilon j}$ if n is sufficiently large, for any j.

In view of these observations and (2.1), it follows that

$$n\int_{a_n}^{\infty} \mathbf{P}\{|X| > u\}\,du \le n\sum_{k=0}^{\infty} \mathbf{P}\{|X| > a_n 2^j\}a_n 2^j$$
$$\le C\,\frac{nL(a_n)}{a_n^2}\,a_n\sum_{j=0}^{\infty} 2^{-j}\,\frac{L(a_n 2^j)}{L(a_n)} \le Ca_n.$$

This proves the estimate in this last case. ■

We now prove a preliminary bound concerning correlations. Let $a = \{a_k, k \ge 1\}$ be some increasing unbounded sequence of positive reals. Let also $f : \mathbb{R} \to \mathbb{R}$ be bounded Lipschitz, with norm $\|f\|_{\mathrm{BL}} = \|f\|_{\mathrm{L}} + \|f\|_\infty < \infty$, where $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$ and

$$\|f\|_{\mathrm{L}} = \sup\left\{\frac{|f(x) - f(y)|}{|x-y|} : x, y \in \mathbb{R},\ x \ne y\right\}.$$

We thus have the inequality $|f(x) - f(y)| \le 2\|f\|_{\mathrm{BL}}(|x-y| \wedge 1)$ for $x, y \in \mathbb{R}$. We now need a suitable version of the correlation inequality in [4].

PROPOSITION 2.4. *For any integers $k \le l$, for every Borel subset A of $\mathbb{R}$ and every bounded Lipschitz function f, we have*

$$\left|\mathbf{Cov}\left(\mathbf{1}_A\left(\frac{S_k}{a_k}\right), f\left(\frac{S_l}{a_l}\right)\right)\right| \le 4\|f\|_{\mathrm{BL}}\mathbf{E}\left(\frac{|S_k|}{a_l} \wedge 1\right). \tag{2.3}$$

Proof. Without loss of generality we can assume $H = \{S_k/a_k \in A\}$ to be not negligible. Let $\mathbf{E}_H$ denote the expectation with respect to the conditional probability $\mathbf{P}(\cdot|H)$, and $(X'_n)_n$ an independent copy of the sequence $(X_n)_n$. Put

$$V_l = \frac{X'_1 + \cdots + X'_k + X_{k+1} + \cdots + X_l}{a_l}. \tag{2.4}$$

As $\mathbf{E}_H[f(V_l)] = \mathbf{E}[f(S_l/a_l)]$, it follows that

$$\left|\mathbf{Cov}\left(\mathbf{1}_A\left(\frac{S_k}{a_k}\right), f\left(\frac{S_l}{a_l}\right)\right)\right|$$
$$= \left|\int_H f\left(\frac{S_l}{a_l}\right)d\mathbf{P} - \mathbf{P}(H)\int f\left(\frac{S_l}{a_l}\right)d\mathbf{P}\right| = \mathbf{P}(H)\left|\mathbf{E}_H f\left(\frac{S_l}{a_l}\right) - \mathbf{E}f\left(\frac{S_l}{a_l}\right)\right|$$

$$= \mathbf{P}(H)\left|\mathbf{E}_H f\left(\frac{S_l}{a_l}\right) - \mathbf{E}_H f(V_l)\right| \le 2\|f\|_{\mathrm{BL}}\mathbf{P}(H)\mathbf{E}_H\left(\left|\frac{S_l}{a_l} - V_l\right| \wedge 1\right)$$

$$= 2\|f\|_{\mathrm{BL}}\mathbf{E}\left(\left|\frac{S_l}{a_l} - V_l\right| \wedge 1\right) = 2\|f\|_{\mathrm{BL}}\mathbf{E}\left(\frac{|S_k - S_k'|}{a_l} \wedge 1\right)$$

$$\le 4\|f\|_{\mathrm{BL}}\mathbf{E}\left(\frac{|S_k|}{a_l} \wedge 1\right),$$

since $x \mapsto x \wedge 1$ is subadditive on $\mathbb{R}_+$. This establishes the proposition. ∎

For any $\lambda > 0$, introduce the *concentration function* of S_n:

$$Q_n(\lambda) = \sup_{x\in\mathbb{R}} \mathbf{P}(x \le S_n \le x + \lambda). \tag{2.5}$$

We shall now prove the following

PROPOSITION 2.5. *Let $0 < \varepsilon \le 1$. For every Borel set A, any real x and integers $k \le l$, we have*

$$\left|\mathbf{Cov}\left(\mathbf{1}_A\left(\frac{S_k}{a_k}\right), \mathbf{1}_{(-\infty,x]}\left(\frac{S_l}{a_l}\right)\right)\right| \le \frac{8}{\varepsilon}\,\mathbf{E}\left(\frac{|S_k|}{a_l} \wedge 1\right) + 2Q_l(a_l\varepsilon). \tag{2.6}$$

Proof. Let ε and x be fixed, and define the Lipschitz function f_ε as

$$f_\varepsilon(t) = \mathbf{1}_{(-\infty,x]}(t) + g_\varepsilon(t) = \mathbf{1}_{(-\infty,x]}(t) + \left(1 + \frac{x-t}{\varepsilon}\right)\mathbf{1}_{(x,x+\varepsilon)}(t).$$

