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ON THE EXISTENCE OF SUPER-DECOMPOSABLE PURE-INJECTIVE MODULES OVER STRONGLY SIMPLY CONNECTED ALGEBRAS OF NON-POLYNOMIAL GROWTH

BY

STANISŁAW KASJAN (Toruń) and GRZEGORZ PASTUSZAK (Warszawa)

Abstract. Assume that k is a field of characteristic different from 2. We show that if Γ is a strongly simply connected k-algebra of non-polynomial growth, then there exists a special family of pointed Γ-modules, called an independent pair of dense chains of pointed modules. Then it follows by a result of Ziegler that Γ admits a super-decomposable pure-injective module if k is a countable field.

1. Introduction. According to Drozd's fundamental tame-wild dichotomy [D], the class of finite-dimensional algebras (over algebraically closed fields) divides into two classes: tame and wild algebras. A lot of effort has been put into understanding various aspects of representation type (see [Ri], [SS2]). On the finite-dimensional level we have, in particular, results on the shape of connected components of the Auslander–Reiten quiver or the component quiver (see [Sk4], [Sk6]). On the level of infinite-dimensional modules, a basic characterization of representation type is given in terms of generic modules (see [CB]).

One may also hope to understand representation type in terms of the behavior of some matrix reduction algorithms described, for instance, in [GKM], [MZ].

The concept of growth of a tame algebra, introduced by A. Skowroński [Sk2], yields a stratification of the class of tame algebras into domestic algebras, polynomial growth algebras etc. The structure of the module category depends very much on the growth properties of the underlying algebra. The reader is referred to [PS1], [PS2], [SZ] for geometric and homological characterizations of polynomial growth strongly simply connected algebras.

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One of the motivations of our research is to explain how the growth rate is reflected on the level of infinite-dimensional modules. A hint, following from results of [Pu1], [Pr2], is to look at super-decomposable pure-injective modules.

There are problems which can be reduced (using covering techniques and geometric degenerations) to the class of strongly simply connected algebras, this is one of the reasons why that class is of interest. An example of such a problem is the classification of selfinjective algebras of polynomial growth [Sk1]; see also the survey [Sk7].

Assume that R is a ring with a unit. By a *module* we usually mean a left module. Recall that an R-module is *super-decomposable* if it has no indecomposable direct summand. We refer to [K] (see also [HZ] and [JL, Chapter 7]) for the concept of *pure-injectivity*.

The problem of existence of super-decomposable pure-injective R-modules is stated in [Z]. In the same paper M. Ziegler proves a fundamental criterion for such modules to exist: if the ring R is countable, then R has a super-decomposable pure-injective module if and only if the width of the lattice of all pp-formulae is undefined (see [Z] and [Pr1, Section 10.2] for the definitions).

The case when R is a finite-dimensional algebra over a field is studied, in particular, in [Pr1], [Pu1], [Pu2], [H], [Pr2] and [KP].

In [Pr1] M. Prest considers the problem in connection with representation types of finite-dimensional algebras and he proves that super-decomposable pure-injective modules exist over strictly wild algebras (see [Pr1, Theorem 13.7]). In [Pu1] G. Puninski proves that such modules exist over non-polynomial growth string algebras over a countable field. In [Pu2] Puninski refines this result and presents a "concrete" example of a super-decomposable pure-injective module over a certain string algebra of non-polynomial growth. The reader is referred to [Pu3] and [Pu4] for other applications of string combinatorics in investigation of infinite-dimensional modules. In [H] R. Harland proves that super-decomposable pure-injective modules exist over tubular algebras (see also [Pr2]). This is the first class of linear growth, although non-domestic, tame algebras that is known to have super-decomposable pure-injective modules.

In [KP] we prove in particular that there exists a super-decomposable pure-injective module over the *garland of length* 3 (see [Si1, Definition 15.29] and Section 4) if the base field k is countable and its characteristic is different from 2.

The present paper applies the results of [KP] to show the existence of super-decomposable pure-injective modules for all non-polynomial growth strongly simply connected algebras (see [Sk5] and [Sk3]) over countable algebraically closed fields of characteristic different from 2. One of the key elements in the proof are criteria for tame representation type and polynomial growth of strongly simply connected algebras given in [BPS] and [Sk5], respectively.

The paper is organized as follows. In Section 2 we introduce the notation, terminology and basic concepts of representation theory of finite-dimensional algebras that we use.

In Section 3 we collect necessary facts concerning wide subposets of modular lattices and generalized pointed modules. We introduce some natural generalization of the concept of an *independent pair of dense chains of pointed modules* from [PPT1] (see also [KP]). This allows us to formulate a sufficient condition for the existence of a super-decomposable pure-injective module in the context we need (see Theorem 3.9). We also prove that fully-faithful (i.e. full and faithful) exact functors preserve, in a certain sense and under some additional assumptions, independent pairs of dense chains of pointed modules and wide subposets of pointed modules (see Theorem 3.13). This fact is crucial to the subsequent sections of the paper.

In Section 4 we apply the main results of [KP] to prove the existence of an independent pair of dense chains of pointed prinjective modules over the garland algebra satisfying some additional conditions. We further apply this result and Theorem 3.13 to prove the existence of an independent pair of dense chains of pointed prinjective modules over the *diamond* algebra (see [Si2] and [Si3]).

In Section 5 we define some special configurations of modules, called the *$\mathcal{M}'$-configuration* and the *$\mathcal{N}$-configuration* (see Definition 5.1). We prove, in a bit more general context, that the existence of such configurations in certain module categories implies the existence of fully-faithful exact functors from the categories of prinjective modules over the garland or diamond to the categories of modules over *pg-critical algebras* [NS] (see Theorem 5.5).

Section 6 is devoted to showing that $\mathcal{M}'$-configurations exist in module categories over pg-critical algebras of type I (see Section 2 for the definition) and that $\mathcal{N}$-configurations exist in module categories over pg-critical algebras of type II (see Theorems 6.6 and 6.7). We recall that this fact is essentially known from [Si3]. However, in order to make our paper self-contained and more convenient to the reader, we provide explicit constructions of $\mathcal{M}'$-configurations and $\mathcal{N}$-configurations in appropriate categories of modules.

We stress that the results from Sections 5 and 6 follow the lines of the proof that pg-critical algebras are of non-polynomial growth (see [Si3, Theorem 2.2] and [Si2]).

Applying Skowroński's criterion of polynomial growth of strongly simply connected algebras [Sk5], in Section 7 we deduce the main results of the paper concerning the existence of independent pairs of dense chains of pointed modules and super-decomposable pure-injective modules for strongly simply connected algebras of non-polynomial growth (see Theorems 7.1 and 7.2).

2. Notation and preliminary facts. Throughout the paper, k is a fixed algebraically closed field. By an *algebra* we mean a finite-dimensional associative basic k-algebra with a unit. If A is an algebra, we denote by A^{op} the algebra *opposite* to A. By a *module* we usually mean a left module. Clearly any left A-module is also a right A^{op}-module.

We denote by A-mod (mod(A), respectively) the category of finitely generated left A-modules (finitely generated right A-modules, respectively).

Recall that there exists a duality functor $D : A\text{-mod} \to A^{\mathrm{op}}\text{-mod}$ defined by $D(M) = \mathrm{Hom}_k(M, k)$ for any A-module M.

Assume that $Q = (Q_0, Q_1)$ is a finite quiver with the set Q_0 of vertices and the set Q_1 of arrows. Given $\alpha \in Q_1$, the starting and the terminal points of α are denoted by $s(\alpha)$ and $t(\alpha)$ respectively.

Assume that $x, y \in Q_0$. By a *path* from x to y in Q we mean a sequence $c_1 \dots c_n$ in Q_1 such that $s(c_n) = x$, $t(c_1) = y$ and $s(c_i) = t(c_{i+1})$ for $1 \leq i < n$. We associate the *stationary path* e_x to each vertex $x \in Q_0$ and we set $s(e_x) = t(e_x) = x$.

Given a quiver $Q = (Q_0, Q_1)$ we denote by kQ the *path algebra* of Q as usual: the k-basis of kQ is the set of all paths in Q, and multiplication in kQ is induced by concatenation of paths.

A two-sided ideal I in kQ is called *admissible* if $\langle Q_1\rangle^n \subseteq I \subseteq \langle Q_1\rangle^2$ for some $n \in \mathbb{N}$, $n \geq 2$. If I is an admissible ideal in kQ, then the pair (Q, I) is called a *bound quiver* and the associated quotient algebra kQ/I a *bound quiver algebra*. Recall that any admissible ideal I is generated by a finite set of *relations* (see [ASS, II]).

For $x \in Q_0$ we denote by $P(x)$, $I(x)$ and $S(x)$ the indecomposable projective, injective and simple modules, respectively, over the bound quiver algebra kQ/I associated with the vertex x.

Assume that (Q, I) is a bound quiver. We denote by $\mathrm{rep}_k(Q, I)$ the category of *finite-dimensional k-linear representations* of (Q, I) (see [ASS, III]). If $A = kQ/I$, then there is a k-linear equivalence $F : A\text{-mod} \to \mathrm{rep}_k(Q, I)$ (see [ASS, III, Theorem 1.6]). From now on we shall identify $\mathrm{rep}_k(Q, I)$ with A-mod. Our convention of composing paths is opposite to the one applied in [ASS]. Consequently, representations of Q correspond to left modules over "our" kQ.

Recall that a full subquiver Q' of Q is *convex* if for any path $c_1 \dots c_n$ from a vertex x to a vertex y in Q'_0 we have $s(c_i) \in Q'_0$ for $i = 1, \dots, n-1$.

If Q' is a convex subquiver of Q, I is an admissible ideal in kQ, and $I' = I \cap kQ'$, then the *functor of extension by zeros* $\Theta : \operatorname{rep}_k(Q', I') \to \operatorname{rep}_k(Q, I)$ sends a representation $(V_x, V_\alpha)_{x \in Q'_0, \alpha \in Q'_1}$ of (Q', I') to the representation $(\widehat{V}_x, \widehat{V}_\alpha)_{x \in Q_0, \alpha \in Q_1}$, where $\widehat{V}_y = V_y$ for any $y \in Q'_0$ or $y \in Q'_1$ and $\widehat{V}_y$ is zero if y is a vertex or an arrow outside Q'. It is easy to see that Θ is fully-faithful and exact.

Recall that in the situation above the algebra $k(Q', I')$ is called a *convex subcategory* of $k(Q, I)$.

We now collect some facts concerning bipartite algebras and prinjective modules which are used in the paper.

Assume that A and B are k-algebras and $M = {}_BM_A$ is a B-A-bimodule. A *bipartite algebra* (see [Si1, 17.4]) is an algebra

$$R = \begin{bmatrix} A & 0 \\ {}_BM_A & B \end{bmatrix}$$

of matrices $\left[\begin{smallmatrix} a & 0 \\ m & b \end{smallmatrix}\right]$ such that $a \in A$, $b \in B$ and $m \in {}_BM_A$ with multiplication given by the formula

$$\begin{bmatrix} a & 0 \\ m & b \end{bmatrix} \cdot \begin{bmatrix} a' & 0 \\ m' & b' \end{bmatrix} = \begin{bmatrix} aa' & 0 \\ ma' + bm' & bb' \end{bmatrix}.$$

Observe that if R is a bound quiver algebra kQ/I, then there are convex subquivers Q_A and Q_B of Q such that every vertex of Q belongs to exactly one of the subquivers Q_A or Q_B, there are no oriented paths from Q_B to Q_A, and $A \cong kQ_A/(I \cap kQ_A)$, $B \cong kQ_B/(I \cap kQ_B)$. Moreover, the bimodule ${}_BM_A$ can be identified with the vector subspace of kQ/I generated by the cosets of the paths starting from Q_A and terminating in Q_B, equipped with the natural bimodule structure.

Assume that

$$e_A = \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad e_B = \begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix},$$

i.e. $A \cong e_A R e_A$ and $B \cong e_B R e_B$.

A left R-module X is ${}_BM_A$*-prinjective* (see [PS]) if $e_A X$ is a projective A-module and $e_B X$ is an injective B-module. We denote by $\operatorname{prin}_{{}_BM_A}(R)$ (or $\operatorname{prin}_M(R)$, or $\operatorname{prin}(R)$) the category of all ${}_BM_A$-prinjective R-modules.

It is well known that any R-module can be identified with an *R-triple*

$$({}_AX', {}_BX'', \varphi_X : {}_BM_A \otimes {}_AX' \to {}_BX'')$$

where ${}_AX'$ is an A-module, ${}_BX''$ is a B-module and $\varphi_X : {}_BM_A \otimes {}_AX' \to {}_BX''$ is a B-module homomorphism.

A morphism from $(X', X'', \varphi_X : M \otimes X' \to X'')$ to $(Y', Y'', \varphi_Y : M \otimes Y' \to Y'')$ is a pair (f_1, f_2) such that $f_1 : X' \to Y'$ is an A-homomorphism,

$f_2 : X'' \to Y''$ is a B-homomorphism and $f_2\varphi_X = \varphi_Y(\mathrm{id}_M \otimes f_1)$. Therefore the R-triples form a category. It is easy to see that the category of R-modules and the category of R-triples are equivalent.

Thanks to the *adjointness formula*, we can also consider triples of the form $(X', X'', \psi : X' \to \mathrm{Hom}_B(M, X''))$.

It is clear that if X is a prinjective R-module, then X can be identified with a triple $(X', X'', \varphi : M \otimes X' \to X'')$ such that X' is a projective A-module and X'' is an injective B-module [Si2].

Now we recall some special classes of bipartite algebras needed in further considerations.

Assume that A is an algebra and M is a left A-module. For any $n \geq 1$, $M^n = \underbrace{M \oplus \cdots \oplus M}_{n}$ has a natural structure of an A-k^n-bimodule. The algebra

$$[M, n]A = \begin{bmatrix} A & 0 \\ DM^n & k^n \end{bmatrix} \cong \begin{bmatrix} A & 0 & 0 & \cdots & 0 \\ DM & k & 0 & \cdots & 0 \\ DM & 0 & k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ DM & 0 & 0 & \cdots & k \end{bmatrix}$$

is the *n-point coextension of A by M*. If $n = 1$, then we set $[M, 1]A = [M]A$.

The Gabriel quiver $Q_{[M,n]A}$ of the algebra $[M, n]A$ consists of the Gabriel quiver Q_A of A and n additional vertices $x_1, \ldots, x_n$ which are the terminal points of some arrows starting in Q_A. Moreover,

$$I(x_i)/\mathrm{soc}(I(x_i)) \cong I(x_i)/S(x_i) \cong M$$

for any $i = 1, \ldots, n$.

Assume that B is an algebra, N is a left B-module and $m \geq 1$. The algebra

$$B[N, m] = \begin{bmatrix} k^m & 0 \\ N^m & B \end{bmatrix} \cong \begin{bmatrix} k & 0 & \cdots & 0 & 0 \\ 0 & k & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & k & 0 \\ N & N & \cdots & N & B \end{bmatrix}$$

is the *m-point extension of B by N*. If $m = 1$, then we set $B[N, 1] = B[N]$.

The Gabriel quiver $Q_{B[N,m]}$ of the algebra $B[N, m]$ consists of the Gabriel quiver Q_B of B and m additional vertices $y_1, \ldots, y_m$ which are the starting points of some arrows terminating in Q_B. Moreover, $\mathrm{rad}(P(y_i)) \cong N$ for any $i = 1, \ldots, m$.

Assume that A is an algebra, N is a left A-module, $t \geq 1$ and L_t is the path algebra of the canonically oriented Dynkin quiver

$$\Delta(\mathbb{A}_t) = \ 1 \to 2 \to \cdots \to t.$$

Observe that

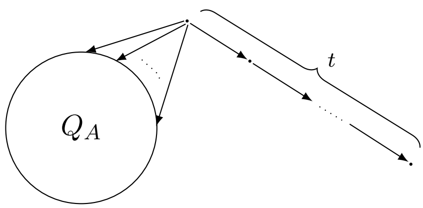
$$L_t \cong \begin{bmatrix} k & 0 & 0 & \cdots & 0 \\ k & k & 0 & \cdots & 0 \\ k & k & k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k & k & k & \cdots & k \end{bmatrix}.$$

Assume that $\overline{N} = \{(n, 0, \ldots, 0);\ n \in N\}$ is an A-L_t-bimodule isomorphic to N as a left A-module with the right structure of L_t-module defined in the following way: $(n, 0, \ldots, 0) \cdot l = [\, n\ 0\ \ldots\ 0 \,] l$ for any $n \in N$, $l \in L_t$, where the right hand side is a matrix product. We shall use the matrix notation $[\, N\ 0\ \ldots\ 0 \,]$ for the module $\overline{N}$.

The *t-linear extension of A by N* is, by definition, the algebra

$$A[N, L_t] = \begin{bmatrix} L_t & 0 \\ \overline{N} & A \end{bmatrix} \cong \begin{bmatrix} k & 0 & \cdots & 0 & 0 \\ k & k & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ k & k & \cdots & k & 0 \\ N & 0 & \cdots & 0 & A \end{bmatrix}.$$

The Gabriel quiver $Q_{A[N,L_t]}$ of the algebra $A[N, L_t]$ consists of the Gabriel quiver Q_A of A and the canonically oriented Dynkin quiver $\Delta(\mathbb{A}_t)$ attached to Q_A by some arrows $\gamma_1, \ldots, \gamma_s$ starting from the vertex 1 and terminating in Q_A. We can visualize the Gabriel quiver $Q_{A[N,L_t]}$ in the following way:

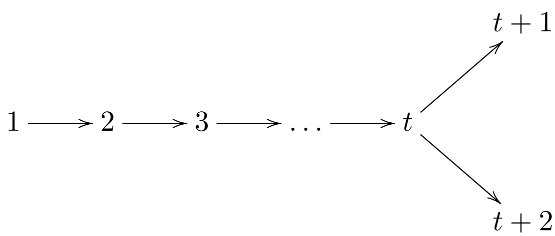
$Q_{A[N,L_t]} =$ 

We shall now define some special 2-point coextensions of t-linear extensions, which we call *$\mathbb{D}_{t+2}$-extensions.*

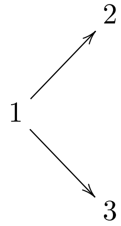
Assume that $A[N, L_t]$ is a t-linear extension of an algebra A by a module N, and $\widehat{\mathcal{L}}$ is an $A[N, L_t]$-module corresponding to the triple $(\mathcal{L}, 0, 0)$ where

$$\mathcal{L} = \ k \xrightarrow{1} k \xrightarrow{1} \cdots \xrightarrow{1} k.$$

Assume that G_t is the canonically oriented Dynkin quiver $\Delta(\mathbb{D}_{t+2})$, i.e.

$G_t =$ 

for $t \geq 2$ and

$G_1 =$ 

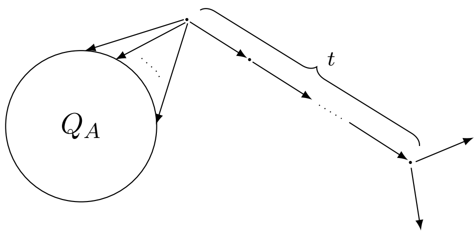
The algebra

$$A[N, G_t] := [\widehat{\mathcal{L}}, 2](A[N, L_t]) = \begin{bmatrix} A[N, L_t] & 0 \\ D\widehat{\mathcal{L}} \oplus D\widehat{\mathcal{L}} & k \oplus k \end{bmatrix} \cong \begin{bmatrix} A[N, L_t] & 0 & 0 \\ D\widehat{\mathcal{L}} & k & 0 \\ D\widehat{\mathcal{L}} & 0 & k \end{bmatrix}$$

$$\cong \begin{bmatrix} k & 0 & \cdots & 0 & 0 & 0 & 0 \\ k & k & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ k & k & \cdots & k & 0 & 0 & 0 \\ N & 0 & \cdots & 0 & A & 0 & 0 \\ k & k & \cdots & k & 0 & k & 0 \\ k & k & \cdots & k & 0 & 0 & k \end{bmatrix} \cong \begin{bmatrix} k & 0 & \cdots & 0 & 0 & 0 & 0 \\ k & k & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ k & k & \cdots & k & 0 & 0 & 0 \\ k & k & \cdots & k & k & 0 & 0 \\ k & k & \cdots & k & 0 & k & 0 \\ N & 0 & \cdots & 0 & 0 & 0 & A \end{bmatrix}$$

is the $\mathbb{D}_{t+2}$-*extension of A by N*.

The Gabriel quiver $Q_{A[N,G_t]}$ of the algebra $A[N, G_t]$ consists of the Gabriel quiver Q_A of A and the quiver G_t attached to Q_A by some arrows $\delta_1, \ldots, \delta_s$ starting from the vertex 1 and terminating in Q_A. We can visualize the Gabriel quiver $Q_{A[N,G_t]}$ in the following way:

$Q_{A[N,G_t]} =$ 

Assume now that $A = kQ/I$ is a *triangular* algebra (or k-category), which means that Q has no oriented cycles. Then A is *strongly simply connected* if the first Hochschild cohomology group $H^1(C, C)$ of any convex subcategory C of A vanishes (see [Sk3] for details).

In the theory of strongly simply connected algebras, and also in this paper, a major role is played by *hypercritical* and *pg-critical* algebras (see [Sk5], [Sk3]). We recall the definitions below. The reader is referred to [ASS] for the background on tilting theory.

Assume that Q is a quiver whose underlying graph is one of the following graphs: T_5, $\widetilde{\widetilde{\mathbb{D}}}_n$ or $\widetilde{\widetilde{\mathbb{E}}}_i$ for $i = 6, 7, 8$ (see for example [Sk5]). A *hypercritical algebra* is a concealed algebra of type Q.

Assume that H is a hereditary algebra of type $\widetilde{\mathbb{D}}_n$, T is a tilting H-module without prinjective direct summands, and $B = \mathrm{End}_H(T)^{\mathrm{op}}$ is a representation-infinite tilted algebra. We denote by $\mathcal{T}(T)$ the torsion class induced by T.

A *pg-critical algebra of type I* is an algebra $\Lambda = B[M, G_t]$, $t \geq 1$, such that:

- $M = \mathrm{Hom}_H(T, S)$ where S is an indecomposable regular H-module of regular length 1 lying in a tube of rank $n-2$ such that $S \in \mathcal{T}(T)$,
- any proper convex subcategory of Λ, viewed as a k-category, is of polynomial growth.

A *pg-critical algebra of type II* is an algebra $\Omega = B[N]$ such that

- $N = \mathrm{Hom}_H(T, R)$ where R is an indecomposable regular H-module of regular length 2 lying in a tube of rank $n-2$ such that $R \in \mathcal{T}(T)$,
- any proper convex subcategory of Ω, viewed as a k-category, is of polynomial growth.

A pg-critical algebra $\Lambda = B[M, G_t]$ of type I is thus, in our notation and terminology, a $\mathbb{D}_{t+2}$-extension of the algebra B by the module M. Clearly this is consistent with [NS], [Sk5] and [Sk3]. A pg-critical algebra $\Omega = B[N]$ of type II is a one-point extension of the algebra B by the module N.

A *pg-critical algebra* is a pg-critical algebra of type I or II.

Recall that hypercritical algebras are strictly wild and have been completely classified by quivers and relations (see [U]). In turn, pg-critical algebras are tame of non-polynomial growth (see [Sk5, Proposition 2.4]) and have been completely classified by quivers and relations in [NS].

The following theorem collects the fundamental results of [BPS] and [Sk5], and shows the importance of hypercritical and pg-critical algebras in the theory of strongly simply connected algebras. It is also crucial for our paper.

THEOREM 2.1. *Assume that A is a strongly simply connected algebra.*

(1) *A is wild if and only if it is strictly wild; and A is strictly wild if and only if it has a convex hypercritical subcategory* [BPS, Corollary 1].
(2) *A is wild or tame of non-polynomial growth if and only if it has a convex pg-critical subcategory* [Sk5, Theorem 4.1]. ∎

Observe that the conditions (1) and (2) of the above theorem imply that a tame strongly simply connected algebra A is of non-polynomial growth if and only if A has a convex pg-critical subcategory.

3. Wide posets of generalized pointed modules and a sufficient existence condition. Assume that R is a ring with a unit. This section is devoted to presenting a sufficient condition for the existence of a superdecomposable pure-injective R-module in terms of some special family of pointed finitely presented R-modules, called an *independent pair of dense chains of pointed R-modules*. This condition can be considered as a generalization of [PPT1, Proposition 5.4].

We start with some preliminaries on modular lattices.

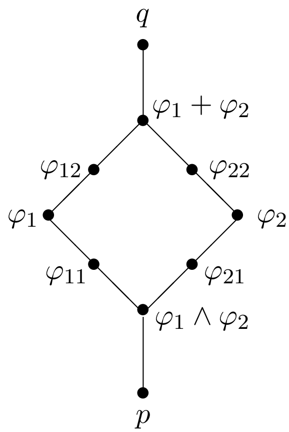
Recall that a poset $(K, \leq)$ is a *lattice* if for any $x, y \in K$ there are $x \wedge y, x + y \in K$ such that $x \wedge y = \inf\{x, y\}$ and $x + y = \sup\{x, y\}$.

A lattice $(K, \leq)$ is *modular* if $x \geq z$ implies $(x \wedge y) + z = x \wedge (y + z)$ for any $x, y, z \in K$. A lattice $(K, \leq)$ is *distributive* if $(x \wedge y) + z = (x + z) \wedge (y + z)$ for any $x, y, z \in K$. Observe that by [G, I.4, Lemma 10] this condition is equivalent to $x \wedge (y + z) = (x \wedge y) + (x \wedge z)$ for any $x, y, z \in K$. It is clear that any distributive lattice is modular.

Assume that $(P, \leq)$ is an arbitrary poset and $p, q \in P$. We write $p < q$ if $p \leq q$ and $p \neq q$. If neither $p \leq q$ nor $q \leq p$, we say that p and q are *incomparable*.

DEFINITION 3.1. Assume that $(K, \leq)$ is a lattice. A *wide subposet* of K is a subposet P of K such that for any $p < q \in P$ there are incomparable elements $\varphi_1, \varphi_2 \in P$ with $p < \varphi_1, \varphi_2 < q$ and $\varphi_1 \wedge \varphi_2 \leq \varphi_{11} < \varphi_1 < \varphi_{12} \leq \varphi_1 + \varphi_2$ and $\varphi_1 \wedge \varphi_2 \leq \varphi_{21} < \varphi_2 < \varphi_{22} \leq \varphi_1 + \varphi_2$ for some $\varphi_{11}, \varphi_{12}, \varphi_{21}, \varphi_{22} \in P$.

The inequalities from the definition above can be visualized as follows:



Note that φ_{11} and φ_{21} may coincide with $\varphi_1 \wedge \varphi_2$ (and with p). Similarly, φ_{12} and φ_{22} may coincide with $\varphi_1 + \varphi_2$ (and with q).

Observe that if P is a sublattice of K, then P is a wide subposet of K if and only if for any $p < q \in P$ there are incomparable elements $q_1, q_2 \in P$ such that $p < q_1, q_2 < q$ and

$$p \leq q_1 \wedge q_2 < q_1 + q_2 \leq q.$$

One can show that a lattice K contains a wide subposet if and only if the width of K is undefined or, according to another convention, the width of K is equal to infinity (see [Pr1, Section 10.2], [Pu1] and [Z] for details).

Our aim is to present a sufficient condition for a modular lattice to contain a wide subposet.

We call any countable dense chain without end points a *$\mathbb{Q}$-chain*. Although it is clear that any $\mathbb{Q}$-chain is isomorphic as a poset to the poset $\mathbb{Q}$ of rational numbers, we work with the more general definition for technical reasons.

Assume that $(K, \leq)$ is a lattice. The following definition generalizes the notions of [PPT1, Proposition 5.4] to the case of arbitrary lattices.

Definition 3.2.

(1) A *primitive element* of K is an element $l \in K$ such that $l = l_1 + l_2$ implies $l = l_1$ or $l = l_2$ for any $l_1, l_2 \in K$.
(2) An *independent pair of $\mathbb{Q}$-chains* in K is a pair $(\mathcal{L}, \mathcal{K})$ of $\mathbb{Q}$-chains in K such that:
  (a) $x \wedge y$ is primitive for any $x \in \mathcal{L}$ and $y \in \mathcal{K}$,
  (b) $x \wedge y \neq x' \wedge y$ and $x \wedge y \neq x \wedge y'$ for any $x \neq x' \in \mathcal{L}$ and $y \neq y' \in \mathcal{K}$.

Observe that (b) implies that if $x \in \mathcal{L}$ and $y \in \mathcal{K}$, then x and y are incomparable. Indeed, if $x \leq y$, then $x \wedge y = x$. Since there is $y' \in \mathcal{K}$ such that $y < y'$, we get

$$x \wedge y = x \wedge (y \wedge y') = (x \wedge y) \wedge y' = x \wedge y',$$

which contradicts (b).

Assume that $(K, \leq)$ is a lattice and X is an arbitrary subset of K. The lattice $\mathrm{Gen}(X)$ *generated by* X is the intersection of all sublattices of K containing X. Obviously $\mathrm{Gen}(X)$ is the smallest sublattice of K containing X.

Assume that $(K, \leq)$ is a modular lattice. We shall now outline the argument that the lattice $\mathrm{Gen}(\mathcal{L} \cup \mathcal{K})$ is wide if $(\mathcal{L}, \mathcal{K})$ is an independent pair of $\mathbb{Q}$-chains in K. Our proof is based on the following criterion due to G. Puninski (see [PPT2], [Pu1] and [Pu2]).

Proposition 3.3. *Assume that $(K, \leq)$ is a modular lattice and L_1, L_2 are $\mathbb{Q}$-chains in K. Assume that*

$$(*) \qquad a \wedge b \leq x + y \quad \textit{if and only if} \quad a \leq x \textit{ or } b \leq y$$

for any $a, x \in L_1$ and $b, y \in L_2$. Then the lattice $\mathrm{Gen}(L_1 \cup L_2)$ is wide.

Proof. It follows from [PPT2, Lemma 5.4] (see also Section 2 of [Pu2]) that $(*)$ implies that $\mathrm{Gen}(L_1 \cup L_2)$ is isomorphic to the lattice $L_1 \otimes L_2$ freely generated by L_1 and L_2 (see [G] and [Pu1] for the definition). Therefore, by the arguments from [Pu1, Corollary 3.2], we conclude that $\mathrm{Gen}(L_1 \cup L_2)$ is wide. ∎

The literal formulation of [Pu1, Corollary 3.2] states that the lattice $L_1 \otimes L_2$ has infinite width. This only implies that $L_1 \otimes L_2$ contains a wide subposet. However, the fact that $L_1 \otimes L_2$ itself is wide is an easy consequence of the proof of [Pu1, Corollary 3.2].

In the following theorem we generalize [PPT1, Proposition 5.4] to arbitrary modular lattices.

THEOREM 3.4. *Assume that $(K, \leq)$ is a modular lattice. If $(\mathcal{L}, \mathcal{K})$ is an independent pair of $\mathbb{Q}$-chains in K, then the lattice* $\mathrm{Gen}(\mathcal{L} \cup \mathcal{K})$ *is wide.*

Proof. It is enough to show that the condition $(*)$ from Proposition 3.3 is satisfied. Recall that by the classical theorem of G. Birkhoff the lattice $\mathrm{Gen}(\mathcal{L} \cup \mathcal{K})$ is distributive (see [G, IV.1, Theorem 13]).

Assume that $a \wedge b \leq x + y$ for some $a, x \in \mathcal{L}$ and $b, y \in \mathcal{K}$. Then $a \wedge b = (a \wedge b) \wedge (x + y) = (a \wedge b \wedge x) + (a \wedge b \wedge y)$ and thus $a \wedge b = a \wedge b \wedge x$ or $a \wedge b = a \wedge b \wedge y$ since $a \wedge b$ is primitive. Consequently, $a \wedge b \leq x$ or $a \wedge b \leq y$.

Assume in turn $a \wedge b \leq x$ and $a > x$. Then $a \wedge b \leq x \wedge b$, $a \wedge b \geq x \wedge b$ and hence $a \wedge b = x \wedge b$, which contradicts the independence of the pair $(\mathcal{L}, \mathcal{K})$.

It follows that $a \wedge b \leq x$ implies $a \leq x$ and similarly $a \wedge b \leq y$ implies $b \leq y$. Consequently, $a \wedge b \leq x + y$ implies $a \leq x$ or $b \leq y$, and since the converse is obvious, the condition $(*)$ is satisfied. Hence the lattice $\mathrm{Gen}(\mathcal{L} \cup \mathcal{K})$ is wide by Proposition 3.3. ∎

We shall now apply Theorem 3.4 to the lattice of *generalized pointed modules.* This will give us a handy version of Ziegler's criterion for the existence of super-decomposable pure-injective module in terms of independent pairs of $\mathbb{Q}$-chains (see [Z, Lemma 7.8 (2)] for the original version of the criterion).

Assume that R is a ring with a unit and Θ is a finitely presented R-module. By a *Θ-pointed R-module* (or a *generalized pointed R-module*) we mean a pair (M, χ_M) where M is a finitely presented R-module and $\chi_M : \Theta \to M$ is an R-module homomorphism. If $\Theta = R$, then χ_M is uniquely determined by the element $\chi_M(1)$ of M. If this is the case, we identify (M, χ_M) with $(M, \chi_M(1))$ and call it a *pointed R-module.*

Assume that (M, χ_M) and (N, χ_N) are Θ-pointed R-modules. By a *Θ-pointed R-homomorphism* from (M, χ_M) to (N, χ_N) we mean an R-homomorphism $f : M \to N$ such that $f\chi_M = \chi_N$. In this case we also write $f : (M, \chi_M) \to (N, \chi_N)$. If $f : M \to N$ is an isomorphism, we call $f : (M, \chi_M) \to (N, \chi_N)$ a *Θ-pointed isomorphism* and the corresponding Θ-pointed modules (M, χ_M) and (N, χ_N) are said to be *Θ-isomorphic.*

It is easy to see that an R-homomorphism $f : M \to N$ is a pointed homomorphism from (M, m) to (N, n) if and only if $f(m) = n$. If this is the case we call $f : (M, m) \to (N, n)$ a *pointed homomorphism*.

Let P_R^Θ be the set of all Θ-isomorphism classes of Θ-pointed R-modules. Let $\equiv$ be the relation on P_R^Θ defined by $(M, \chi_M) \equiv (N, \chi_N)$ if and only if there exist pointed homomorphisms $f : (M, \chi_M) \to (N, \chi_N)$ and $g : (N, \chi_N) \to (M, \chi_M)$. It is clearly an equivalence relation and the quotient set $\mathcal{P}_R^\Theta = P_R^\Theta/\equiv$ is a poset with respect to the relation $\le$ defined by $\overline{(M, \chi_M)} \le \overline{(N, \chi_N)}$ if and only if there exists a pointed homomorphism $f : (N, \chi_N) \to (M, \chi_M)$. We denote by $\overline{(S, \chi_S)}$ the $\equiv$-class of a Θ-pointed R-module (S, χ_S).