Then it is easily checked that $\|f_\varepsilon\|_{\mathrm{BL}} = 1 + 1/\varepsilon$. Let H be the event $\{S_k/a_k \in A\}$; we can assume that H is not negligible. Let $\mathbf{C}$ be the conditional probability $\mathbf{P}(\cdot|H)$. Then we have

$$\left|\mathbf{Cov}\left(\mathbf{1}_A\left(\frac{S_k}{a_k}\right), \mathbf{1}_{(-\infty,x]}\left(\frac{S_l}{a_l}\right)\right)\right| = \mathbf{P}(H)\left|\mathbf{C}\left(\frac{S_l}{a_l} \le x\right) - \mathbf{P}\left(\frac{S_l}{a_l} \le x\right)\right|.$$

But

$$\begin{aligned} \mathbf{C}\left(\frac{S_l}{a_l} \le x\right) - \mathbf{P}\left(\frac{S_l}{a_l} \le x\right) &= \mathbf{E}^{\mathbf{C}}\left[(f_\varepsilon - g_\varepsilon)\left(\frac{S_l}{a_l}\right)\right] - \mathbf{E}^{\mathbf{P}}\left[(f_\varepsilon - g_\varepsilon)\left(\frac{S_l}{a_l}\right)\right] \\ &= \mathbf{E}^{\mathbf{C}}\left[(f_\varepsilon - g_\varepsilon)\left(\frac{S_l}{a_l}\right)\right] - \mathbf{E}^{\mathbf{C}}[(f_\varepsilon - g_\varepsilon)(V_l)] \\ &= \mathbf{E}^{\mathbf{C}}\left[f_\varepsilon\left(\frac{S_l}{a_l}\right) - f_\varepsilon(V_l)\right] - \mathbf{E}^{\mathbf{C}}\left[g_\varepsilon\left(\frac{S_l}{a_l}\right) - g_\varepsilon(V_l)\right], \end{aligned} \tag{2.7}$$

where V_l is the random variable defined in (2.4). By arguing as in the proof of Proposition 2.4, we get

$$\left|\mathbf{E}^{\mathbf{C}}\left[f_\varepsilon\left(\frac{S_l}{a_l}\right) - f_\varepsilon(V_l)\right]\right| \le 4(1 + 1/\varepsilon)\,\frac{1}{\mathbf{P}(H)}\,\mathbf{E}\left(\frac{|S_k|}{a_l} \wedge 1\right), \tag{2.8}$$

while trivially

$$\left|\mathbf{E}^{\mathbf{C}}\left[g_\varepsilon\left(\frac{S_l}{a_l}\right)-g_\varepsilon(V_l)\right]\right|\le\frac{2Q_l(a_l\varepsilon)}{\mathbf{P}(H)}. \tag{2.9}$$

From (2.8) and (2.9), we deduce the claimed inequality by summing and multiplying by $\mathbf{P}(H)$. ∎

Proposition 2.6. *Assume that $F\in\mathcal{F}_2$. Then there exist a constant C and $k_0>0$ such that for any Borel subset A of $\mathbb{R}$, any real x, and integers $l\ge k\ge k_0$, we have*

$$\left|\mathbf{Cov}\left(\mathbf{1}_A\left(\frac{S_k}{\sqrt{k}}\right),\mathbf{1}_{(-\infty,x]}\left(\frac{S_l}{\sqrt{l}}\right)\right)\right|\le C\left(\frac{k}{l}\right)^{1/4}.$$

Proof. Let $D(\widetilde{X},\lambda)=\lambda^{-2}\mathbf{E}\widetilde{X}^2\mathbf{1}_{|\widetilde{X}|<\lambda}+\mathbf{P}\{|\widetilde{X}|\ge\lambda\}$ define the *censored variance* of a symmetrized version $\widetilde{X}$ of X. Since $\mathcal{X}$ is an i.i.d. sequence, in view of Esseen's inequality ([11, Theorem 3, p. 43]), there exists an absolute constant C_* such that $Q_n(\lambda)\le C_*[nD(\widetilde{X},\lambda)]^{-1/2}$ for any $\lambda>0$. We apply Proposition 2.5 with $a_k=k^{1/2}$ to obtain

$$\begin{aligned}\left|\mathbf{Cov}\left(\mathbf{1}_A\left(\frac{S_k}{a_k}\right),\mathbf{1}_{(-\infty,x]}\left(\frac{S_l}{a_l}\right)\right)\right|&\le\frac{8}{\varepsilon}\,\mathbf{E}\left(\frac{|S_k|}{a_l}\wedge 1\right)+2C_*[lD(\widetilde{X},a_l\varepsilon)]^{-1/2}\\&\le\frac{8}{\varepsilon}\,\frac{\mathbf{E}|S_k|}{a_l}+\frac{2C_*\varepsilon}{(\mathbf{E}\widetilde{X}^2\mathbf{1}_{\{|\widetilde{X}|\le l^{1/2}\varepsilon\}})^{1/2}}.\end{aligned}$$