It is known (see [Pr1] for example) that the poset $\mathcal{P}_R^\Theta$ is a modular lattice with respect to the operations $\oplus$ and $*$ defined below.

Assume that (M, χ_M), (N, χ_N) are Θ-pointed R-modules. A Θ-pointed R-module $(M \oplus N, \chi_{M\oplus N})$ where $\chi_{M\oplus N}(l) = (\chi_M(l), \chi_N(l))$ for any $l \in \Theta$ is the *pointed direct sum* of (M, χ_M) and (N, χ_N). We set $(M, \chi_M) \oplus (N, \chi_N) = (M \oplus N, \chi_{M\oplus N})$.

Assume that $M * N$ is the *pushout* of χ_M and χ_N, that is,

$$M * N = M \oplus N/\{(\chi_M(l), -\chi_N(l));\ l \in \Theta\}.$$

Moreover, let $\epsilon_M : M \to M * N$ and $\epsilon_N : N \to M * N$ be the R-module homomorphisms given by $\epsilon_M(m) = \overline{(m, 0)}$ and $\epsilon_N(n) = \overline{(0, n)}$ for any $m \in M$ and $n \in N$. The Θ-pointed R-module $(M * N, \chi_{M*N})$ where $\chi_{M*N} = \epsilon_M \chi_M = \epsilon_N \chi_N$ is the *pointed pushout* of (M, χ_M) and (N, χ_N). We set $(M, \chi_M) * (N, \chi_N) = (M * N, \chi_{M*N})$.

We have

$$\sup\{\overline{(M, \chi_M)}, \overline{(N, \chi_N)}\} = \overline{(M \oplus N, \chi_{M\oplus N})},$$
$$\inf\{\overline{(M, \chi_M)}, \overline{(N, \chi_N)}\} = \overline{(M * N, \chi_{M*N})}.$$

Recall that if $\Theta = R^t$, then the lattice $\mathcal{P}_R^\Theta$ is equivalent to the lattice of all *pp-formulae* with t free variables (see [Pr1]).

We are now ready to present a version of Ziegler's criterion in terms of independent pairs of $\mathbb{Q}$-chains in $\mathcal{P}_R^\Theta$.

THEOREM 3.5. *Assume that R is a countable ring with a unit and Θ is a finitely presented R-module. If there exists an independent pair $(\mathcal{L}, \mathcal{K})$ of $\mathbb{Q}$-chains in $\mathcal{P}_R^\Theta$, then there exists a super-decomposable pure-injective R-module.*

Proof. Since $\mathcal{P}_R^\Theta$ is a modular lattice, Theorem 3.4 implies that $\mathrm{Gen}(\mathcal{L} \cup \mathcal{K})$ is wide. Hence $\mathcal{P}_R^\Theta$ contains a wide subposet and it follows from [PP1, Theorem 7.1, Proposition 7.2] that there exists a super-decomposable pure-injective R-module. ∎

Our next aim is to present a sufficient condition for the existence of an independent pair of $\mathbb{Q}$-chains in $\mathcal{P}_R^{\Theta}$. The following two definitions are crucial.

Assume that R is a ring with a unit and Θ is a finitely presented R-module.

DEFINITION 3.6. Assume that C is a $\mathbb{Q}$-chain. A *dense chain of Θ-pointed R-modules* is a family $(M_q, \chi_{M_q})_{q\in C}$ of Θ-pointed R-modules such that:

(a) the endomorphism ring $\mathrm{End}_R(M_q)$ is local and $\chi_{M_q} \neq 0$ for any $q \in C$,
(b) there exist Θ-pointed homomorphisms $\mu_{q,q'} : (M_q, \chi_{M_q}) \to (M_{q'}, \chi_{M_{q'}})$ for any $q < q' \in C$,
(c) the pointed modules (M_q, χ_{M_q}) and $(M_{q'}, \chi_{M_{q'}})$ are not isomorphic for any $q \neq q' \in C$.

DEFINITION 3.7. An *independent pair of dense chains of Θ-pointed R-modules* is a pair $((M_q, \chi_{M_q})_{q\in C_1}, (N_t, \chi_{N_t})_{t\in C_2})$ of dense chains of Θ-pointed R-modules such that:

(a) the endomorphism ring $\mathrm{End}_R(M_q * N_t)$ is local for any $q \in C_1$, $t \in C_2$ where $(M_q * N_t, \chi_{M_q * N_t}) = (M_q, \chi_{M_q}) * (N_t, \chi_{N_t})$,
(b) the pointed module $(M_q, \chi_{M_q}) * (N_t, \chi_{N_t})$ is not Θ-isomorphic to $(M_{q'}, \chi_{M_{q'}}) * (N_t, \chi_{N_t})$ or to $(M_q, \chi_{M_q}) * (N_{t'}, \chi_{N_{t'}})$ for any $q \neq q'$ in C_1 and $t \neq t'$ in C_2.

We show that an independent pair of dense chains of Θ-pointed R-modules induces an independent pair of $\mathbb{Q}$-chains in $\mathcal{P}_R^{\Theta}$.

PROPOSITION 3.8. *Assume that $(M\chi_M)$ and (N, χ_N) are Θ-pointed R-modules such that the endomorphism rings $\mathrm{End}_R(M)$ and $\mathrm{End}_R(N)$ are local and $\chi_M, \chi_N \neq 0$.*

(1) *If $(M, \chi_M) \equiv (N, \chi_N)$, then (M, χ_M) and (N, χ_N) are Θ-isomorphic.*
(2) *$\overline{(M, \chi_M)}$ is a primitive element of $\mathcal{P}_R^{\Theta}$.*

Proof. (1) Since $(M, \chi_M) \equiv (N, \chi_N)$, there are pointed homomorphisms $f : (M, \chi_M) \to (N, \chi_N)$ and $g : (N, \chi_N) \to (M, \chi_M)$ such that $f\chi_M = \chi_N$ and $g\chi_N = \chi_M$. Then $\chi_N = f\chi_M = f(g\chi_N) = (fg)\chi_N$ and $\chi_M = g\chi_N = g(f\chi_M) = (gf)\chi_M$, hence $(\mathrm{id}_N - fg)\chi_N = 0$ and $(\mathrm{id}_M - gf)\chi_M = 0$. This implies that $\mathrm{id}_N - fg$ and $\mathrm{id}_M - gf$ are not invertible since $\chi_M, \chi_N \neq 0$. Because the rings $\mathrm{End}_R(M)$ and $\mathrm{End}_R(N)$ are local, we conclude that fg and gf are invertible. It follows that f and g are also invertible and thus (M, χ_M) and (N, χ_N) are Θ-isomorphic.

(2) Assume that $(M, \chi_M) \equiv (S, \chi_S) \oplus (T, \chi_T) = (S \oplus T, \chi_{S\oplus T})$ for some pointed modules (S, χ_S) and (T, χ_T). We show that $(M, \chi_M) \equiv (S, \chi_S)$ or $(M, \chi_M) \equiv (T, \chi_T)$.

Assume that

$$f = \begin{bmatrix} f_S \\ f_T \end{bmatrix} : (M, \chi_M) \to (S \oplus T, \chi_{S \oplus T}),$$

$$g = [g_S \quad g_T] : (S \oplus T, \chi_{S\oplus T}) \to (M, \chi_M)$$

are pointed homomorphisms. Then

$$\begin{bmatrix} f_S \chi_M \\ f_T \chi_M \end{bmatrix} = \begin{bmatrix} f_S \\ f_T \end{bmatrix} \chi_M = \chi_{S\oplus T} = \begin{bmatrix} \chi_S \\ \chi_T \end{bmatrix},$$

$$g_S \chi_S + g_T \chi_T = [g_S \quad g_T] \begin{bmatrix} \chi_S \\ \chi_T \end{bmatrix} = \chi_M,$$

which implies that $g_S f_S \chi_M + g_T f_T \chi_M = \chi_M$. It follows that $(\mathrm{id}_M - (g_S f_S + g_T f_T))\chi_M = 0$ and hence $\mathrm{id}_M - (g_S f_S + g_T f_T)$ is not invertible, because $\chi_M \neq 0$. Since $\mathrm{End}_R(M)$ is local, $g_S f_S + g_T f_T$ is invertible and thus $g_S f_S$ is invertible or $g_T f_T$ is invertible, because the sum of non-invertible elements of a local ring is non-invertible.

Assume that $g_S f_S$ is invertible. Then there exists $\alpha \in \mathrm{End}_R(M)$ such that $\alpha g_S f_S = \mathrm{id}_M$. Hence $\chi_M = \alpha g_S f_S \chi_M = \alpha g_S \chi_S$ and so $\alpha g_S : (S, \chi_S) \to (M, \chi_M)$ is a pointed homomorphism. Since $f_S : (M, \chi_M) \to (S, \chi_S)$ is also a pointed homomorphism, we get $(M, \chi_M) \equiv (S, \chi_S)$. Similarly, $(M, \chi_M) \equiv (T, \chi_T)$ if $g_T f_T$ is invertible. ∎

As an easy conclusion from our previous observations we get the following generalization of [PPT1, Proposition 5.4].

Theorem 3.9.

(1) *Assume* $((M_q, \chi_{M_q})_{q\in C_1}, (N_t, \chi_{N_t})_{t\in C_2})$ *is an independent pair of dense chains of* Θ*-pointed* R*-modules. Then* $(\overline{(M_q, \chi_{M_q})}_{q\in C_1}, \overline{(N_t, \chi_{N_t})}_{t\in C_2})$ *is an independent pair of* $\mathbb{Q}$*-chains in* $\mathcal{P}_R^{\Theta}$.

(2) *Assume that* R *is a countable ring with a unit and* Θ *is a finitely presented* R*-module. If there exists an independent pair* $((M_q, \chi_{M_q})_{q\in C_1}, (N_t, \chi_{N_t})_{t\in C_2})$ *of dense chains of* Θ*-pointed* R*-modules, then there exists a super-decomposable pure-injective* R*-module.*

Proof. (1) is an easy consequence of Proposition 3.8, and (2) follows from (1) and Theorem 3.5. ∎

We shall apply Theorem 3.9 only to the situation when R is a finite-dimensional algebra over a field k and all the modules M_q, N_t are finite-dimensional R-modules. In this case we can replace locality of endomorphism rings from Definitions 3.6 and 3.7 by indecomposability of the corresponding modules. Moreover, if R is a bound quiver k-algebra, then clearly R is countable if and only if k is.

We will make use of the following technical lemma. We omit its (routine) proof.

LEMMA 3.10. *Assume that* $((M_q, \chi_{M_q})_{q\in C_1}, (N_t, \chi_{N_t})_{t\in C_2})$ *is an independent pair of dense chains of* Θ*-pointed* R*-modules,* $\widehat{\Theta}$ *is a finitely presented* R*-module and* $\pi : \Theta \to \widehat{\Theta}$ *is an epimorphism such* $\widehat{\chi}_{M_q}\pi = \chi_{M_q}$, $\widehat{\chi}_{N_t}\pi = \chi_{N_t}$ *for some* R*-homomorphisms* $\widehat{\chi}_{M_q} : \widehat{\Theta} \to M_q$, $\widehat{\chi}_{N_t} : \widehat{\Theta} \to N_t$, *for any* $q \in C_1$ *and* $t \in C_2$. *Then* $((M_q, \widehat{\chi}_{M_q})_{q\in C_1}, (N_t, \widehat{\chi}_{N_t})_{t\in C_2})$ *is an independent pair of dense chains of* $\widehat{\Theta}$*-pointed* R*-modules.* ∎

We are now heading to the proof that fully-faithful right-exact or exact functors preserve, in a certain sense and under some additional assumptions, independent pairs of dense chains of pointed modules and wide posets of pointed modules.

Assume that R, S are rings, $\mathcal{R}$ is a full subcategory of R-mod closed under direct sums, direct summands and isomorphic images, and $F : \mathcal{R} \to S$-mod is a covariant functor. Moreover, let Θ be a finitely presented R-module such that $\Theta \in \mathrm{ob}(\mathcal{R})$ and $F(\Theta)$ is a finitely presented S-module. If (M, χ_M) is a Θ-pointed R-module such that $M \in \mathrm{ob}(\mathcal{R})$, we set $\chi_{F(M)} = F(\chi_M)$ and

$$F(M, \chi_M) := (F(M), F(\chi_M)) = (F(M), \chi_{F(M)}).$$

Then, by our assumptions, $F(M, \chi_M)$ is an $F(\Theta)$-pointed S-module. We recall that if $\Theta = R$, then a Θ-pointed module (M, χ_M) is identified with the pointed module $(M, \chi_M(1))$. Therefore we set $F(M, m) := F(M, \chi_M)$ where $\chi_M : R \to M$ is a homomorphism defined by $\chi_M(1) = m$, for any pointed R-module (M, m).

In this paper we consider right-exact functors and exact functors in the following sense. A functor $F : \mathcal{R} \to S$-Mod is *right-exact* (*exact*, respectively) if for any exact sequence of R-modules $X \to Y \to Z \to 0$ ($0 \to X \to Y \to Z \to 0$, respectively) such that $X, Y, Z \in \mathrm{ob}(\mathcal{R})$, the induced sequence $F(X) \to F(Y) \to F(Z) \to 0$ ($0 \to F(X) \to F(Y) \to F(Z) \to 0$, respectively) of S-modules is exact.

Observe that in general, that is, when $\mathcal{R}$ is not an abelian category, the exactness of the functor F does not imply that F is right-exact.

Let us fix a finitely presented R-module Θ. From now on we consider functors $F : \mathcal{R} \to S$-Mod such that $\Theta \in \mathrm{ob}(\mathcal{R})$ and $F(\Theta)$ is a finitely presented S-module. This implies that $F(M, \chi_M)$ if a well defined $F(\Theta)$-pointed S-module for any Θ-pointed R-module (M, χ_M).

Proposition 3.11. *Assume that* $\mathcal{R}$ *is a full subcategory of* R-mod *closed under direct sums, direct summands and isomorphic images,* $F : \mathcal{R} \to S$-mod *is an additive functor and* (M, χ_M), (N, χ_N) *are* Θ*-pointed* R*-modules such that* $M, N, M * N \in \mathrm{ob}(\mathcal{R})$ *where* $(M, \chi_M) * (N, \chi_N) = (M * N, \chi_{M*N})$.

(1) *If the functor* $F : \mathcal{R} \to S$-mod *is right-exact, then*
$$F((M, \chi_M) * (N, \chi_N)) \cong F(M, \chi_M) * F(N, \chi_N)$$
as Θ*-pointed modules.*

(2) *If the functor* $F : \mathcal{R} \to S$-mod *is exact and the module* Θ *is semisimple, then* $F((M, \chi_M) * (N, \chi_N)) \cong F(M, \chi_M) * F(N, \chi_N)$ *as* Θ*-pointed modules.*

Proof. (1) follows easily from the fact that right-exact functors commute with finite colimits (see for example [Pop] and [Ro]). More precisely, assume that $\alpha : M \to M \oplus N$ and $\beta : N \to M \oplus N$ are defined by $\alpha(m) = (m, 0)$ and $\beta(n) = (0, n)$ for any $m \in M$ and $n \in N$. If $\eta = \alpha\chi_M - \beta\chi_N$, then $M * N \cong \mathrm{Coker}(\eta)$ and thus there exists an exact sequence
$$(*) \qquad \Theta \xrightarrow{\eta} M \oplus N \xrightarrow{\epsilon} M * N \to 0$$
such that $\chi_{M*N} = \epsilon\alpha\chi_M = \epsilon\beta\chi_N$. Since $(*)$ is an exact sequence in $\mathcal{R}$ and the functor F is right-exact, the sequence
$$(**) \qquad F(\Theta) \xrightarrow{F(\eta)} F(M \oplus N) \xrightarrow{F(\epsilon)} F(M * N) \to 0$$
of S-modules is exact. Then the isomorphism $F((M, \chi_M) * (N, \chi_N)) \cong F(M, \chi_M) * F(N, \chi_N)$ follows by standard arguments.

Therefore to prove (2) it is enough to show that the exactness of $(*)$ implies the exactness of $(**)$. Indeed, since Θ is semisimple, the epimorphism $\overline{\eta} : \Theta \to \mathrm{Im}(\eta)$ given by $\overline{\eta}(x) = \eta(x)$ for any $x \in \Theta$ splits. Consequently, $F(\overline{\eta})$ is an epimorphism and $\mathrm{Ker}(F(\epsilon)) = \mathrm{Im}(F(u)) = \mathrm{Im}(F(\eta))$, where u is the canonical embedding of $\mathrm{Im}(\eta)$ into $M \oplus N$. ∎

Lemma 3.12. *Assume that* $\mathcal{R}$ *is a full subcategory of* R-mod, $F : \mathcal{R} \to S$-mod *is a fully-faithful functor and* $(M, \chi_M), (N\chi_N)$ *are* Θ*-pointed* R*-modules such that* $M, N \in \mathrm{ob}(\mathcal{R})$. *Then* $\overline{(M, \chi_M)} \leq \overline{(N, \chi_N)}$ *if and only if* $\overline{F(M, \chi_M)} \leq \overline{F(N, \chi_N)}$.

Proof. This follows easily from the fact that $F : \mathcal{R} \to S$-mod is a fully-faithful functor. ∎

Clearly, under the conditions of the lemma, the equivalence $\overline{(M, \chi_M)} < \overline{(N, \chi_N)} \Leftrightarrow \overline{F(M, \chi_M)} < \overline{F(N, \chi_N)}$ also holds.

Theorem 3.13. *Assume that* $\mathcal{R}$ *is a full subcategory of* R-mod *closed under direct sums, direct summands and isomorphic images,* $\Theta \in \mathrm{ob}(\mathcal{R})$ *is a finitely presented* R*-module and* $F : \mathcal{R} \to S$-mod *is an additive fully-faithful* *right-exact functor or an additive fully-faithful exact functor. If* $F : \mathcal{R} \to S\text{-Mod}$ *is an additive fully-faithful exact functor, then we assume that the module* Θ *is semisimple.*

(1) *If* $((M_q, \chi_{M_q})_{q\in C_1}, (N_t, \chi_{N_t})_{t\in C_2})$ *is an independent pair of dense chains of* Θ*-pointed* R*-modules such that* $\Theta, M_q, N_t, M_q * N_t \in \mathrm{ob}(\mathcal{R})$ *for any* $q \in C_1$ *and* $t \in C_2$*, then* $(F(M_q, \chi_{M_q})_{q\in C_1}, F(N_t, \chi_{N_t})_{t\in C_2})$ *is an independent pair of dense chains of* $F(\Theta)$*-pointed* S*-modules.*

(2) *If* $\{\overline{(M_p, \chi_{M_p})}\}_{p\in P}$ *is a wide poset of* Θ*-pointed modules in* $\mathcal{P}_R^{\Theta}$ *such that* $M_p, M_p * M_q \in \mathrm{ob}(\mathcal{R})$ *for any* $p, q \in P$*, then* $\{\overline{F(M_p, \chi_{M_p})}\}_{p\in P}$ *is a wide poset of* $F(\Theta)$*-pointed modules in* $\mathcal{P}_S^{F(\Theta)}$*.*

Proof. (1) Since F is fully-faithful and additive, we have $\mathrm{End}_S(F(M)) \cong \mathrm{End}_R(M)$ for any R-module M such that $M \in \mathrm{ob}(\mathcal{R})$. So it follows directly from Lemma 3.12 that $F(M_q, \chi_{M_q})_{q\in C_1}$ and $F(N_t, \chi_{N_t})_{t\in C_2}$ are dense chains of $F(\Theta)$-pointed S-modules. Their independence is an easy consequence of Proposition 3.11, Lemma 3.12, Proposition 3.8 and the assumption that F is fully-faithful and additive.

(2) is an easy consequence of Lemma 3.12 and Proposition 3.11. ∎

If the category $\mathcal{R}$ is abelian, in particular if $\mathcal{R} = R$-mod, then any fully-faithful exact functor $F : \mathcal{R} \to S$-Mod is also right-exact and we can simplify the assumptions of Proposition 3.11 and Theorem 3.13. However, in this paper we are interested in the case when $\mathcal{R}$ is a category of prinjective modules. That category is not abelian in general.

Observe that the existence of an independent pair $((M_q, \chi_{M_q})_{q\in C_1}, (N_t, \chi_{N_t})_{t\in C_2})$ of dense chains of Θ-pointed modules in $\mathcal{R}$ such that $M_q * N_t \in \mathrm{ob}(\mathcal{R})$ for any $q \in C_1$ and $t \in C_2$ does not imply the existence of a wide poset $\{\overline{(M_p, \chi_{M_p})}\}_{p\in P}$ of Θ-pointed modules in $\mathcal{P}_R^{\Theta}$ such that $M_p, M_p * M_q \in \mathrm{ob}(\mathcal{R})$ for any $p, q \in P$.

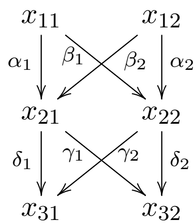
Indeed, we only know by Theorems 3.4 and 3.9 that the lattice

$$Q = \mathrm{Gen}(\overline{(M_q, \chi_{M_q})}_{q\in C_1} \cup \overline{(N_t, \chi_{N_t})}_{t\in C_2})$$

is wide in $\mathcal{P}_R^{\Theta}$, but if $\overline{(K, \chi_K)} \in Q$, then K does not necessarily belong to $\mathrm{ob}(\mathcal{R})$, since $\mathcal{R}$ is not closed under formation of pushouts in general. Moreover, it is clear that the converse implication does not hold either. Hence Theorem 3.13 allows one to prove the existence of a super-decomposable pure-injective S-module in two different situations. Both of them appear in the study of super-decomposable pure-injective modules.

4. Independent pairs of dense chains of pointed modules over the garland $\mathcal{G}_3$ and the diamond $\mathcal{D}$. We recall from [Si2] and [Si3] that the *garland* $\mathcal{G}_3$ *of length* 3 is the bound quiver k-algebra (k-category) kQ'/I'

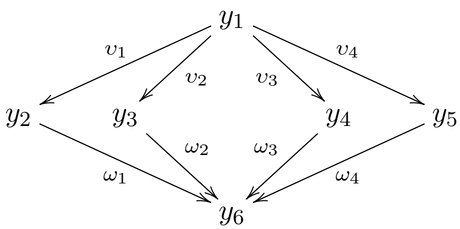
where

$Q' =$ 

and

$$I' = \langle \gamma_1\alpha_1 - \delta_2\beta_1, \delta_1\alpha_1 - \gamma_2\beta_1, \gamma_2\alpha_2 - \delta_1\beta_2, \delta_2\alpha_2 - \gamma_1\beta_2 \rangle,$$

and the *diamond* is the bound quiver k-algebra (k-category) kQ''/I'' where

$Q'' =$ 

and

$$I'' = \langle \omega_1 v_1 + \omega_3 v_3 + \omega_4 v_4, \omega_2 v_2 + \omega_3 v_3 + \omega_4 v_4 \rangle.$$

Let us remark that our definition of the diamond algebra is slightly different from the one in [Si2], but the definitions coincide if the characteristic of k is other than 2.

In this section we present a technical refinement of [KP, Theorem 6.3]; in that result we proved the existence of independent pairs of dense chains of pointed modules over $\mathcal{G}_3$. This refinement allows us to prove the existence of a certain independent pair of dense chains of pointed modules over $\mathcal{D}$ by using Theorem 3.13(1).

We stress that the results of this section are fundamental to our proof of the existence of super-decomposable pure-injective modules for strongly simply connected algebras of non-polynomial growth.

Let us start with the observation that the garland $\mathcal{G}_3$ and the diamond $\mathcal{D}$ can be treated as bipartite algebras of the following form:

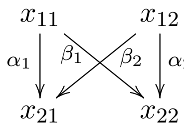
$$\mathcal{G}_3 \cong \begin{bmatrix} \mathcal{A}_1 & 0 \\ \mathcal{M} & \mathcal{B}_1 \end{bmatrix} \quad \text{and} \quad \mathcal{D} \cong \begin{bmatrix} \mathcal{A}_2 & 0 \\ \mathcal{N} & \mathcal{B}_2 \end{bmatrix}$$

where

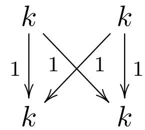
$$\mathcal{A}_1 = (e_{x_{11}} + e_{x_{12}} + e_{x_{21}} + e_{x_{22}})\mathcal{G}_3(e_{x_{11}} + e_{x_{12}} + e_{x_{21}} + e_{x_{22}}),$$
$$\mathcal{B}_1 = (e_{x_{31}} + e_{x_{32}})\mathcal{G}_3(e_{x_{31}} + e_{x_{32}}),$$
$$\mathcal{M} = (e_{x_{31}} + e_{x_{32}})\mathcal{G}_3(e_{x_{11}} + e_{x_{12}} + e_{x_{21}} + e_{x_{22}}),$$

$$\mathcal{A}_2 = e_{y_1}\mathcal{D}e_{y_1},$$
$$\mathcal{B}_2 = (e_{y_2}+e_{y_3}+e_{y_4}+e_{y_5}+e_{y_6})\mathcal{D}(e_{y_2}+e_{y_3}+e_{y_4}+e_{y_5}+e_{y_6}),$$
$$\mathcal{N} = (e_{y_2}+e_{y_3}+e_{y_4}+e_{y_5}+e_{y_6})\mathcal{D}e_{y_1}.$$

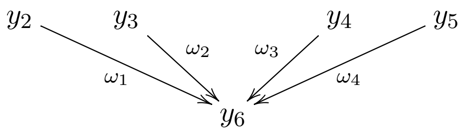
The algebra $\mathcal{A}_1$ is the path algebra of the quiver



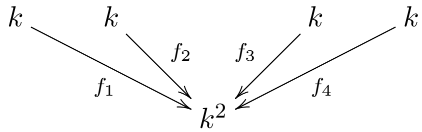
We have $\mathcal{B}_1 \cong k\oplus k$ and the $\mathcal{B}_1$-$\mathcal{A}_1$-bimodule $\mathcal{M}$ is a direct sum of two copies of the right $\mathcal{A}_1$-module $\mathcal{M}'$ where we identify $\mathcal{M}'$ with the representation



The algebra $\mathcal{A}_2$ is isomorphic to k, the algebra $\mathcal{B}_2$ is the path algebra of the quiver



and the $\mathcal{B}_2$-$\mathcal{A}_2$-bimodule $\mathcal{N}$ as a left $\mathcal{B}_2$-module is identified with the representation



where $f_1 = f_2 = \begin{bmatrix}1\\0\end{bmatrix}$, $f_3 = \begin{bmatrix}-1\\-1\end{bmatrix}$ and $f_4 = \begin{bmatrix}0\\1\end{bmatrix}$. Since any left module over a k-algebra is a right k-module in a natural way, we identify the above representation with the bimodule $\mathcal{N}$.

It follows from the above considerations that

$$\mathcal{G}_3 \cong \begin{bmatrix}\mathcal{A}_1 & 0 & 0\\ \mathcal{M}' & k & 0\\ \mathcal{M}' & 0 & k\end{bmatrix} \quad \text{and} \quad \mathcal{D} \cong \begin{bmatrix}k & 0\\ \mathcal{N} & \mathcal{B}_2\end{bmatrix},$$

hence the algebra $\mathcal{G}_3$ is a two-point coextension of the algebra $\mathcal{A}_1$ by the left $\mathcal{A}_1$-module $D\mathcal{M}'$, and the algebra $\mathcal{D}$ is a one-point extension of the algebra $\mathcal{B}_2$ by the left $\mathcal{B}_2$-module $\mathcal{N}$ (see Section 2 for the definitions).

We assume the above notation for the rest of the paper and we set $\operatorname{prin}(\mathcal{G}_3) = \operatorname{prin}_{\mathcal{M}}(\mathcal{G}_3)$ and $\operatorname{prin}(\mathcal{D}) = \operatorname{prin}_{\mathcal{N}}(\mathcal{D})$.

The following theorem is a consequence of [KP, Corollary 7.1(a)] and Lemma 3.10.

THEOREM 4.1. *Assume that* $\mathrm{char}(k) \neq 2$. *There exists a semisimple* $\mathcal{G}_3$-*module* Θ *and an independent pair* $((\widetilde{M}_q, \widehat{\chi}_{\widetilde{M}_q})_{q\in L_1}, (\widetilde{N}_t, \widehat{\chi}_{\widetilde{N}_t})_{t\in L_2})$ *of dense chains of* Θ-*pointed* $\mathcal{G}_3$-*modules such that the modules* Θ, $\widetilde{M}_q$, $\widetilde{N}_t$ *and* $\widetilde{M}_q * \widetilde{N}_t$ *belong to* $\mathrm{prin}(\mathcal{G}_3)$ *for any* $q \in L_1$ *and* $t \in L_2$.

Proof. It follows from [KP, Corollary 7.1(a)] that there exists an independent pair $((\widetilde{M}_q, \widetilde{m}_q)_{q\in L_1}, (\widetilde{N}_t, \widetilde{n}_t)_{t\in L_2})$ of dense chains of pointed modules over $\mathcal{G}_3$ such that $\widetilde{M}_q$ and $\widetilde{N}_t$ belong to $\mathrm{prin}(\mathcal{G}_3)$ for any $q \in L_1$ and $t \in L_2$. The fact that $\widetilde{M}_q * \widetilde{N}_t$ also belongs to $\mathrm{prin}(\mathcal{G}_3)$ is an easy consequence [KP, Corollary 7.1(a)] (see Section 5 of [KP] for details). From the same result we have

$$(*) \qquad \widetilde{m}_q \in (e_{x_{31}} + e_{x_{32}})\widetilde{M}_q \quad \text{and} \quad \widetilde{n}_t \in (e_{x_{31}} + e_{x_{32}})\widetilde{N}_t$$

where $e_{x_{31}}$ and $e_{x_{32}}$ are primitive orthogonal idempotents associated with the vertices x_{31} and x_{32}, respectively, of the quiver Q' of the algebra $\mathcal{G}_3$.

We set $\Theta = P(x_{31}) \oplus P(x_{32})$ where $P(x_{31})$ and $P(x_{32})$ are simple projective $\mathcal{G}_3$-modules associated with the vertices x_{31} and x_{32}, respectively. Hence the module Θ is semisimple and prinjective. Since Θ is a direct summand of $\mathcal{G}_3$ as a $\mathcal{G}_3$-module, there is a canonical epimorphism $\pi : \mathcal{G}_3 \to \Theta$.

Assume $\chi_{\widetilde{M}_q} : \mathcal{G}_3 \to \widetilde{M}_q$ and $\chi_{\widetilde{N}_t} : \mathcal{G}_3 \to \widetilde{N}_t$ are defined by $\chi_{\widetilde{M}_q}(1) = \widetilde{m}_q$ and $\chi_{\widetilde{N}_t}(1) = \widetilde{n}_t$. Then $(*)$ implies that there are $\mathcal{G}_3$-homomorphisms $\widehat{\chi}_{\widetilde{M}_q} : \Theta \to \widetilde{M}_q$ and $\widehat{\chi}_{\widetilde{N}_t} : \Theta \to \widetilde{N}_t$ with $\widehat{\chi}_{\widetilde{M}_q}\pi = \chi_{\widetilde{M}_q}$ and $\widehat{\chi}_{\widetilde{N}_t}\pi = \chi_{\widetilde{N}_t}$.

It follows from Lemma 3.10 that $((\widetilde{M}_q, \widehat{\chi}_{\widetilde{M}_q})_{q\in L_1}, (\widetilde{N}_t, \widehat{\chi}_{\widetilde{N}_t})_{t\in L_2})$ is an independent pair of dense chains of Θ-pointed $\mathcal{G}_3$-modules. ∎

THEOREM 4.2. *Assume that* $\mathrm{char}(k) \neq 2$. *There exists a semisimple* $\mathcal{D}$-*module* Ξ *and an independent pair* $((M'_q, \chi_{M'_q})_{q\in L_1}, (N'_t, \chi_{N'_t})_{t\in L_2})$ *of dense chains of* Ξ-*pointed* $\mathcal{D}$-*modules such that the modules* Ξ, M'_q, N'_t *and* $M'_q * N'_t$ *belong to* $\mathrm{prin}(\mathcal{D})$ *for any* $q \in L_1$ *and* $t \in L_2$.

Proof. We know from Theorem 4.1 that there is an independent pair $((\widetilde{M}_q, \widehat{\chi}_{\widetilde{M}_q})_{q\in L_1}, (\widetilde{N}_t, \widehat{\chi}_{\widetilde{N}_t})_{t\in L_2})$ of dense chains of Θ-pointed $\mathcal{G}_3$-modules where $\Theta = P(x_{31}) \oplus P(x_{32})$ such that the $\mathcal{G}_3$-modules Θ, $\widetilde{M}_q$, $\widetilde{N}_t$ and $\widetilde{M}_q * \widetilde{N}_t$ belong to $\mathrm{prin}(\mathcal{G}_3)$ for any $q \in L_1$, $t \in L_2$.

By [Si2, Theorem 5.2] there exists a fully-faithful exact functor $F : \mathrm{prin}(\mathcal{G}_3) \to \mathrm{prin}(\mathcal{D})$. A detailed analysis of the proof of that theorem yields $F(\Theta) \cong S(y_2) \oplus S(y_3)$ where $S(y_2), S(y_3)$ are simple modules associated with the vertices y_2, y_3, respectively, of the quiver Q'' of the algebra $\mathcal{D}$. For the convenience of the reader we present an explicit description of a functor with these properties in the Appendix.

We set $\varXi = F(\varTheta)$. Since $\mathrm{prin}(\mathcal{G}_3)$ is a full subcategory of $\mathcal{G}_3$-mod closed under direct sums and direct summands, and the module $\varTheta$ is semisimple, we conclude from Theorem 3.13(1) that

$$((M'_q, \chi_{M'_q})_{q\in L_1}, (N'_t, \chi_{N'_t})_{t\in L_2}) := (F(\widetilde{M}_q, \widehat{\chi}_{\widetilde{M}_q})_{q\in L_1}, F(\widetilde{N}_t, \widehat{\chi}_{\widetilde{N}_t})_{t\in L_2})$$

is an independent pair of dense chains of $\varXi$-pointed $\mathcal{D}$-modules.