Let $\widetilde{X}_1,\widetilde{X}_2,\dots$ be i.i.d. copies of $\widetilde{X}$, and consider the sequence of partial sums $\widetilde{S}_n=\widetilde{X}_1+\cdots+\widetilde{X}_n$, $n=1,2,\dots$. Then, by centering, $\mathbf{E}|S_n|\le\mathbf{E}|\widetilde{S}_n|$. Let $\widetilde{F}$ be the distribution function of $\widetilde{X}$. Then $F\in\mathcal{F}_2$ implies that $\widetilde{F}\in\mathcal{F}_2$. Now, by Lemma 2.2, $\mathbf{E}|\widetilde{S}_n|\le C(n\mathbf{E}\widetilde{X}^2\mathbf{1}_{\{|\widetilde{X}|\le n^{1/2}\}})^{1/2}$. We can thus continue our estimates with

$$\le C\left(\frac{1}{\varepsilon}\left(\frac{k}{l}\right)^{1/2}(\mathbf{E}\widetilde{X}^2\mathbf{1}_{\{|\widetilde{X}|\le k^{1/2}\}})^{1/2}+\frac{\varepsilon}{(\mathbf{E}\widetilde{X}^2\mathbf{1}_{\{|\widetilde{X}|\le l^{1/2}\varepsilon\}})^{1/2}}\right).$$

Choose $\varepsilon=(k/l)^{1/4}(\mathbf{E}\widetilde{X}^2\mathbf{1}_{\{|\widetilde{X}|\le k^{1/2}\}})^{1/2}$, and observe that

$$l^{1/2}\varepsilon=(kl)^{1/4}(\mathbf{E}\widetilde{X}^2\mathbf{1}_{\{|\widetilde{X}|\le k^{1/2}\}})^{1/2}\ge k^{1/2}$$

whenever k is large enough, say $k\ge k_0$. Then $(\mathbf{E}\widetilde{X}^2\mathbf{1}_{\{|\widetilde{X}|\le l^{1/2}\varepsilon\}})^{1/2}\ge(\mathbf{E}\widetilde{X}^2\mathbf{1}_{\{|\widetilde{X}|\le k^{1/2}\}})^{1/2}$ for $k\ge k_0$ and the above is thus

$$\begin{aligned}&\le C\left(\frac{1}{\varepsilon}\left(\frac{k}{l}\right)^{1/2}(\mathbf{E}\widetilde{X}^2\mathbf{1}_{\{|X|\le k^{1/2}\}})^{1/2}+\frac{\varepsilon}{(\mathbf{E}\widetilde{X}^2\mathbf{1}_{\{|\widetilde{X}|\le k^{1/2}\}})^{1/2}}\right)\\&=2C\left(\frac{k}{l}\right)^{1/4}.\end{aligned}$$ ∎

Proof of Theorem 1.1. We combine Proposition 2.6 with Lemma 7.4.3 of [13] that we recall for convenience.

LEMMA 2.7. *Let H be a Hilbert space, and $\Phi = \{f_n, n \geq 1\} \subset H$ with correlations $a_{j,k} = \langle f_j, f_k \rangle$. In order that Φ be a quasi-orthogonal system, it is enough that* $\sup_{j\geq 1} \sum_k |a_{j,k}| < \infty$.

Applying Proposition 2.6 with $A =]-\infty, s_k/\sqrt{k}]$, $x = s_l/\sqrt{l}$, we get

$$|\mathbf{Cov}(\mathbf{1}_{\{S_k<s_k\}}, \mathbf{1}_{\{S_l<s_l\}})| \leq C\left(\frac{k}{l}\right)^{1/4}.$$

Thus for $m \geq n$,

$$\begin{aligned}|\mathbf{Cov}(Y_n, Y_m)| &= \Big| \sum_{\substack{2^n\leq k<2^{n+1}\\ 2^m\leq l<2^{m+1}}} \frac{1}{kl}\,\mathbf{Cov}(\mathbf{1}_{\{S_k<s_k\}}, \mathbf{1}_{\{S_l<s_l\}})\Big| \\ &\leq C \sum_{\substack{2^n\leq k<2^{n+1}\\ 2^m\leq l<2^{m+1}}} \frac{1}{kl}\left(\frac{k}{l}\right)^{1/4} \\ &= C\Big(\sum_{2^n\leq k<2^{n+1}} k^{-3/4}\Big)\Big(\sum_{2^m\leq l<2^{m+1}} l^{-5/4}\Big) \leq C2^{-(m-n)/4}.\end{aligned}$$

Therefore

$$\sum_{m\geq n} |\mathbf{Cov}(Y_n, Y_m)| \leq C\sum_{m\geq n} 2^{-(m-n)/4} \leq C\sum_{u\geq 0} 2^{-u/4} = C',$$
$$\sum_{m\leq n} |\mathbf{Cov}(Y_n, Y_m)| \leq C\sum_{m\leq n} 2^{-(n-m)/4} \leq C'.$$

We get $\sup_n \sum_m |\mathbf{Cov}(Y_n, Y_m)| \leq 2C'$. The result thus follows from Lemma 2.7. ■

To prove Theorem 1.2, we need a suitable estimate of $Q_n(\varepsilon)$.

LEMMA 2.8. *Assume that $F \in \mathcal{G}_p$ with $p > 0$. Then there exists λ_0 such that for any $\lambda \geq \lambda_0$,*

$$Q_n(\lambda) \leq Cn^{-1/2}\lambda^{p/2}.$$

Proof. We use again Esseen's inequality, and the notation from the proof of Proposition 2.6. Since $X \in \mathcal{G}_p$ and $D(\widetilde{X}, \lambda) \geq \frac{1}{2}\mathbf{P}\{|X| \geq \lambda\}$, it follows that $D(\widetilde{X}, \lambda) \geq C\lambda^{-p}$ for λ sufficiently large, say $\lambda \geq \lambda_0$. This proves our claim. ■

PROPOSITION 2.9. *Let $F \in \mathcal{G}_p$. Assume that*

$$\liminf_{k\to\infty} k^{1/2} \min(k/a_k^p, 1) = \infty. \tag{2.10}$$

Then there exists $k_0 > 0$ and a constant C_1 depending on the constant C of Lemma 2.8 and p only, such that for any Borel subset A of $\mathbb{R}$, any real x and integers $l \geq k \geq k_0$, we have

$$\left|\mathbf{Cov}\left(\mathbf{1}_A\left(\frac{S_k}{a_k}\right), \mathbf{1}_{(-\infty,x]}\left(\frac{S_l}{a_l}\right)\right)\right| \leq C\left\{\mathbf{E}\left(\frac{|S_k|}{l^{1/p}} \wedge \frac{a_l}{l^{1/p}}\right)\right\}^{\frac{p}{p+2}}.$$

Proof. In view of inequality (2.6) and Lemma 2.8,

$$\left|\mathbf{Cov}\left(\mathbf{1}_A\left(\frac{S_k}{a_k}\right), \mathbf{1}_{(-\infty,x]}\left(\frac{S_l}{a_l}\right)\right)\right| \leq \frac{8}{\varepsilon}\mathbf{E}\left(\frac{|S_k|}{a_l} \wedge 1\right) + 2Cl^{-1/2}(a_l\varepsilon)^{p/2}, \tag{2.11}$$

for any Borel set $A \subset \mathbb{R}$, any real x, any integers $l \geq k$ and $0 < \varepsilon \leq 1$, provided that $a_l\varepsilon \geq \lambda_0$. Choose

$$\varepsilon = l^{\frac{1}{p+2}} a_l^{-\frac{p}{p+2}} \left(\frac{8}{pC}\right)^{\frac{2}{p+2}} \left(\mathbf{E}\left(\frac{|S_k|}{a_l} \wedge 1\right)\right)^{\frac{2}{p+2}}.$$

Then

$$\begin{aligned} \frac{8}{\varepsilon}\mathbf{E}\left(\frac{|S_k|}{a_l} \wedge 1\right) &= 8\left(\frac{pC}{8}\right)^{\frac{2}{p+2}}\left\{\mathbf{E}\left(\frac{|S_k|}{l^{1/p}} \wedge \frac{a_l}{l^{1/p}}\right)\right\}^{\frac{p}{p+2}}, \\ 2Cl^{-1/2}(a_l\varepsilon)^{p/2} &= 2C\left(\frac{8}{pC}\right)^{\frac{2}{p+2}}\left\{\mathbf{E}\left(\frac{|S_k|}{l^{1/p}} \wedge \frac{a_l}{l^{1/p}}\right)\right\}^{\frac{p}{p+2}}. \end{aligned} \tag{2.12}$$

It remains to verify the condition $a_l\varepsilon \geq \lambda_0$. But

$$a_l\varepsilon = \left(\frac{8}{pC}\right)^{\frac{2}{p+2}} l^{\frac{1}{p+2}} a_l^{\frac{2}{p+2}} \left(\mathbf{E}\left(\frac{|S_k|}{a_l} \wedge 1\right)\right)^{\frac{2}{p+2}},$$

so

$$(a_l\varepsilon)^{\frac{p+2}{2}} \geq \left(\frac{8k^{1/2}}{pC}\right)\mathbf{E}(|S_k| \wedge a_k).$$

The required condition will be certainly satisfied if we show that

$$\liminf_{k\to\infty} k^{1/2}\mathbf{E}(|S_k| \wedge a_k) = \infty.$$

But, by [11, Theorem 10, p. 50], and by using the Paley–Zygmund inequality and independence,

$$\begin{aligned} \mathbf{E}(|S_k| \wedge a_k) &= \int_0^{a_k} \mathbf{P}\{|S_k| > t\}\, dt \geq a_k\mathbf{P}\{|S_k| > a_k\} \\ &\geq \frac{1}{2}\,\mathbf{P}\{\max_{j=1}^{k} |S_j| > a_k\} \geq \frac{1}{2}\mathbf{P}\{\max_{j=1}^{k} |X_j| > 2a_k\} \end{aligned}$$

$$\geq \frac{1}{2}\,\frac{(\sum_{j=1}^{k}\mathbf{P}\{|X_j|>2a_k\})^2}{\sum_{j=1}^{k}\mathbf{P}\{|X_j|>2a_k\}+(\sum_{j=1}^{k}\mathbf{P}\{|X_j|>2a_k\})^2}$$

$$\geq \frac{1}{4}\min\Big(\sum_{j=1}^{k}\mathbf{P}\{|X_j|>2a_k\},1\Big)$$

$$\geq \frac{1}{4}\min(k\mathbf{P}\{|X|>2a_k\},1)\geq \frac{1}{4}\min\Big(\frac{Ck}{a_k^p},1\Big).$$

Thus,

$$k^{1/2}\mathbf{E}(|S_k|\wedge a_k)\geq \frac{k^{1/2}}{4}\min\Big(\frac{Ck}{a_k^p},1\Big)\to\infty$$

as $k\to\infty$ by assumption. This proves, for our choice of ε, that the condition $a_l\varepsilon\geq\lambda_0$ is satisfied as soon as k is large enough, say $k\geq k_0$. Combining (2.11) with (2.12) then completes the proof. ■