Obviously $\varXi$, M'_q and N'_t are objects of the category $\mathrm{prin}(\mathcal{D})$ for any $q \in L_1$ and $t \in L_2$. Since $F : \mathrm{prin}(\mathcal{G}_3) \to \mathrm{prin}(\mathcal{D})$ is an exact functor and $\varTheta$ is semisimple, it follows from Proposition 3.11(2) that

$$(M'_q, \chi_{M'_q}) * (N'_t, \chi_{N'_t}) = F(\widetilde{M}_q, \widehat{\chi}_{\widetilde{M}_q}) * F(\widetilde{N}_t, \widehat{\chi}_{\widetilde{N}_t}) \cong F((\widetilde{M}_q, \widehat{\chi}_{\widetilde{M}_q}) * (\widetilde{N}_t, \widehat{\chi}_{\widetilde{N}_t})).$$

This implies that $M'_q * N'_t = F(\widetilde{M}_q) * F(\widetilde{N}_t) \cong F(\widetilde{M}_q * \widetilde{N}_t)$ and hence $M'_q * N'_t$ also belongs to the category $\mathrm{prin}(\mathcal{D})$. ■

5. $\mathcal{M}'$-configurations and $\mathcal{N}$-configurations of modules. Assume that $A[M, G_t]$ is a $\mathbb{D}_{t+2}$-extension of the algebra A by the A-module M and $B[N]$ is a one-point extension of the algebra B by the B-module N (see Section 2 for the definitions).

In this section we show that the existence of some special families of modules, which we call *$\mathcal{M}'$-configurations* and *$\mathcal{N}$-configurations* in A-mod and B-mod, respectively (in line with the notation introduced in Section 4), implies the existence of fully-faithful exact functors $\mathrm{prin}(\mathcal{G}_3) \to A[M, G_t]$-mod and $\mathrm{prin}(\mathcal{D}) \to B[N]$-mod. Our approach is based on ideas of D. Simson as presented in [Si2] and [Si3]. The main result of the section can be considered as a very special case of [Si2, Theorem 3.12] or [Si3, Theorem 3.1].

We start with the following crucial definition of $\mathcal{M}'$-configuration of modules (of type I and type II) and $\mathcal{N}$-configuration of modules. We stress that these are analogues of the *garland of modules* and the *diamond of modules*, respectively, introduced in [Si3] and [Si2].

Recall from Section 4 that

$$\mathcal{G}_3 \cong \begin{bmatrix} \mathcal{A}_1 & 0 & 0 \\ \mathcal{M}' & k & 0 \\ \mathcal{M}' & 0 & k \end{bmatrix} \quad \text{and} \quad \mathcal{D} \cong \begin{bmatrix} k & 0 \\ \mathcal{N} & \mathcal{B}_2 \end{bmatrix}.$$

DEFINITION 5.1. Assume that A and B are k-algebras.

(1) An *$\mathcal{M}'$-configuration of type I* (or *$\mathcal{M}'$-configuration*) in A-mod is a tuple $(X_1, \ldots, X_4, M)$ of A-modules such that $\mathrm{End}_A(X_i) \cong k$ for $i = 1, \ldots, 4$ and there are A-homomorphisms $f_{ij} : X_i \to X_j$ for

$j \leq 2 < i$ and $h_i : M \to X_i$ for $i = 1, \dots, 4$ satisfying the following conditions:

(a) $\operatorname{Hom}_A(X_i, X_j) = \begin{cases} \langle f_{ij} \rangle \cong k & \text{if } j \leq 2 < i, \\ 0 & \text{otherwise,} \end{cases}$

(b) $h_i \neq 0$ for $i = 1, \dots, 4$ and $\operatorname{Hom}_A(M, f_{ij})(h_i) = f_{ij} h_i = h_j$ for $j \leq 2 < i$.

(2) An $\mathcal{M}'$-*configuration of type II* in A-mod is a tuple $(X_1, \dots, X_4, M)$ of A-modules such that $\operatorname{End}_A(X_i) \cong k$ for $i = 1, \dots, 4$ and there are A-homomorphisms $f_{ij} : X_i \to X_j$ for $j \leq 2 < i$ and $h_i : X_i \to M$ for $i = 1, \dots, 4$ satisfying:

(a) $\operatorname{Hom}_A(X_i, X_j) = \begin{cases} \langle f_{ij} \rangle \cong k & \text{if } j \leq 2 < i, \\ 0 & \text{otherwise,} \end{cases}$

(b) $\operatorname{Hom}_A(X_i, M) = \langle h_i \rangle \cong k$ and $\operatorname{Hom}_A(f_{ij}, M)(h_j) = h_j f_{ij} = h_i$ for $j \leq 2 < i$.

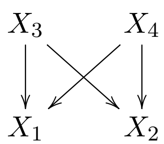
(3) An $\mathcal{N}$-*configuration* in B-mod is a tuple $(Y_1, \dots, Y_5, N)$ of B-modules such that $\operatorname{End}_B(Y_i) \cong k$ for $i = 1, \dots, 5$ and there are B-homomorphisms $g_j : Y_5 \to Y_j$, $h_i : N \to Y_i$ for $i, j = 1, \dots, 4$ and $h_5^1, h_5^2 : N \to Y_5$ satisfying:

(a) $\operatorname{Hom}_B(Y_i, Y_j) = \begin{cases} \langle g_j \rangle \cong k & \text{if } i = 5,\ j = 1, \dots, 4, \\ 0 & \text{otherwise,} \end{cases}$

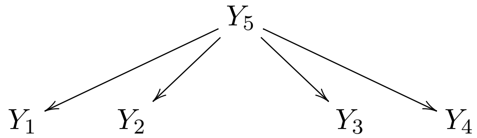
(b) $\operatorname{Hom}_B(N, Y_i) = \langle h_i \rangle \cong k$ for $i = 1, \dots, 4$, $\operatorname{Hom}_B(N, Y_5) = \langle h_5^1, h_5^2 \rangle \cong k^2$ and

$$\begin{aligned}
&\operatorname{Hom}_B(N, g_1)(h_5^1) = g_1 h_5^1 = h_1, && \operatorname{Hom}_B(N, g_1)(h_5^2) = g_1 h_5^2 = 0, \\
&\operatorname{Hom}_B(N, g_2)(h_5^1) = g_2 h_5^1 = h_2, && \operatorname{Hom}_B(N, g_2)(h_5^2) = g_2 h_5^2 = 0, \\
&\operatorname{Hom}_B(N, g_3)(h_5^1) = g_3 h_5^1 = -h_3, && \operatorname{Hom}_B(N, g_3)(h_5^2) = g_3 h_5^2 = -h_3, \\
&\operatorname{Hom}_B(N, g_4)(h_5^1) = g_4 h_5^1 = 0, && \operatorname{Hom}_B(N, g_4)(h_5^2) = g_4 h_5^2 = h_4.
\end{aligned}$$

Observe that the conditions (1)(a) and (2)(a) of the above definition can be visualized in the following way:



where we draw an arrow from X_i to X_j if and only if $\operatorname{Hom}_A(X_i, X_j) \neq 0$. Similarly, the condition (3)(a) can be visualized as



with an arrow from Y_i to Y_j if and only if $\mathrm{Hom}_B(Y_i, Y_j) \neq 0$.

PROPOSITION 5.2. *Assume that A and B are k-algebras.*

(1) *Assume that $(X_1, \ldots, X_4, M)$ is an $\mathcal{M}'$-configuration of type II in* A-mod *and let $X = X_1 \oplus \cdots \oplus X_4$. There exists an isomorphism* $\mathrm{End}_A(X)^{\mathrm{op}} \cong \mathcal{A}_1$ *of algebras and an isomorphism* $D\mathrm{Hom}_A(X, M) \cong \mathcal{M}'$ *of right $\mathcal{A}_1$-modules. Consequently,*

$$\mathcal{G}_3 \cong \begin{bmatrix} \mathrm{End}_A(X)^{\mathrm{op}} & 0 \\ D\mathrm{Hom}_A(X, M) \oplus D\mathrm{Hom}_A(X, M) & k \oplus k \end{bmatrix}.$$

(2) *Assume that $(Y_1, \ldots, Y_5, N)$ is an $\mathcal{N}$-configuration in* B-mod *and let $Y = Y_1 \oplus \cdots \oplus Y_5$. There exists an isomorphism* $\mathrm{End}_B(Y)^{\mathrm{op}} \cong \mathcal{B}_2$ *of algebras and an isomorphism* $D\mathrm{Hom}_B(N, Y) \cong \mathcal{N}$ *of left $\mathcal{B}_2$-modules. Consequently,*

$$\mathcal{D} \cong \begin{bmatrix} k & 0 \\ D\mathrm{Hom}_B(N, Y) & \mathrm{End}_B(Y)^{\mathrm{op}} \end{bmatrix}.$$

Proof. By direct calculations. ■

In the following theorem we show that there exist fully-faithful exact functors $\mathrm{prin}(\mathcal{L}) \to \mathit{\Lambda}$-mod and $\mathrm{prin}(\mathcal{K}) \to \mathit{\Omega}$-mod where $\mathcal{L}, \mathit{\Lambda}$ are certain two-point coextensions and $\mathcal{K}, \mathit{\Omega}$ are certain one-point extensions. This can be considered as a special case of [Si3, Theorem 3.1] or [Si2, Theorem 3.12], adjusted to our purposes.

THEOREM 5.3.

(1) *Assume that A is a k-algebra and X, M are left A-modules. If*

$$\mathit{\Lambda} = \begin{bmatrix} A & 0 \\ DM \oplus DM & k \oplus k \end{bmatrix},$$

$$\mathcal{L} = \begin{bmatrix} \mathrm{End}_A(X)^{\mathrm{op}} & 0 \\ D\mathrm{Hom}_A(X, M) \oplus D\mathrm{Hom}_A(X, M) & k \oplus k \end{bmatrix},$$

then there exists a fully-faithful exact functor $F : \mathrm{prin}(\mathcal{L}) \to \mathit{\Lambda}$-mod. *Thus if there are A-modules $X_1, \ldots, X_4$ such that $(X_1, \ldots, X_4, M)$ is an $\mathcal{M}'$-configuration of type II, then there exists a fully-faithful exact functor* $\mathrm{prin}(\mathcal{G}_3) \to \mathit{\Lambda}$-mod.

(2) *Assume that B is a k-algebra and Y, N are left B-modules. If*

$$\Omega = \begin{bmatrix} k & 0 \\ N & B \end{bmatrix}, \qquad \mathcal{K} = \begin{bmatrix} k & 0 \\ D\mathrm{Hom}_B(N,Y) & \mathrm{End}_B(Y)^{\mathrm{op}} \end{bmatrix},$$

then there exists a fully-faithful exact functor $G : \mathrm{prin}(\mathcal{K}) \to \Omega\text{-mod}$. Thus if there are B-modules $Y_1, \ldots, Y_5$ such that $(Y_1, \ldots, Y_5, N)$ is an $\mathcal{N}$-configuration, then there exists a fully-faithful exact functor $\mathrm{prin}(\mathcal{D}) \to \Omega\text{-mod}$.

Proof. The assertions follow from [Si3, Theorem 3.1] applied to suitably chosen categories $\mathbb{K}$ and $\mathbb{L}$ (we refer to the notation from [Si3]) and from Proposition 5.2. For example, to obtain (1) we set $\mathbb{K}$ (resp. $\mathbb{L}$) to be the smallest additive category containing the Λ-modules $(X_i, 0, 0)$, $i = 1, \ldots, 4$ (resp. the simple projective Λ-modules annihilated by A). The case of (2) is similar. ∎

Assume that A is an algebra, N is a left A-module, $t \geq 1$ and L_t is a path algebra of the canonically oriented Dynkin quiver $\Delta(\mathbb{A}_t)$. We recall from Section 2 that the t-linear extension of A by N is the algebra

$$A[N, L_t] = \begin{bmatrix} L_t & 0 \\ \overline{N} & A \end{bmatrix}$$

where $\overline{N} = [N \quad 0 \quad \ldots \quad 0]$ is an A-L_t-bimodule. Let

$$A[N, G_t] = \begin{bmatrix} A[N, L_t] & 0 \\ D\widehat{\mathcal{L}} \oplus D\widehat{\mathcal{L}} & k \oplus k \end{bmatrix}$$

be the $\mathbb{D}_{t+2}$-extension of A by N. Here $\widehat{\mathcal{L}}$ denotes the $A[N, L_t]$-module corresponding to the triple $(\mathcal{L}, 0, 0)$ where

$$\mathcal{L} = \; k \xrightarrow{1} k \xrightarrow{1} \cdots \xrightarrow{1} k.$$

Lemma 5.4. *Assume that A is an algebra, M is a left A-module and $(X_1, \ldots, X_4, M)$ is an $\mathcal{M}'$-configuration of type I in A-mod. There exist $A[M, L_t]$-modules $\widetilde{X}_1, \ldots, \widetilde{X}_4$ such that $(\widetilde{X}_1, \ldots, \widetilde{X}_4, \widehat{\mathcal{L}})$ is an $\mathcal{M}'$-configuration of type II in $A[M, L_t]$-mod. Hence there exists a fully-faithful exact functor $\mathrm{prin}(\mathcal{G}_3) \to A[M, G_t]\text{-mod}$.*

Proof. Since $(X_1, \ldots, X_4, M)$ is an $\mathcal{M}'$-configuration of type I, we have $\mathrm{End}_A(X_i) \cong k$ for $i = 1, \ldots, 4$ and there are A-homomorphisms $f_{ij} : X_i \to X_j$ for $j \leq 2 < i$ and $h_i : M \to X_i$, $h_i \neq 0$, for $i = 1, \ldots, 4$ such that

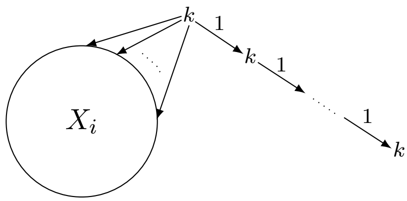
$$\mathrm{Hom}_A(X_i, X_j) = \begin{cases} \langle f_{ij} \rangle \cong k & \text{if } j \leq 2 < i, \\ 0 & \text{otherwise,} \end{cases}$$

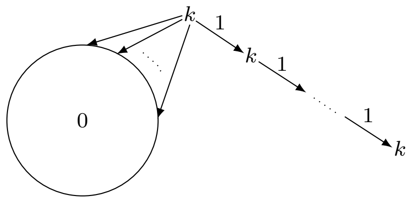
and $\mathrm{Hom}_A(M, f_{ij})(h_i) = f_{ij} h_i = h_j$ for $j \leq 2 < i$. Note that the L_t-

module $\mathrm{Hom}_A(\overline{M}, X_i)$ corresponds to the representation $\mathrm{Hom}_A(M, X_i) \to 0 \to \cdots \to 0$ of the quiver $\Delta(\mathbb{A}_t)$. Let $\varphi_i : \mathcal{L} \to \mathrm{Hom}_A(\overline{M}, X_i)$ be an L_t-homomorphism determined by the vector space homomorphism $k \to \mathrm{Hom}_A(M, X_i)$ such that $1 \mapsto h_i$. We set

$$\widetilde{X}_i = (\mathcal{L}, X_i, \varphi_i : \mathcal{L} \to \mathrm{Hom}_A(\overline{M}, X_i)).$$

We can visualize the representations corresponding to the modules $\widetilde{X}_i$ and $\widehat{\mathcal{L}}$ in the following way:

$\widetilde{X}_i =$ 

$\widehat{\mathcal{L}} =$ 

We set $\widetilde{f}_{ij} = (\mathrm{id}_{\mathcal{L}}, f_{ij}) : \widetilde{X}_i \to \widetilde{X}_j$, $\widetilde{h}_i = (\mathrm{id}_{\mathcal{L}}, 0) : \widetilde{X}_i \to \widehat{\mathcal{L}}$. It is easy to see that these are $A[M, L_t]$-module homomorphisms. Since $\mathrm{End}_L(\mathcal{L}) \cong k$, we get $\mathrm{Hom}_{A[M,L_t]}(\widetilde{X}_i, \widehat{\mathcal{L}}) = \langle \widetilde{h}_i \rangle \cong k$. Obviously $\mathrm{Hom}_{A[M,L_t]}(\widetilde{f}_{ij}, \widehat{\mathcal{L}})(\widetilde{h}_j) = \widetilde{h}_j \widetilde{f}_{ij} = \widetilde{h}_i$.

We show that $\mathrm{Hom}_{A[M,L_t]}(\widetilde{X}_i, \widetilde{X}_j) \cong \langle \widetilde{f}_{ij} \rangle$ for $j \leq 2 < i$. Indeed, if $0 \neq g \in \mathrm{Hom}_{A[M,L_t]}(\widetilde{X}_i, \widetilde{X}_j)$, then $g = (\lambda\, \mathrm{id}_{\mathcal{L}}, \mu f_{ij})$ for some $\lambda, \mu \in k$, because $\mathrm{End}_L(\mathcal{L}) \cong k$ and $\mathrm{Hom}_A(X_i, X_j) \cong \langle f_{ij} \rangle$. Hence $\lambda h_j = \mathrm{Hom}_A(M, \mu f_{ij})(h_i) = \mu f_{ij} h_i = \mu h_j$, so $\lambda = \mu$ since $h_j \neq 0$. This implies that $g = \lambda \widetilde{f}_{ij} \in \langle \widetilde{f}_{ij} \rangle$. Similarly, $\mathrm{End}_{A[M,L_t]}(\widetilde{X}_i) \cong k$ and $\mathrm{Hom}_{A[M,L_t]}(\widetilde{X}_i, \widetilde{X}_j) = 0$ for $i \neq j$ such that the condition $j \leq 2 < i$ is not satisfied.

The above arguments show that the tuple $(\widetilde{X}_1, \ldots, \widetilde{X}_4, \widehat{\mathcal{L}})$ is an $\mathcal{M}'$-configuration of type II in $A[M, L_t]$-mod and hence Theorem 5.3(1) yields a fully-faithful exact functor $\mathrm{prin}(\mathcal{G}_3) \to A[M, G_t]$-mod. ■

We summarize our considerations from this section in the following theorem. From now on we abbreviate the term *$\mathcal{M}'$-configuration of type I* to *$\mathcal{M}'$-configuration.*

Theorem 5.5.