The following statement corresponds to Proposition 2.6 for the case $F\in\mathcal{F}_{p_1}\cap\mathcal{G}_{p_2}$ with $0<p_1\leq p_2<2$:

Corollary 2.10. *Assume that $F\in\mathcal{F}_{p_1}\cap\mathcal{G}_{p_2}$ with $0<p_1\leq p_2<2$. Let also $0<\alpha<p_1$ with the convention that $\alpha=1$ if $1<p_1<2$. Then there exists a constant C_{α,p_1} (depending on α, p_1, F only) and $k_0>0$ such that for any Borel set $A\subset\mathbb{R}$, any real x and integers $l\geq k\geq k_0$, we have*

$$\Big|\mathbf{Cov}\Big(\mathbf{1}_A\Big(\frac{S_k}{k^{1/p_2}}\Big),\mathbf{1}_{(-\infty,x]}\Big(\frac{S_l}{l^{1/p_2}}\Big)\Big)\Big|$$
$$\leq\begin{cases} C_{1,p_1}\Big(\dfrac{k^{p_2/p_1}}{l}\wedge 1\Big)^{\frac{1}{p_2+2}} & \text{if } 1<p_1<2,\\[2ex] C_{\alpha,p_1}\Big(\dfrac{k^{p_2/p_1}}{l}\wedge 1\Big)^{\frac{\alpha}{p_2+2}} & \text{if } 0<\alpha<p_1\leq 1.\end{cases}$$

Proof. Case 1: $1<p_1<2$. With the choice $a_k=k^{1/p_2}$, condition (2.10) is satisfied. Next, by Lemma 2.2, $\mathbf{E}|S_n|\leq C_{p_1}n^{1/p_1}$. The claimed inequality thus follows directly from Proposition 2.9, since

$$\Big|\mathbf{Cov}\Big(\mathbf{1}_A\Big(\frac{S_k}{a_k}\Big),\mathbf{1}_{(-\infty,x]}\Big(\frac{S_l}{a_l}\Big)\Big)\Big|\leq C_{p_1}\Big\{\mathbf{E}\Big(\frac{|S_k|}{l^{1/p_2}}\wedge 1\Big)\Big\}^{\frac{p_2}{p_2+2}}$$
$$\leq C_{p_1}\Big(\frac{k^{p_2/p_1}}{l}\Big)^{\frac{1}{p_2+2}}.$$

Case 2: $0<p_1\leq 1$. Let $0<\alpha<p_1$. By Lemma 2.2, $\mathbf{E}|S_n|^\alpha\leq C_{\alpha,p_1}n^{\alpha/p_1}$. Again, we apply Proposition 2.9 with $a_k=k^{1/p_2}$ to get

$$\left|\mathbf{Cov}\left(\mathbf{1}_A\left(\frac{S_k}{a_k}\right),\mathbf{1}_{(-\infty,x]}\left(\frac{S_l}{a_l}\right)\right)\right|$$
$$\le C_{\alpha,p_1}\left\{\mathbf{E}\left(\frac{|S_k|}{l^{1/p_2}}\wedge 1\right)\right\}^{\frac{p_2}{p_2+2}} \le C_{\alpha,p_1}\left\{\mathbf{E}\left(\frac{|S_k|}{l^{1/p_2}}\wedge 1\right)^{\alpha}\right\}^{\frac{p_2}{p_2+2}}$$
$$\le C_{\alpha,p_1}\left\{\frac{\mathbf{E}\,|S_k|^{\alpha}}{l^{\alpha/p_2}}\right\}^{\frac{p_2}{p_2+2}} \le C_{\alpha,p_1}\left(\frac{k^{\frac{p_2}{p_1}}}{l}\right)^{\frac{\alpha}{p_2+2}}. \blacksquare$$

Proof of Theorem 1.2. Applying Corollary 2.10 with $p_1 = p_2 = p$, $A =]-\infty, s_k/\sqrt{k}]$ and $x = s_l/\sqrt{l}$ gives

$$|\mathbf{Cov}(\mathbf{1}_{\{S_k<s_k\}},\mathbf{1}_{\{S_l<s_l\}})| \le \begin{cases} C_p(k/l)^{\frac{1}{p+2}} & \text{if } 1<p<2,\\ C_{\alpha,p}(k/l)^{\frac{\alpha}{p+2}} & \text{if } 0<\alpha<p\le 1.\end{cases}$$

Let $m \ge n$. On the one hand, if $1 < p < 2$, then

$$|\mathbf{Cov}(Y_n,Y_m)| = \left|\sum_{\substack{2^n\le k<2^{n+1}\\ 2^m\le l<2^{m+1}}}\frac{1}{kl}\,\mathbf{Cov}(\mathbf{1}_{\{S_k<s_k\}},\mathbf{1}_{\{S_l<s_l\}})\right|$$
$$\le C\sum_{\substack{2^n\le k<2^{n+1}\\ 2^m\le l<2^{m+1}}}\frac{1}{kl}\left(\frac{k}{l}\right)^{1/(p+2)} \le C2^{-(m-n)/(p+2)},$$

implying that $\sup_n \sum_m |\mathbf{Cov}(Y_n,Y_m)| \le 2C\sum_{h=0}^{\infty} 2^{-h/(p+2)}$. On the other hand, if $0<\alpha<p\le 1$, then

$$|\mathbf{Cov}(Y_n,Y_m)| = \left|\sum_{\substack{2^n\le k<2^{n+1}\\ 2^m\le l<2^{m+1}}}\frac{1}{kl}\,\mathbf{Cov}(\mathbf{1}_{\{S_k<s_k\}},\mathbf{1}_{\{S_l<s_l\}})\right|$$
$$\le C_\alpha\sum_{\substack{2^n\le k<2^{n+1}\\ 2^m\le l<2^{m+1}}}\frac{1}{kl}\left(\frac{k}{l}\right)^{\alpha/(p+2)} \le C_\alpha 2^{-\alpha(m-n)/(p+2)},$$

leading to $\sup_n \sum_m |\mathbf{Cov}(Y_n,Y_m)| \le 2C_\alpha\sum_{h=0}^{\infty} 2^{-h/(p+2)}$. The result thus follows from Lemma 2.7. $\blacksquare$

Proof of Theorem 1.3. Recall that

$$Z_n^{(s)} = \sum_{v_n\le k<v_{n+1}}\frac{1}{k\log k}\,(\mathbf{1}_{A_k^{(s)}} - \mathbf{P}(A_k^{(s)})),$$

where we have defined $v_n = [e^{r^n}]$ for any integer $n \ge 1$, and note that

$$\sum_{v_n\le k<v_{n+1}}\frac{1}{k\log k} = O(1).$$

Let $n \le m$. Applying once again Corollary 2.10 with $A =]-\infty, s_k/k^{1/p_2}]$ and $x = s_l/l^{1/p_2}$ gives, for $1 < p_1 < 2$,

$$\begin{aligned}|\mathbf{Cov}(Z_n^{(s)}, Z_m^{(s)})| &= \Bigg| \sum_{\substack{v_n \le k < v_{n+1} \\ v_m \le l < v_{m+1}}} \frac{1}{(k\log k)(l\log l)} \mathbf{Cov}(\mathbf{1}_{\{S_k < s_k\}}, \mathbf{1}_{\{S_l < s_l\}}) \Bigg| \\ &\le C_{1,p_1} \sum_{\substack{v_n \le k < v_{n+1} \\ v_m \le l < v_{m+1}}} \frac{1}{(k\log k)(l\log l)} \left(\frac{k^r}{l}\right)^{\frac{1}{p_2+2}} \\ &\le C_{1,p_1} e^{(r^{n+1}-r^m)/(p_2+2)}.\end{aligned}$$

As $e^{(r^{n+1}-r^m)/(p_2+2)} \le e^{-r^m(1-r^{-1})/(p_2+2)}$ if $m \ge n+2$, it follows that

$$\sup_n \sum_m |\mathbf{Cov}(Z_n, Z_m)| \le 2C'_{1,p_1}\Big(1 + \sum_{h=0}^{\infty} e^{-r^h(1-r^{-1})/(p_2+2)}\Big).$$

Now if $0 < p_1 \le 1$, let $0 < \alpha < p_1$. By Corollary 2.10 again, with the same choices of A and x,

$$\begin{aligned}|\mathbf{Cov}(Z_n^{(s)}, Z_m^{(s)})| &= \Bigg| \sum_{\substack{v_n \le k < v_{n+1} \\ v_m \le l < v_{m+1}}} \frac{1}{(k\log k)(l\log l)} \mathbf{Cov}(\mathbf{1}_{\{S_k < s_k\}}, \mathbf{1}_{\{S_l < s_l\}}) \Bigg| \\ &\le C_{\alpha,p_1} \sum_{\substack{v_n \le k < v_{n+1} \\ v_m \le l < v_{m+1}}} \frac{1}{(k\log k)(l\log l)} \left(\frac{k^r}{l}\right)^{\frac{\alpha}{p_2+2}} \\ &\le C_{\alpha,p_1} e^{(r^{n+1}-r^m)\alpha/(p_2+2)}.\end{aligned}$$

As $e^{(r^{n+1}-r^m)\alpha/(p_2+2)} \le e^{-r^m(1-r^{-1})\alpha/(p_2+2)}$ if $m \ge n+2$, it follows that

$$\sup_n \sum_m |\mathbf{Cov}(Z_n, Z_m)| \le 2C_{\alpha,p_1}\Big(1 + \sum_{h=0}^{\infty} e^{-r^h(1-r^{-1})\alpha/(p_2+2)}\Big).$$

One concludes by applying Lemma 2.7.

Proof of Theorem 1.4. By (2.1), $nL(a_n)/a_n^p \sim c > 0$, where $L(\cdot)$ is slowly varying. We use the following fact (see [3, p. 579]). We may define $a_n = \inf\{x : nx^{-2}L(x) \le c\}$. This definition easily implies that (a_n) is non-decreasing. By Lemma 2.3 and Proposition 2.5,

$$\left|\mathbf{Cov}\left(\mathbf{1}_A\left(\frac{S_k}{a_k}\right), \mathbf{1}_{(-\infty,x]}\left(\frac{S_l}{a_l}\right)\right)\right| \le \frac{8}{\varepsilon}\,\mathbf{E}\left(\frac{|S_k|}{a_l} \wedge 1\right) + 2Q_l(a_l\varepsilon).$$

Choose $\varepsilon = (a_k/a_l)^{\frac{2}{p+2}}$. Then $a_l\varepsilon = a_l^{\frac{p}{p+2}} a_k^{\frac{2}{p+2}}$ $(\ge a_k)$. We use the notation from the proof of Lemma 2.3 and the properties of F mentioned therein.