(1) *Assume that A is an algebra and M is a left A-module. If there exists an $\mathcal{M}'$-configuration $(X_1, \ldots, X_4, M)$ in* A-mod *for some A-modules*

$X_1, \ldots, X_4$, *then there exists a fully-faithful exact functor*

$$\operatorname{prin}(\mathcal{G}_3) \to A[M, G_t]\text{-mod}.$$

(2) *Assume that B is an algebra and N is a left B-module. If there exists an $\mathcal{N}$-configuration $(Y_1, \ldots, Y_5, N)$ in B-*mod *for some B-modules $Y_1, \ldots, Y_5$, then there exists a fully-faithful exact functor*

$$\operatorname{prin}(\mathcal{D}) \to B[N]\text{-mod}.$$

Proof. (1) follows from Lemma 5.4, and (2) from Theorem 5.3(2). ∎

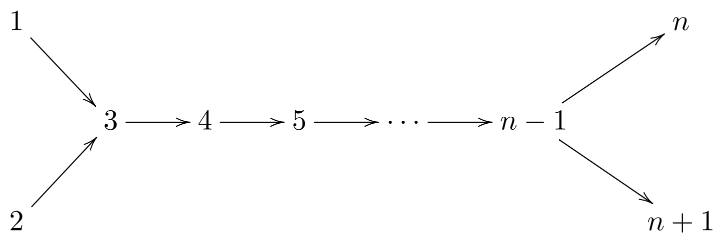
6. $\mathcal{M}'$-configurations and $\mathcal{N}$-configurations in categories of modules over pg-critical algebras. Assume that $\Lambda = B[M, G_t]$ is a pg-critical algebra of type I and $\Omega = B[N]$ is a pg-critical algebra of type II (see Section 2 for the definitions). This section is devoted to showing that there exist B-modules $X_1, \ldots, X_4$ and $Y_1, \ldots, Y_5$ such that $(X_1, \ldots, X_4, M)$ is an $\mathcal{M}'$-configuration in B-mod and $(Y_1, \ldots, Y_5, N)$ is an $\mathcal{N}$-configuration in B-mod. Therefore Theorem 5.5 will imply that there exist fully-faithful exact functors $\operatorname{prin}(\mathcal{G}_3) \to \Lambda$-mod and $\operatorname{prin}(\mathcal{D}) \to \Omega$-mod.

The proof of the existence of the above configurations in appropriate module categories is rather long and divided into two main parts. In the first part, which consists of Lemmas 6.1 and 6.2, we prove the existence of such configurations when B is a fixed hereditary algebra of type $\widetilde{\mathbb{D}}_n$, namely the path algebra of a canonically oriented Euclidean quiver of type $\widetilde{\mathbb{D}}_n$. In the second part we apply tilting theory (see Lemmas 6.4 and 6.5) to generalize the previous result to an arbitrary tilted algebra $B = \operatorname{End}_H(T)^{\mathrm{op}}$ where H is a hereditary algebra of type $\widetilde{\mathbb{D}}_n$ and T is a tilting H-module without prinjective direct summands (see Theorems 6.6 and 6.7).

This section follows the idea of the proof that pg-critical algebras are of non-polynomial growth (see [Si3, Theorem 2.2]). In order to make the paper self-contained, we present suitable configurations of modules explicitly, in full detail.

We start by introducing the notation. Assume that $n \geq 4$ and $\widetilde{H}_n$ is the path algebra of the canonically oriented Euclidean quiver of type $\widetilde{\mathbb{D}}_n$, that is, $\widetilde{H}_n = k\Delta(\widetilde{\mathbb{D}}_n)$ where

$\Delta(\widetilde{\mathbb{D}}_n) =$



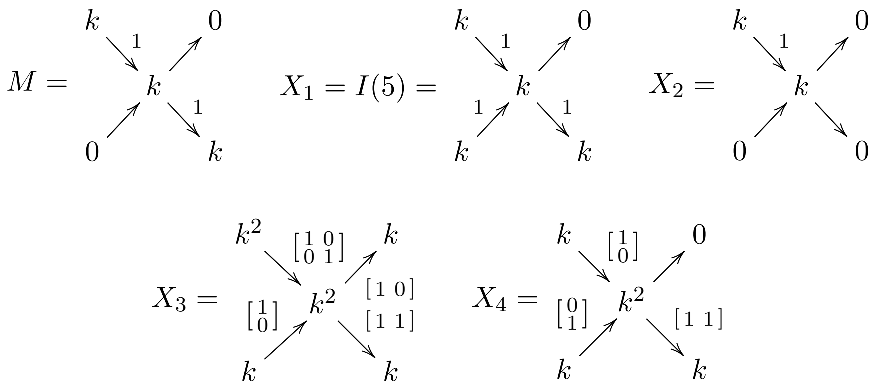
Assume $X, Y \in \widetilde{H}_n$-mod and $\phi \in \mathrm{Hom}_{\widetilde{H}_n}(X, Y)$. In what follows we identify X and Y with representations of the quiver $\Delta(\widetilde{\mathbb{D}}_n)$. The homomorphism $\phi : X \to Y$ is identified with the sequence $(\phi_1, \ldots, \phi_{n+1})$ of linear homomorphisms $\phi_i : e_i X \to e_i Y$, $\phi_i = \phi|_{e_i X}$ for $i = 1, \ldots, n+1$, where $e_1, \ldots, e_{n+1}$ are primitive idempotents of the algebra $\widetilde{H}_n$ associated with the vertices of $\Delta(\widetilde{\mathbb{D}}_n)$. If $e_i X \cong k^{s_i}$ and $e_i Y \cong k^{t_i}$ for some natural numbers s_i, t_i, then we identify the homomorphism ϕ_i with its matrix in appropriate standard bases.

Recall from [SS1, XII, Theorem 2.1] that the algebra $\widetilde{H}_n$ is of tubular type $(2, 2, n-2)$. Hence the regular part of its Auslander–Reiten quiver $\Gamma_{\widetilde{H}_n}$ consists of two tubes $\mathcal{T}_0^{\Delta(\widetilde{\mathbb{D}}_n)}$ and $\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{D}}_n)}$ of rank 2, one tube $\mathcal{T}_1^{\Delta(\widetilde{\mathbb{D}}_n)}$ of rank $n-2$, and the remaining tubes in $\Gamma_{\Delta(\widetilde{\mathbb{D}}_n)}$ are homogeneous.

In the following lemma we show that for any tube $\mathcal{T}$ of rank $n-2$ in $\Gamma_{\widetilde{H}_n}$ there is a regular $\widetilde{H}_n$-module $M \in \mathcal{T}$ of regular length 1 and preinjective $\widetilde{H}_n$-modules $X_1, \ldots, X_4$ such that $(X_1, \ldots, X_4, M)$ is an $\mathcal{M}'$-configuration. Note that for $n = 4$ we have three tubes of rank $n - 2 = 2$ and for $n \geq 5$ we have one tube of rank $n - 2 \geq 3$.

Lemma 6.1.

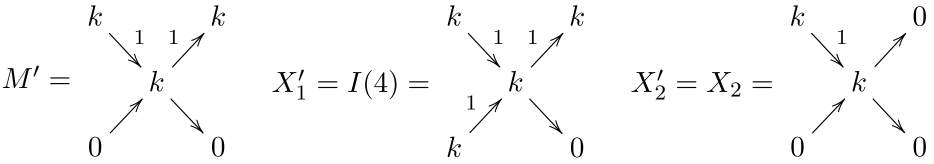
(1) *Assume $n = 4$. The tuple $(X_1, \ldots, X_4, M)$ of $\widetilde{H}_n$-modules is an $\mathcal{M}'$-configuration, where*



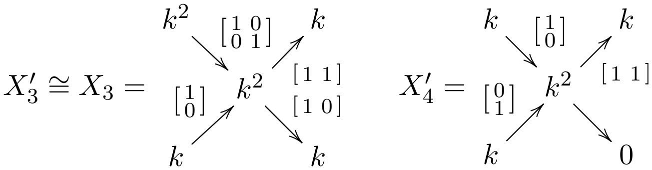
Moreover, the module M is regular of regular length 1 and lies in the tube $\mathcal{T}_\infty^{\Delta(\widetilde{\mathbb{D}}_n)}$ of rank $n - 2 = 2$ in the quiver $\Gamma_{\widetilde{H}_n}$, and the modules $X_1, \ldots, X_4$ are preinjective.

(2) *Assume $n = 4$. The tuple $(X'_1, \ldots, X'_4, M')$ of $\widetilde{H}_n$-modules is an $\mathcal{M}'$-configuration, where*

$M' =$, $X_1' = I(4) =$, $X_2' = X_2 =$



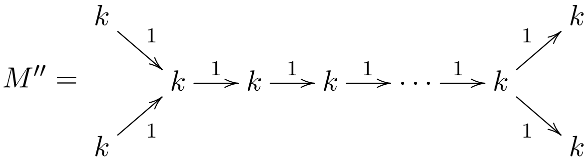
$X_3' \cong X_3 =$, $X_4' =$



Moreover, the module M' is regular of regular length 1 *and lies in the tube $\mathcal{T}_0^{\Delta(\widetilde{\mathbb{D}}_n)}$ of rank $n-2=2$ in $\Gamma_{\widetilde{H}_n}$, and the modules $X_1', \ldots, X_4'$ are preinjective.*

(3) *Assume that $n > 4$. The tuple $(X_1'', \ldots, X_4'', M'')$ of $\widetilde{H}_n$-modules is an $\mathcal{M}'$-configuration, where*

$M'' =$



and $X_1'' = I(1)$, $X_2'' = I(2)$, $X_3'' = I(n-1)$, $X_4'' = I(n)$. Moreover, the module M'' is regular of regular length 1 *and lies in the tube $\mathcal{T}_1^{\Delta(\widetilde{\mathbb{D}}_n)}$ of rank $n-2 \geq 3$ in $\Gamma_{\widetilde{H}_n}$, and the modules $X_1'', \ldots, X_4''$ are preinjective, being injective.*

Proof. (1) It is easy to check by direct calculations that $\operatorname{End}_{\widetilde{H}_n}(X_i) \cong k$ for $i = 1, \ldots, 4$ and $\operatorname{Hom}_{\widetilde{H}_n}(X_1, X_i) = 0$, $\operatorname{Hom}_{\widetilde{H}_n}(X_2, X_j) = 0$ for $i = 2, 3, 4$, $j = 1, 3, 4$.

Observe that $\operatorname{Hom}_{\widetilde{H}_n}(X_i, X_j) = \langle f_{ij} \rangle \cong k$ for $j \leq 2 < i$ where

$$f_{31} = ([\,1\ 1\,], 1, [\,1\ 1\,], 0, 1), \quad f_{32} = ([\,0\ 1\,], 0, [\,0\ 1\,], 0, 0),$$
$$f_{41} = (1, 1, [\,1\ 1\,], 0, 1), \quad f_{42} = (1, 0, [\,1\ 0\,], 0, 0),$$

and $\operatorname{Hom}_{\widetilde{H}_n}(M, X_i) = \langle h_i \rangle \cong k$ for $i = 1, \ldots, 4$ where

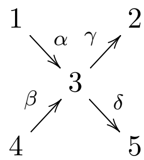
$$h_1 = (1, 0, 1, 0, 1), \quad h_2 = (1, 0, 1, 0, 0),$$
$$h_3 = \left(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], 0, \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], 0, 1\right), \quad h_4 = \left(1, 0, \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], 0, 1\right).$$

Hence $\operatorname{Hom}_{\widetilde{H}_n}(M, f_{ij})(h_i) = f_{ij} h_i = h_j$, and $(X_1, \ldots, X_4, M)$ is an $\mathcal{M}'$-configuration in $\widetilde{H}_n$-mod.

The module M is regular of regular length 1 and lies in the tube $\mathcal{T}_{\infty}^{\Delta(\widetilde{\mathbb{D}}_n)}$ of rank $n-2=2$ in $\Gamma_{\widetilde{H}_n}$ by [SS1, XIII, Table 2.6].

The modules $X_1,\dots,X_4$ are preinjective, because $\partial_{\widetilde{H}_n}(\underline{\dim}(X_i)) = x_1^i + x_2^i - x_4^i - x_5^i > 0$ where $\partial_{\widetilde{H}_n}$ denotes the defect of the algebra $\widetilde{H}_n$ and $\underline{\dim}(X_i) = (x_1^i,\dots,x_5^i)$ is the dimension vector of X_i, for $i=1,\dots,4$ (see [SS1, XI, XIII, Lemma 1.3]).

(2) Name the arrows in the quiver $\Delta(\widetilde{\mathbb{D}}_4)$ in the following way:

$\Delta(\widetilde{\mathbb{D}}_4) =$ 

The homomorphism $\eta : \widetilde{H}_4 \to \widetilde{H}_4$ of algebras (or k-categories) defined by $\eta(\alpha)=\alpha$, $\eta(\beta)=\beta$, $\eta(\gamma)=\delta$, $\eta(\delta)=\gamma$ is an isomorphism. This isomorphism induces a k-category isomorphism $\eta_\bullet : \widetilde{H}_4\text{-mod} \to \widetilde{H}_4\text{-mod}$. Since $\eta_\bullet(M) = M'$ and $\eta_\bullet(X_i) = X_i'$ for $i=1,\dots,4$, the tuple $(X_1',\dots,X_4',M')$ is an $\mathcal{M}'$-configuration in $\widetilde{H}_4$-mod by (1).

The module M' is regular of regular length 1 and lies in the tube $\mathcal{T}_{0}^{\Delta(\widetilde{\mathbb{D}}_n)}$ of rank $n-2=2$ in $\Gamma_{\widetilde{H}_n}$ by [SS1, XIII, Table 2.6].

The modules $X_1',\dots,X_4'$ are preinjective by (1) and the fact that $\eta_\bullet : \widetilde{H}_4\text{-mod} \to \widetilde{H}_4\text{-mod}$ is an isomorphism.

(3) Since all the modules X_i'' are indecomposable injective, it is easy to check that

$$\operatorname{Hom}_{\widetilde{H}_n}(X_i'', X_j'') = \begin{cases} k & \text{if } j \le 2 < i, \\ 0 & \text{otherwise,} \end{cases}$$

and $\operatorname{Hom}_{\widetilde{H}_n}(M'', X_i'') \cong k$ for $i=1,\dots,4$.

Let $f_{31} = (1,0,\dots,0)$, $f_{41} = (1,0,\dots,0)$, $f_{32} = (0,1,0,\dots,0)$, $f_{42} = (0,1,0,\dots,0)$ and observe that $f_{ij} \in \operatorname{Hom}_{\widetilde{H}_n}(X_i'', X_j'')$ for $i=3,4$ and $j=1,2$.

Let $h_1 = (1,0,\dots,0)$, $h_2 = (0,1,0,\dots,0)$, $h_3 = (1,\dots,1,0)$, $h_4 = (1,\dots,1,0,1)$ and observe that $h_i \in \operatorname{Hom}_{\widetilde{H}_n}(M'', X_i'')$ for $i=1,\dots,4$.

It follows easily that $\operatorname{Hom}_{\widetilde{H}_n}(M'', f_{ij})(h_i) = f_{ij}h_i = h_j$ and hence the tuple $(X_1'',\dots,X_4'',M'')$ is an $\mathcal{M}'$-configuration in $\widetilde{H}_n$-mod.

The module M'' is regular of regular length 1 and lies in the tube $\mathcal{T}_{1}^{\Delta(\widetilde{\mathbb{D}}_n)}$ of rank $n-2 \ge 2$ in $\Gamma_{\widetilde{H}_n}$ by [SS1, XIII, Table 2.6].

The modules $X_1'',\dots,X_4''$ are preinjective since they are injective. ∎

In the following lemma we show that for any tube $\mathcal{T}$ of rank $n-2$ in $\Gamma_{\widetilde{H}_n}$ there is a regular $\widetilde{H}_n$-module $N \in \mathcal{T}$ of regular length 2 and preinjective $\widetilde{H}_n$-modules $Y_1, \dots, Y_5$ such that $(Y_1, \dots, Y_5, N)$ is an $\mathcal{N}$-configuration.

LEMMA 6.2.

(1) *Assume that $H = kQ$ is a hereditary algebra of type $\widetilde{\mathbb{D}}_4$, where*

$$Q = \begin{array}{ccccccc} 1 & & 2 & & 3 & & 4 \\ & \searrow & \searrow & & \swarrow & \swarrow & \\ & & & 5 & & & \end{array}$$

The tuple $(Y_1, \dots, Y_5, N)$ of H-modules is an $\mathcal{N}$-configuration, where

$$N = \begin{array}{ccccccc} k & & k & & k & & k \\ & \overset{f_1}{\searrow} & \overset{f_2}{\searrow} & & \overset{f_3}{\swarrow} & \overset{f_4}{\swarrow} & \\ & & & k^2 & & & \end{array}$$

for $f_1 = f_2 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $f_3 = \left[\begin{smallmatrix}-1\\-1\end{smallmatrix}\right]$, $f_4 = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ and $Y_1 = I(1), \dots, Y_5 = I(5)$. Moreover, the module N is regular of regular length 2 and lies in the tube $\mathcal{T}_1^Q$ of rank 2 in the quiver Γ_H.

(2) *Assume that*

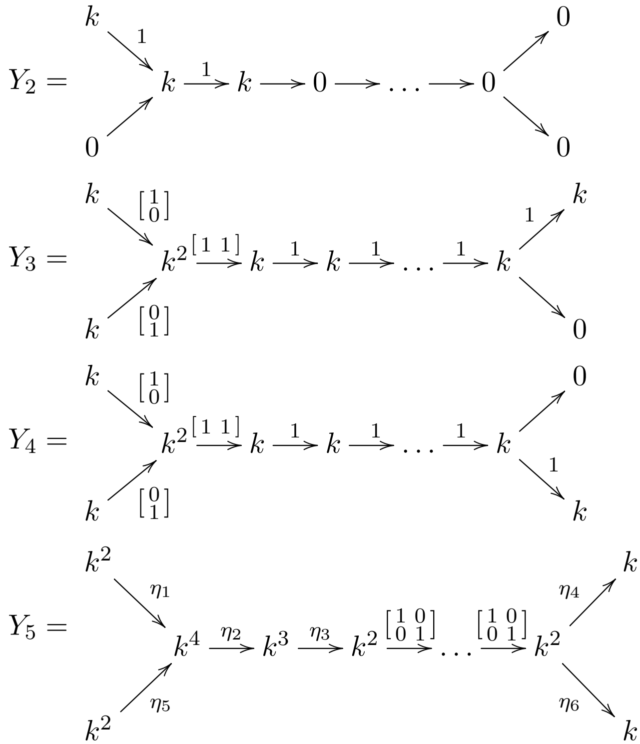
$$N' = \begin{array}{ccccccc} k & & k & & k & & k \\ & \overset{g_1}{\searrow} & \overset{g_2}{\searrow} & & \overset{g_3}{\swarrow} & \overset{g_4}{\swarrow} & \\ & & & k^2 & & & \end{array} \quad \text{and} \quad N'' = \begin{array}{ccccccc} k & & k & & k & & k \\ & \overset{h_1}{\searrow} & \overset{h_2}{\searrow} & & \overset{h_3}{\swarrow} & \overset{h_4}{\swarrow} & \\ & & & k^2 & & & \end{array}$$

for $g_1 = g_4 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $g_2 = \left[\begin{smallmatrix}-1\\-1\end{smallmatrix}\right]$, $g_3 = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ and $h_2 = h_4 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $h_1 = \left[\begin{smallmatrix}-1\\-1\end{smallmatrix}\right]$, $h_3 = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. There exist H-modules $Y_1', \dots, Y_5'$ and $Y_1'', \dots, Y_5''$ such that the tuples $(Y_1', \dots, Y_5', N')$ and $(Y_1'', \dots, Y_5'', N'')$ are $\mathcal{N}$-configurations. Moreover, the modules N', N'' are regular of regular length 2 and lie in the tubes $\mathcal{T}_0^Q$, $\mathcal{T}_\infty^Q$, respectively, of rank 2 in Γ_H, and the modules $Y_1', \dots, Y_5'$ and $Y_1'', \dots, Y_5''$ are preinjective.

(3) *Assume that $n \geq 5$ and $\widetilde{H}_n = k\Delta(\widetilde{\mathbb{D}}_n)$. The tuple $(Y_1, \dots, Y_5, N)$ of $\widetilde{H}_n$-modules is an $\mathcal{N}$-configuration, where*

$$N = \begin{array}{ccccccccccc} 0 & & & & & & & & & & 0 \\ & \searrow & & & & & & & & \nearrow & \\ & & k & \xrightarrow{1} & k & \longrightarrow & 0 & \longrightarrow \dots \longrightarrow & 0 & & \\ & \nearrow & & & & & & & & \searrow & \\ 0 & & & & & & & & & & 0 \end{array}$$

$$Y_1 = \begin{array}{ccccccccccc} 0 & & & & & & & & & & 0 \\ & \searrow & & & & & & & & \nearrow & \\ & & k & \xrightarrow{1} & k & \longrightarrow & 0 & \longrightarrow \dots \longrightarrow & 0 & & \\ & \overset{1}{\nearrow} & & & & & & & & \searrow & \\ k & & & & & & & & & & 0 \end{array}$$

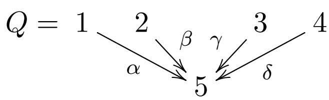


for $\eta_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\eta_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $\eta_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $\eta_4 = [\,1\ 0\,]$, $\eta_5 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\eta_6 = [\,0\ 1\,]$. *(In the case* $n = 5$, *i.e.* $\widetilde{H}_n = \widetilde{H}_5$, *we set* $\eta_4 = [\,1\ 0\ 0\,]$ *and* $\eta_6 = [\,0\ 1\ 0\,]$.*) The module* N *is regular of regular length* 2 *and lies in the tube* $\mathcal{T}_1^{\Delta(\widetilde{\mathbb{D}}_n)}$ *of rank* $n-2$ *in* $\Gamma_{\widetilde{H}_n}$, *and the modules* $Y_1, \ldots, Y_5$ *are preinjective.*

Proof. (1) It is easy to check that $(Y_1, \ldots, Y_5, N)$ is an $\mathcal{N}$-configuration. Indeed, the modules $Y_1, \ldots, Y_5$ are indecomposable injective and hence the description of $\mathrm{End}_H(Y_i)$, $\mathrm{Hom}_H(Y_i, Y_j)$ and $\mathrm{Hom}_H(N, X_i)$ for $i, j = 1, \ldots, 5$ is immediate.

The module N is regular of regular length 2 and lies in the tube $\mathcal{T}_1^{Q}$ of rank 2 in Γ_H by [SS1, XIII, Table 3.12].

(2) Assume that



and let $H = kQ$. The homomorphisms $\omega, \xi : H \to H$ of algebras defined by $\omega(\alpha) = \alpha$, $\omega(\beta) = \delta$, $\omega(\gamma) = \beta$, $\omega(\delta) = \gamma$, $\xi(\alpha) = \beta$, $\xi(\beta) = \delta$,

$\xi(\gamma) = \alpha$, $\xi(\delta) = \gamma$ are isomorphisms. They induce k-category isomorphisms $\omega_\bullet, \xi_\bullet : H\text{-mod} \to H\text{-mod}$. Hence $(\omega_\bullet(Y_1), \dots, \omega_\bullet(Y_5), \omega_\bullet(N) = N')$ and $(\xi_\bullet(Y_1), \dots, \xi_\bullet(Y_5), \xi_\bullet(N) = N'')$ are $\mathcal{N}$-configurations and the modules N', N'' are regular of regular length 2 by (1). Moreover, we have $N' \cong R_1^{(0)}[2]$ and $N'' \cong R_1^{(\infty)}[2]$ (see [SS1, XIII, Table 3.12]), so these modules lie in the tubes $\mathcal{T}_0^Q$, $\mathcal{T}_\infty^Q$, respectively, of rank 2 in Γ_H. The modules $\omega_\bullet(Y_i)$ and $\xi_\bullet(Y_i)$ are obviously injective for $i = 1, \dots, 5$.

(3) It follows by direct calculations that $\mathrm{End}_{\widetilde{H}_n}(Y_i) \cong k$ for $i = 1, \dots, 5$.

From the location of the modules Y_i in the preinjective component of the quiver $\Gamma_{\widetilde{H}_n}$ it is easy to see that $\mathrm{Hom}_{\widetilde{H}_n}(Y_i, Y_j) = 0$ for $i \neq j$, $i \leq 4$, $j \leq 5$. This can also be checked by direct calculations.

Let

$$\begin{aligned} f_1 &= (0, [1\ 0], [1\ 0\ 0\ 0], [1\ 1\ {-1}], 0, \dots, 0), \\ f_2 &= ([1\ 0], 0, [0\ 0\ 1\ 0], [0\ 0\ 1], 0, \dots, 0), \\ f_3 &= \Big([1\ 1], [1\ 1], \left[\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{smallmatrix}\right], [1\ 0\ 0], [1\ 0], \dots, [1\ 0], 1, 0\Big), \\ f_4 &= \Big([0\ {-1}], [0\ {-1}], \left[\begin{smallmatrix} 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{smallmatrix}\right], [0\ 1\ 0], [0\ 1], \dots, [0\ 1], 0, 1\Big) \end{aligned}$$

and observe that $f_i \in \mathrm{Hom}_{\widetilde{H}_n}(Y_5, Y_i)$ for $i = 1, \dots, 4$. Direct calculations show that $\mathrm{Hom}_{\widetilde{H}_n}(Y_5, Y_i) = \langle f_i \rangle \cong k$ for $i = 1, \dots, 4$.

Let

$$\begin{aligned} g_1 &= (0, 0, 1, 1, 0, \dots, 0), & g_2 &= (0, 0, -1, -1, 0, \dots, 0), \\ g_3 &= (0, 0, \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right], 0, \dots, 0), & g_4 &= (0, 0, \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right], 0, \dots, 0), \\ g_5^1 &= \left(0, 0, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, 0, \dots, 0\right), & g_5^2 &= \left(0, 0, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, 0, \dots, 0\right), \end{aligned}$$

and observe $g_i \in \mathrm{Hom}_{\widetilde{H}_n}(N, Y_i)$ for $i = 1, \dots, 4$ and $g_5^1, g_5^2 \in \mathrm{Hom}_{\widetilde{H}_n}(N, Y_5)$. Direct calculations show that $\mathrm{Hom}_{\widetilde{H}_n}(N, Y_i) = \langle g_i \rangle \cong k$ for $i = 1, \dots, 4$ and $\mathrm{Hom}_{\widetilde{H}_n}(N, Y_5) = \langle g_5^1, g_5^2 \rangle \cong k^2$.

Note that $f_1 g_5^1 = g_1$, $f_1 g_5^2 = 0$; $f_2 g_5^1 = g_2$, $f_2 g_5^2 = 0$; $f_3 g_5^1 = -g_3$, $f_3 g_5^2 = -g_3$; $f_4 g_5^1 = 0$, $f_4 g_5^2 = g_4$. Hence $(Y_1, \dots, Y_5, N)$ is an $\mathcal{N}$-configuration.

The modules $Y_1, \dots, Y_5$ are preinjective by [SS1, XIII, Lemma 1.3] since $\partial_{\widetilde{H}}(\underline{\dim}(Y_i)) = y_1^i + y_2^i - y_n^i - y_{n+1}^i > 0$ where $\underline{\dim}(Y_i) = (y_1^i, \dots, y_{n+1}^i)$ for $i = 1, \dots, 5$.

The module N is regular of regular length 2 and lies in the tube $\mathcal{T}_1^{\Delta(\widetilde{\mathbb{D}}_n)}$ of rank $n-2$. Indeed, N is the direct predecessor of $F_1^{(1)}$ and the direct successor of $F_2^{(1)}$ which lie on the mouth of the tube $\mathcal{T}_1^{\Delta(\widetilde{\mathbb{D}}_n)}$ of rank $n-2$ in $\Gamma_{\widetilde{H}_n}$ (see [SS1, XIII, Lemma 2.8]).

This ends the proof of (3) in the case $n > 5$. The proof $n = 5$ is similar. ∎

In the two lemmas above we show the existence of an $\mathcal{M}'$-configuration or $\mathcal{N}$-configuration in the categories of modules over fixed hereditary algebras of type $\widetilde{\mathbb{D}}_n$. Now we shall generalize this result to an arbitrary tilted algebra $B = \mathrm{End}_H(T)^{\mathrm{op}}$ where H is a hereditary algebra of type $\widetilde{\mathbb{D}}_n$ and T is a tilting H-module without preinjective direct summands.

We start with the following useful lemma.

Lemma 6.3. *Assume that C_1 and C_2 are k-algebras, $\mathcal{C}_1$ and $\mathcal{C}_2$ are full additive subcategories of C_1-mod and C_2-mod, respectively, and $F : \mathcal{C}_1 \to \mathcal{C}_2$ is a fully-faithful additive functor.*

(1) *If $(X_1, \dots, X_4, M)$ is an $\mathcal{M}'$-configuration such that $X_1, \dots, X_4, M$ are objects of $\mathcal{C}_1$, then $(F(X_1), \dots, F(X_4), F(M))$ is an $\mathcal{M}'$-configuration.*

(2) *If $(Y_1, \dots, Y_5, N)$ is an $\mathcal{N}$-configuration such that $Y_1, \dots, Y_5, N$ are objects of $\mathcal{C}_1$, then $(F(Y_1), \dots, F(Y_5), F(N))$ is an $\mathcal{N}$-configuration.*

Proof. The assertion follows easily from the fact that $F : \mathcal{C}_1 \to \mathcal{C}_2$ is fully-faithful and additive. ■

Assume that A is an arbitrary k-algebra. We denote by $\mathcal{K}(A)$ the full additive subcategory of A-mod generated by all regular and preinjective A-modules. If T is a tilting A-module, we denote by $\mathcal{T}(T)$ the torsion class associated with T.

Assume that $H = kQ$ is an arbitrary hereditary algebra and $a \in Q_0$ is a sink in Q. We denote by $T[a] = \tau_H^{-1} S(a) \oplus \bigoplus_{b \neq a} P(b)$ the APR-tilting H-module and by $\mathcal{S}_a^+ = \mathrm{Hom}_H(T[a], -)$ one of the reflection functors. Note that $\mathrm{End}_H(T[a]) \cong k\sigma_a Q$ where $\sigma_a Q$ is the quiver with the same underlying graph as Q and where all the arrows of Q having a as source or as target are reversed, while all the other arrows remain unchanged. We refer to [ASS] for the setup of reflection functors.

Lemma 6.4. *Assume that $H = kQ$ is a hereditary algebra of type $\widetilde{\mathbb{D}}_n$ and $(X_1, \dots, X_4, M)$ is an $\mathcal{M}'$-configuration in H-mod such that the modules $X_1, \dots, X_4$ are preinjective and the module M is regular.*

(1) *The tuple*
$$(\tau_H^i(X_1), \dots, \tau_H^i(X_4), \tau_H^i(M))$$
is an $\mathcal{M}'$-configuration in H-mod for any $i \in \mathbb{N}$. Moreover, the modules $\tau_H^i(X_1), \dots, \tau_H^i(X_4)$ are preinjective and the module $\tau_H^i(M)$ is regular.

(2) *The tuple*
$$(\mathcal{S}_{i_t}^+ \dots \mathcal{S}_{i_1}^+(X_1), \dots, \mathcal{S}_{i_t}^+ \dots \mathcal{S}_{i_1}^+(X_4), \mathcal{S}_{i_t}^+ \dots \mathcal{S}_{i_1}^+(M))$$
is an $\mathcal{M}'$-configuration in H'-mod where $i_1, \dots, i_t$ is an admissible sequence of vertices in Q and $H' = k\sigma_{i_t} \dots \sigma_{i_1} Q$. Moreover, the modules

$\mathcal{S}^+_{i_t}\dots\mathcal{S}^+_{i_1}(X_1),\dots,\mathcal{S}^+_{i_t}\dots\mathcal{S}^+_{i_1}(X_4)$ are preinjective and $\mathcal{S}^+_{i_t}\dots\mathcal{S}^+_{i_1}(M)$ is regular.

(3) *Assume that T is a tilting H-module without preinjective direct summands, $B=\mathrm{End}_H(T)^{\mathrm{op}}$ and $M\in\mathcal{T}(T)$. The tuple*

$$(\mathrm{Hom}_H(T,X_1),\dots,\mathrm{Hom}_H(T,X_4),\mathrm{Hom}_H(T,M))$$

is an $\mathcal{M}'$-configuration.

Proof. (1) It is known that the Auslander–Reiten translation $\tau_H : \underline{H\text{-mod}} \to \overline{H\text{-mod}}$ between the stable module categories (here $\underline{H\text{-mod}}$ and $\overline{H\text{-mod}}$ denote the factors of H-mod modulo projectives and modulo injectives, respectively) induces a fully-faithful functor $\tau_H:\mathcal{K}(H)\to\mathcal{K}(H)$, and hence (1) follows from Lemma 6.3. Obviously, the modules $\tau_H^i(X_1),\dots,\tau_H^i(X_4)$ are preinjective and $\tau_H^i(M)$ is regular, for any $i\in\mathbb{N}$.

(2) Assume that $a\in Q_0$ is a sink in Q, $T[a]$ is the APR-tilting H-module and $A=\mathrm{End}_H(T[a])\cong k\sigma_aQ$. Then $T[a]$ is postprojective and hence $\mathcal{K}(H)\subseteq\mathcal{T}(T[a])$ and $\mathcal{S}^+_a(\mathcal{K}(H))\subseteq\mathcal{K}(A)$ by [ASS, VIII, Theorem 4.5]. So the restriction of the reflection functor $\mathcal{S}^+_a : H\text{-mod}\to A\text{-mod}$ to $\mathcal{K}(H)$ induces a fully-faithful functor $\mathcal{S}^+_a:\mathcal{K}(H)\to\mathcal{K}(A)$. Since $X_1,\dots,X_4,M$ are objects of $\mathcal{K}(H)$, (2) follows from Lemma 6.3.