Then $D(\widetilde{X},\lambda) \geq CL(\lambda)\lambda^{-p}$ for any $\lambda \geq \lambda_0$, where λ_0 depends on F only. And by Esseen's estimate, for $\lambda \geq \lambda_0$,

$$Q_l(\lambda) \leq C[lD(\widetilde{X},\lambda)]^{-1/2} \leq C\left(\frac{\lambda^p}{lL(\lambda)}\right)^{1/2}. \tag{2.10}$$

Choose k_0 sufficiently large to have $a_{k_0} \geq \lambda_0$. Applying (2.10) with $\lambda = a_l\varepsilon$ gives

$$Q_l(a_l\varepsilon) \leq C\,\frac{a_l^{\frac{p^2}{2(p+2)}} a_k^{\frac{p}{p+2}}}{l^{1/2}L(a_l\varepsilon)^{1/2}} \leq C\left(\frac{a_k}{a_l}\right)^{\frac{p}{p+2}}\left(\frac{a_l^p}{lL(a_l)}\right)^{1/2}\left(\frac{L(a_l)}{L(a_l\varepsilon)}\right)^{1/2}$$

for $l \geq k \geq k_0$, where k_0 depends on F only. Let $0 < \eta < p$. By using again Karamata's representation of slowly varying functions, we find that

$$\begin{aligned} \frac{L(a_l)}{L(a_l\varepsilon)} &\leq C\exp\left\{\eta\log\frac{1}{\varepsilon}\right\} = C\exp\left\{\eta\cdot\frac{-2}{p+2}\log\frac{a_k}{a_l}\right\} \\ &= C\left(\frac{a_k}{a_l}\right)^{-\eta\cdot\frac{2}{p+2}}, \end{aligned} \tag{2.11}$$

assuming k large enough, say $k \geq k_\eta$. By using this with relation (2.1), we obtain: there exists a constant C_η, depending on F and η only, and $k_\eta < \infty$ such that for any integers $l \geq k \geq k_\eta$,

$$Q_l(a_l\varepsilon) \leq C_\eta\left(\frac{a_k}{a_l}\right)^{\frac{p-\eta}{p+2}}. \tag{2.12}$$

- If $1 < p \leq 2$, then by Lemma 2.3, $\mathbf{E}|S_k| \leq Ca_k$, and so

$$\left|\mathbf{Cov}\left(\mathbf{1}_A\left(\frac{S_k}{a_k}\right), \mathbf{1}_{(-\infty,x]}\left(\frac{S_l}{a_l}\right)\right)\right| \leq C\left\{\frac{1}{\varepsilon}\left(\frac{a_k}{a_l}\right) + Q_l(a_l\varepsilon)\right\}.$$

By substituting estimate (2.12) into that inequality, we get

$$\begin{aligned} &\left|\mathbf{Cov}\left(\mathbf{1}_A\left(\frac{S_k}{a_k}\right), \mathbf{1}_{(-\infty,x]}\left(\frac{S_l}{a_l}\right)\right)\right| \\ &\qquad \leq C\left(\frac{a_k}{a_l}\right)^{\frac{p}{p+2}} + C_\eta\left(\frac{a_k}{a_l}\right)^{\frac{(1+\eta)p}{p+2}} \leq C_\eta\left(\frac{a_k}{a_l}\right)^{\frac{p}{p+2}}. \end{aligned} \tag{2.13}$$

One then deduces Theorem 1.4 from the combination of (2.13) with Lemma 2.7 in that case.

- If $0 < p \leq 1$, let $p/(p+2) < \alpha < p$. Then, by Lemma 2.3,

$$\mathbf{E}\left(\frac{|S_k|}{a_l}\wedge 1\right) \leq \mathbf{E}\left(\frac{|S_k|}{a_l}\wedge 1\right)^\alpha \leq \frac{\mathbf{E}|S_k|^\alpha}{a_l^\alpha} \leq C_\alpha\left(\frac{a_k}{a_l}\right)^\alpha,$$

and so

$$\left|\mathbf{Cov}\left(\mathbf{1}_A\left(\frac{S_k}{a_k}\right),\mathbf{1}_{(-\infty,x]}\left(\frac{S_l}{a_l}\right)\right)\right| \le \frac{1}{\varepsilon}\,C_\alpha\left(\frac{a_k}{a_l}\right)^\alpha + 2Q_l(a_l\varepsilon).$$

Using (2.12) gives

$$\left|\mathbf{Cov}\left(\mathbf{1}_A\left(\frac{S_k}{a_k}\right),\mathbf{1}_{(-\infty,x]}\left(\frac{S_l}{a_l}\right)\right)\right| \le C_\alpha\left(\frac{a_k}{a_l}\right)^{\frac{2}{p+2}+\alpha-1} + C_\eta\left(\frac{a_k}{a_l}\right)^{\frac{(1+\eta)p}{p+2}}. \tag{2.14}$$

Note that $\frac{2}{p+2}+\alpha-1>0$. Summarizing, from (2.13) and (2.14) we have

$$\left|\mathbf{Cov}\left(\mathbf{1}_A\left(\frac{S_k}{a_k}\right),\mathbf{1}_{(-\infty,x]}\left(\frac{S_l}{a_l}\right)\right)\right| \le C_\tau\left(\frac{a_k}{a_l}\right)^\tau, \tag{2.15}$$

where $\tau>0$ is some constant. We have already used the fact that L can be represented, as $x\to\infty$, as

$$L(x)=C(1+o(1))\exp\left\{\int_1^x\frac{\varepsilon(u)}{u}\,du\right\},$$

where $C>0$ and $\lim_{x\to\infty}\varepsilon(u)=0$. Let $0<\varepsilon<1$. Then, for some $k_\varepsilon<\infty$ and every $l\ge k\ge k_\varepsilon$,

$$\begin{aligned}\frac{L(a_k)}{L(a_l)} &\le C'\exp\left\{-\int_{a_k}^{a_l}\frac{\varepsilon(u)}{u}\,du\right\}\le C'\exp\left\{\int_{a_k}^{a_l}\frac{|\varepsilon(u)|}{u}\,du\right\}\\ &\le C'\exp\{\varepsilon\log(a_l/a_k)\}=C'(a_l/a_k)^\varepsilon,\end{aligned}$$

and invoking this time (2.1) we obtain

$$\frac{a_k}{a_l}\le(1+\varepsilon)\,\frac{k^{1/p}L^{1/p}(a_k)}{l^{1/p}L^{1/p}(a_l)}.$$

Combining both estimates gives

$$\frac{a_k}{a_l}\le C'(1+\varepsilon)\left(\frac{k}{l}\right)^{1/p}\left(\frac{a_l}{a_k}\right)^{\varepsilon/p}$$

and so

$$\left(\frac{a_k}{a_l}\right)^{1+\varepsilon/p}\le C'(1+\varepsilon)\left(\frac{k}{l}\right)^{1/p}. \tag{2.16}$$

From (2.15) and (2.16) it follows that

$$\left|\mathbf{Cov}\left(\mathbf{1}_A\left(\frac{S_k}{a_k}\right),\mathbf{1}_{(-\infty,x]}\left(\frac{S_l}{a_l}\right)\right)\right| \le C_\tau\left(\frac{a_k}{a_l}\right)^\tau\le C_{\tau,\varepsilon}\left(\frac{k}{l}\right)^{\tau/p(1+\varepsilon/p)}. \tag{2.17}$$

Put $\tau' = \tau/p(1+\varepsilon/p)$, $C_{\tau'} = C_{\tau,\varepsilon}$. Applying (2.17) with $A =]-\infty, s_k/a_k]$ and $x = s_l/a_l$ gives

$$|\mathbf{Cov}(\mathbf{1}_{\{S_k<s_k\}}, \mathbf{1}_{\{S_l<s_l\}})| \le C_{\tau'} \left(\frac{k}{l}\right)^{\tau'}.$$

Thus for $m \ge n$,

$$\begin{aligned} |\mathbf{Cov}(Y_n, Y_m)| &= \Bigg| \sum_{\substack{2^n \le k < 2^{n+1} \\ 2^m \le l < 2^{m+1}}} \frac{1}{kl} \mathbf{Cov}(\mathbf{1}_{\{S_k<s_k\}}, \mathbf{1}_{\{S_l<s_l\}}) \Bigg| \\ &\le C_{\tau'} \sum_{\substack{2^n \le k < 2^{n+1} \\ 2^m \le l < 2^{m+1}}} \frac{1}{kl} \left(\frac{k}{l}\right)^{\tau'} \le C'_{\tau'} 2^{-\tau'(m-n)}. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{m \ge n} |\mathbf{Cov}(Y_n, Y_m)| &\le C'_{\tau'} \sum_{m \ge n} 2^{-\tau'(m-n)} \le C'_{\tau'} \sum_{u \ge 0} 2^{-\tau' u}, \\ \sum_{m \le n} |\mathbf{Cov}(Y_n, Y_m)| &\le C'_{\tau'} \sum_{m \le n} 2^{-(n-m)/2} \le C'_{\tau'} \sum_{u \ge 0} 2^{-\tau' u}. \end{aligned}$$

We get $\sup_n \sum_m |\mathbf{Cov}(Y_n, Y_m)| \le 2C'_{\tau'} \sum_{u \ge 0} 2^{-\tau' u}$. The result thus follows from Lemma 2.7. ∎

3. Concluding remarks. It is also possible, by using Proposition 2.6 or Corollary 2.10 for instance, to obtain results concerning the weighted series $\sum_{k=1}^{\infty} w_k(\mathbf{1}_{A_k} - \mathbf{P}(A_k))$. When the sequence of weights is sufficiently regular, one can indeed directly control the convergence almost everywhere of the above series, by invoking some classical theorems from the theory of orthogonal series, like the Rademacher–Men'shov Theorem. The attentive reader will have observed, in many of the situations considered here, for instance, that the conditions $\sum_k w_k k^{-\alpha} < \infty$, for a suitable $\alpha > 0$, and $\sum_k w_k^2 \log^2 k < \infty$ are enough to get the convergence. The statements are however less satisfactory than those obtained with the sequence of blocks Y_n, where we could work with universal sequences, via the notion of quasi-orthogonal systems of functions.

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