(3) Since the modules X_i are preinjective over the hereditary algebra H and the tilting module T does not have preinjective direct summands, we have $X_i\in\mathcal{T}(T)=\{X\in H\text{-mod};\ \mathrm{Ext}^1_H(T,X)=0\}$ for $i=1,\dots,4$ (see [ASS, Section VIII] for details). Consequently, $X_1,\dots,X_4,M$ all belong to $\mathcal{T}(T)$ and since the tilting functor $\mathrm{Hom}_H(T,-)$ restricted to $\mathcal{T}(T)$ is an equivalence, (3) follows from Lemma 6.3. ∎

Lemma 6.5. *Assume that $H=kQ$ is a hereditary algebra of type $\widetilde{\mathbb{D}}_n$ and $(Y_1,\dots,Y_5,N)$ is an $\mathcal{N}$-configuration in* H-mod *such that the modules $Y_1,\dots,Y_5$ are preinjective and the module N is regular.*

(1) *The tuple*

$$(\tau_H^i(Y_1),\dots,\tau_H^i(Y_5),\tau_H^i(N))$$

is an $\mathcal{N}$-configuration in H-mod *for any $i\in\mathbb{N}$. Moreover, the modules $\tau_H^i(Y_1),\dots,\tau_H^i(Y_5)$ are preinjective and the module $\tau_H^i(N)$ is regular.*

(2) *The tuple*

$$(\mathcal{S}^+_{i_t}\dots\mathcal{S}^+_{i_1}(Y_1),\dots,\mathcal{S}^+_{i_t}\dots\mathcal{S}^+_{i_1}(Y_5),\mathcal{S}^+_{i_t}\dots\mathcal{S}^+_{i_1}(N))$$

is an $\mathcal{N}$-configuration in H'-mod *where $i_1,\dots,i_t$ is an admissible sequence of vertices in Q and $H'=k\sigma_{i_t}\dots\sigma_{i_1}Q$. Moreover, the modules $\mathcal{S}^+_{i_t}\dots\mathcal{S}^+_{i_1}(Y_1),\dots,\mathcal{S}^+_{i_t}\dots\mathcal{S}^+_{i_1}(Y_5)$ are preinjective and the module $\mathcal{S}^+_{i_t}\dots\mathcal{S}^+_{i_1}(N)$ is regular.*

(3) *Assume that T is a tilting H-module without preinjective direct summands, $B = \mathrm{End}_H(T)^{\mathrm{op}}$ and $N \in \mathcal{T}(T)$. The tuple*

$$(\mathrm{Hom}_H(T, Y_1), \ldots, \mathrm{Hom}_H(T, Y_5), \mathrm{Hom}_H(T, N))$$

is an $\mathcal{N}$-configuration.

Proof. Analogous to the proof of Lemma 6.4. ∎

We are now ready to present the main results of this section.

THEOREM 6.6. *Assume that H is a hereditary algebra of type $\widetilde{\mathbb{D}}_n$, $n \geq 4$, T is a tilting H-module without preinjective direct summands, $B = \mathrm{End}_H(T)^{\mathrm{op}}$ and $\widetilde{M} = \mathrm{Hom}_H(T, S)$ where S is an indecomposable regular H-module of regular length 1 lying in a tube of rank $n-2$ such that $S \in \mathcal{T}(T)$. There exist B-modules $\widetilde{X}_1, \ldots, \widetilde{X}_4$ such that $(\widetilde{X}_1, \ldots, \widetilde{X}_4, \widetilde{M})$ is an $\mathcal{M}'$-configuration in B-mod.*

Proof. Assume that Q is a quiver of type $\widetilde{\mathbb{D}}_n$, $H = kQ$ and $\widetilde{H}_n = k\Delta(\widetilde{\mathbb{D}}_n)$. It follows from [ASS, VII, Lemma 5.1] that there is an admissible sequence of vertices $i_1, \ldots, i_t$ such that $\sigma_{i_t} \ldots \sigma_{i_1} \Delta(\widetilde{\mathbb{D}}_n) = Q$. Since the functor $\mathcal{S}^+_{i_t} \ldots \mathcal{S}^+_{i_1} : \widetilde{H}_n\text{-mod} \to H\text{-mod}$ is an equivalence between the categories of regular modules over H and over $\widetilde{H}_n$, we get $S \cong \mathcal{S}^+_{i_t} \ldots \mathcal{S}^+_{i_1}(U)$ for some regular $\widetilde{H}_n$-module U of regular length 1 lying in a tube $\mathcal{T}$ of rank $n-2$ in $\Gamma_{\widetilde{H}_n}$.

If $n = 4$, then $\mathcal{T}$ is one of three tubes of rank $n - 2 = 2$ in the regular part of $\Gamma_{\widetilde{H}_n}$. If $n \geq 5$, then $\mathcal{T} = \mathcal{T}_1^{\Delta(\widetilde{\mathbb{D}}_n)}$.

It follows from Lemma 6.1 that there is a regular module $M^{\mathcal{T}} \in \mathcal{T}$ of regular length 1 and preinjective $\widetilde{H}_n$-modules $X_1^{\mathcal{T}}, \ldots, X_4^{\mathcal{T}}$ such that $(X_1^{\mathcal{T}}, \ldots, X_4^{\mathcal{T}}, M^{\mathcal{T}})$ is an $\mathcal{M}'$-configuration in $\widetilde{H}_n$-mod. Note that $U \cong \tau^i_{\widetilde{H}_n}(M^{\mathcal{T}})$ for some $i \in \mathbb{N}$.

We conclude from Lemma 6.4(1) that

$$(\tau^i_{\widetilde{H}_n}(X_1^{\mathcal{T}}), \ldots, \tau^i_{\widetilde{H}_n}(X_4^{\mathcal{T}}), \tau^i_{\widetilde{H}_n}(M^{\mathcal{T}}) \cong U)$$

is an $\mathcal{M}'$-configuration in $\widetilde{H}_n$-mod. Since $\tau^i_{\widetilde{H}_n}(X_1^{\mathcal{T}}), \ldots, \tau^i_{\widetilde{H}_n}(X_4^{\mathcal{T}})$ are preinjective and $\tau^i_{\widetilde{H}_n}(M^{\mathcal{T}})$ is regular, Lemma 6.4(2) shows that

$$\big(\mathcal{S}^+_{i_t} \ldots \mathcal{S}^+_{i_1}(\tau^i_{\widetilde{H}_n}(X_1^{\mathcal{T}})), \ldots, \mathcal{S}^+_{i_t} \ldots \mathcal{S}^+_{i_1}(\tau^i_{\widetilde{H}_n}(X_4^{\mathcal{T}})), \mathcal{S}^+_{i_t} \ldots \mathcal{S}^+_{i_1}(\tau^i_{\widetilde{H}_n}(M^{\mathcal{T}})) \cong S\big)$$

is an $\mathcal{M}'$-configuration in H-mod.

Since $\mathcal{S}^+_{i_t} \ldots \mathcal{S}^+_{i_1}(\tau^i_{\widetilde{H}_n}(X_1^{\mathcal{T}})), \ldots, \mathcal{S}^+_{i_t} \ldots \mathcal{S}^+_{i_1}(\tau^i_{\widetilde{H}_n}(X_4^{\mathcal{T}}))$ are preinjective, the module $\mathcal{S}^+_{i_t} \ldots \mathcal{S}^+_{i_1}(\tau^i_{\widetilde{H}_n}(M^{\mathcal{T}})) \cong S$ is regular and $S \in \mathcal{T}(T)$, we deduce the assertion of the theorem from Lemma 6.4(3). ∎

THEOREM 6.7. *Assume that H is a hereditary algebra of type $\widetilde{\mathbb{D}}_n$, $n \geq 4$, T is a tilting H-module without preinjective direct summands, $B = \mathrm{End}_H(T)^{\mathrm{op}}$ and $\widetilde{N} = \mathrm{Hom}_H(T, R)$ where R is an indecomposable regular H-module of regular length 2 lying in a tube of rank $n-2$ such that $R \in \mathcal{T}(T)$. There exist B-modules $\widetilde{Y}_1, \ldots, \widetilde{Y}_5$ such $(\widetilde{Y}_1, \ldots, \widetilde{Y}_5, \widetilde{N})$ is an $\mathcal{N}$-configuration in B-mod.*

Proof. We consider two cases: $n = 4$ and $n \geq 5$. In both cases the argument is analogous to the proof of Theorem 6.6, but we apply Lemma 6.2 and Lemma 6.5. ∎

7. Independent pairs of dense chains of pointed modules over strongly simply connected algebras of non-polynomial growth. In this section we apply the results of the previous sections to prove the two main theorems of the paper.

THEOREM 7.1. *Assume that k is a field such that* $\mathrm{char}(k) \neq 2$.

(1) *Assume that $\Lambda = B[M, G_t]$ is a pg-critical k-algebra of type I. There exists an independent pair of dense chains of Θ^Λ-pointed Λ-modules for some finite-dimensional Λ-module Θ^Λ.*
(2) *Assume that $\Omega = B[N]$ is a pg-critical k-algebra of type II. There exists an independent pair of dense chains of Θ^Ω-pointed Ω-modules for some finite-dimensional Ω-module Θ^Ω.*
(3) *Assume that Π is a hypercritical k-algebra. There exists an independent pair of dense chains of Θ^Π-pointed Π-modules for some finite-dimensional Π-module Θ^Π.*

Proof. (1) It follows by Theorem 6.6 that there are B-modules $X_1, \ldots, X_4$ such that $(X_1, \ldots, X_4, M)$ is an $\mathcal{M}'$-configuration in B-mod. There exists a fully-faithful exact functor $\mathcal{F}^\Lambda : \mathrm{prin}(\mathcal{G}_3) \to \Lambda$-mod by Theorem 5.5(1).

Theorem 4.1 yields a semisimple $\mathcal{G}_3$-module Θ and an independent pair $((\widetilde{M}_q, \widehat{\chi}_{\widetilde{M}_q})_{q\in L_1}, (\widetilde{N}_t, \widehat{\chi}_{\widetilde{N}_t})_{t\in L_2})$ of dense chains of Θ-pointed modules over $\mathcal{G}_3$ such that Θ, $\widetilde{M}_q$, $\widetilde{N}_t$ and $\widetilde{M}_q * \widetilde{N}_t$ are objects of the category $\mathrm{prin}(\mathcal{G}_3)$ for any $q \in L_1$ and $t \in L_2$. We set $\Theta^\Lambda = \mathcal{F}^\Lambda(\Theta)$. Thus it follows from Theorem 3.13(1) that $(\mathcal{F}^\Lambda(\widetilde{M}_q, \widehat{\chi}_{\widetilde{M}_q})_{q\in L_1}, \mathcal{F}^\Lambda(\widetilde{N}_t, \widehat{\chi}_{\widetilde{N}_t})_{t\in L_2})$ is an independent pair of dense chains of Θ^Λ-pointed Λ-modules.

(2) Apply Theorems 6.7, 5.5(2), 4.2 and 3.13(1) in a similar way to the argument for (1).

(3) The algebra Π is strictly wild since it has the same representation type as a hereditary algebra of a wild quiver (see [SS2, XVIII, Theorem 4.1] and [Ri]). Consequently, there exists a fully-faithful exact functor from A-mod to Π-mod for any k-algebra A. Thus the assertion follows from Theorem 3.13(1) and, for example, from (1) or (2). ∎

THEOREM 7.2. *Assume that k is a field such that* $\mathrm{char}(k) \neq 2$ *and A is a strongly simply connected k-algebra of non-polynomial growth.*

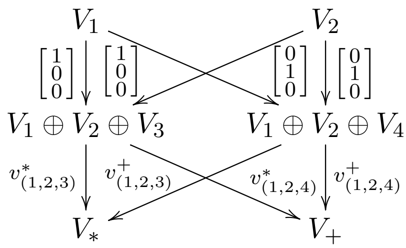
(1) *There exists an independent pair of dense chains of Σ-pointed A-modules for some finite-dimensional A-module Σ.*
(2) *If k is countable, then there exists a super-decomposable pure-injective A-module.*

Proof. (1) It follows from Theorem 2.1 that A has a convex hypercritical or pg-critical subcategory C. Hence there exists a fully-faithful exact functor of extension by zeros $J : C\text{-mod} \to A\text{-mod}$ (see Section 2), and (1) follows from Theorems 7.1 and 3.13(1).

(2) follows from (1) and Theorem 3.9(2). ∎

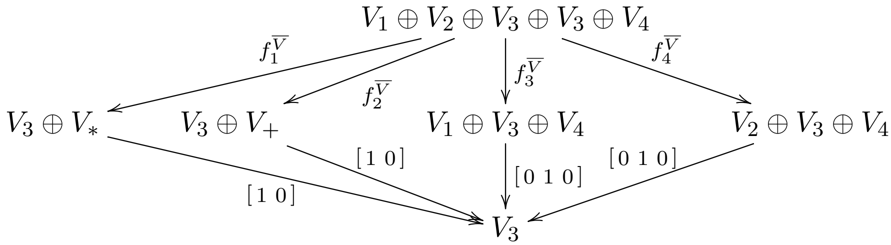
8. Appendix: from the garland to the diamond. In this section we sketch an explicit construction of a functor $F : \mathrm{prin}(\mathcal{G}_3) \to \mathrm{prin}(\mathcal{D})$ satisfying the requirements formulated in the proof of Theorem 4.2.

A left prinjective $\mathcal{G}_3$-module $\overline{V}$ can be represented as



where $v^x_{(1,2,j)} = [v_{x1} \quad v_{x2} \quad v_{xj}]$ and $v_{xi} : V_i \to V_x$ for $j = 3, 4$, $i = 1, 2, 3, 4$ and $x = *, +$.

We define the representation $F(\overline{V})$ as follows:



where

$$f_1^{\overline{V}} = \begin{bmatrix} 0 & 0 & -1 & -1 & 0 \\ v_{*1} & v_{*2} & 0 & v_{*3} & v_{*4} \end{bmatrix}, \qquad f_2^{\overline{V}} = \begin{bmatrix} 0 & 0 & -1 & -1 & 0 \\ v_{+1} & v_{+2} & 0 & v_{+3} & v_{+4} \end{bmatrix},$$

$$f_3^{\overline{V}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad f_4^{\overline{V}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is immediate that $F(\overline{V})$ is a prinjective $\mathcal{D}$-module.

Assume now that $\overline{h} : \overline{V} \to \overline{W}$ is a homomorphism of prinjective $\mathcal{G}_3$-modules. The homomorphism $\overline{h} : \overline{V} \to \overline{W}$ can be identified with the tuple $(h_1, h_2, h_{(1,2,3)}, h_{(1,2,4)}, h_*, h_+)$, where $h_i : V_i \to W_i$ for $i = 1, 2$, $h_j : V_j \to W_j$ for $j = *, +$ and

$$h_{(1,2,3)} : V_1 \oplus V_2 \oplus V_3 \to W_1 \oplus W_2 \oplus W_3,$$
$$h_{(1,2,4)} : V_1 \oplus V_2 \oplus V_4 \to W_1 \oplus W_2 \oplus W_4.$$

Since $\overline{h} : \overline{V} \to \overline{W}$ is a homomorphism of representations, we have

$$h_{(1,2,3)} = \begin{bmatrix} h_1 & 0 & h_{13} \\ 0 & h_2 & h_{23} \\ 0 & 0 & h_3 \end{bmatrix}, \quad h_{(1,2,4)} = \begin{bmatrix} h_1 & 0 & h_{14} \\ 0 & h_2 & h_{24} \\ 0 & 0 & h_4 \end{bmatrix}$$

for some $h_i : V_i \to W_i$, $i = 3, 4$, and $h_{st} : V_t \to W_s$, $s = 1, 2$, $t = 3, 4$.

We define $F(\overline{h}) : F(\overline{V}) \to F(\overline{W})$ as follows:

$$F(\overline{h}) = (f_1^{\overline{h}}, f_2^{\overline{h}}, f_3^{\overline{h}}, f_4^{\overline{h}}, f_5^{\overline{h}}, f_6^{\overline{h}}),$$

where $f_i^{\overline{h}} : e_{y_i} F(\overline{V}) \to e_{y_i} F(\overline{W})$ and

$$f_1^{\overline{h}} = \begin{bmatrix} h_1 & 0 & 0 & h_{13} & h_{14} \\ 0 & h_2 & -h_{23} & 0 & h_{24} \\ 0 & 0 & h_3 & 0 & 0 \\ 0 & 0 & 0 & h_3 & 0 \\ 0 & 0 & 0 & 0 & h_4 \end{bmatrix},$$

$$f_2^{\overline{h}} = \begin{bmatrix} h_3 & 0 \\ w_{*2}h_{23} & h_* \end{bmatrix}, \quad f_3^{\overline{h}} = \begin{bmatrix} h_3 & 0 \\ w_{+2}h_{23} & h_+ \end{bmatrix},$$

$$f_4^{\overline{h}} = \begin{bmatrix} h_1 & h_{13} & h_{14} \\ 0 & h_3 & 0 \\ 0 & 0 & h_4 \end{bmatrix}, \quad f_5^{\overline{h}} = \begin{bmatrix} h_2 & -h_{23} & h_{24} \\ 0 & h_3 & 0 \\ 0 & 0 & h_4 \end{bmatrix}, \quad f_6^{\overline{h}} = h_3.$$

LEMMA 8.1. *The map F defined above extends to a full, faithful and exact functor $F : \mathrm{prin}(\mathcal{G}_3) \to \mathrm{prin}(\mathcal{D})$.*

We omit the proof, which is mostly a straightforward calculation. Let us only stress one detail in the proof of exactness. Namely, one has to show that if a pair $(\overline{h}, \overline{g})$ of homomorphisms forms an exact sequence (which means that $\overline{h}$ is a monomorphism, $\overline{g}$ is an epimorphism and $\mathrm{Im}(\overline{h}) = \mathrm{Ker}(\overline{g})$) in $\mathrm{prin}(\mathcal{G}_3)$, then (h_i, g_i) constitute an exact sequence of vector spaces for $i = 1, 2, 3, 4, *, +$ (we use the notation introduced above).

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Stanisław Kasjan
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
87-100 Toruń, Poland
E-mail: skasjan@mat.umk.pl

Grzegorz Pastuszak
Center for Theoretical Physics of the
Polish Academy of Sciences
Al. Lotników 32/46
02-668 Warszawa, Poland
E-mail: past@mat.umk.pl

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