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COFINITENESS OF TORSION FUNCTORS OF COFINITE MODULES

BY

REZA NAGHIPOUR (Tabriz and Tehran),
KAMAL BAHMANPOUR (Ardabil and Tehran) and
IMANEH KHALILI GORJI (Qazvin)

Abstract. Let R be a Noetherian ring and I an ideal of R. Let M be an I-cofinite and N a finitely generated R-module. It is shown that the R-modules $\operatorname{Tor}_i^R(N,M)$ are I-cofinite for all $i \geq 0$ whenever $\dim \operatorname{Supp}(M) \leq 1$ or $\dim \operatorname{Supp}(N) \leq 2$. This immediately implies that if I has dimension one (i.e., $\dim R/I = 1$) then the R-modules $\operatorname{Tor}_i^R(N, H_I^j(M))$ are I-cofinite for all $i, j \geq 0$. Also, we prove that if R is local, then the R-modules $\operatorname{Tor}_i^R(N,M)$ are I-weakly cofinite for all $i \geq 0$ whenever $\dim \operatorname{Supp}(M) \leq 2$ or $\dim \operatorname{Supp}(N) \leq 3$. Finally, it is shown that the R-modules $\operatorname{Tor}_i^R(N, H_I^j(M))$ are I-weakly cofinite for all $i, j \geq 0$ whenever $\dim R/I \leq 2$.

1. Introduction. Throughout this paper, let R denote a commutative Noetherian ring (with identity) and I an ideal of R. For an R-module M, the ith local cohomology module of M with respect to I is defined as

$$H_I^i(M) = \varinjlim_{n \geq 1} \operatorname{Ext}_R^i(R/I^n, M).$$

We refer the reader to [3] for more details about local cohomology. In [8] Grothendieck conjectured that for any ideal I of R and any finitely generated R-module M, the module $\operatorname{Hom}_R(R/I, H_I^i(M))$ is finitely generated for all i. Two years later, Hartshorne [9] provided a counterexample to Grothendieck's conjecture. He defined an R-module M to be *I-cofinite* if $\operatorname{Supp}(M) \subseteq V(I)$ and $\operatorname{Ext}_R^j(R/I, M)$ is finitely generated for all j and asked:

- *For which rings R and ideals I are the modules $H_I^i(M)$ I-cofinite for all i and all finitely generated modules M?*

Concerning this question, Hartshorne [9] showed that if R is a complete regular local ring and I is a prime ideal such that $\dim R/I = 1$, then $H_I^i(M)$ is I-cofinite for any finitely generated R-module M. This result was later extended to more general Noetherian rings and one-dimensional ideals. Huneke and Koh [10, Theorem 4.1] proved that if R is a complete Goren-

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stein local domain and I is an ideal of R such that $\dim R/I = 1$, then $\mathrm{Ext}^j_R(N, H^i_I(M))$ is finitely generated for any finitely generated R-modules M, N such that $\mathrm{Supp}(N) \subseteq V(I)$ and for all i, j. Furthermore, using [10, Theorem 4.1], Delfino [4] proved that if R is a complete local domain under some mild conditions then the analogous results hold. Also, Delfino and Marley [5, Theorem 1] and Yoshida [18, Theorem 1.1] eliminated the completeness hypothesis entirely. Recently, in a slightly different line of research, Bahmanpour and Naghipour [1, Theorem 2.6] have removed the locality assumption on R.

In this paper we continue the study of modules cofinite with respect to an ideal in a Noetherian ring. Melkersson [16, Theorem 2.1] showed the striking result that cofiniteness of a module M with respect to an ideal I in a Noetherian ring R is actually equivalent to the finiteness of the R-modules $\mathrm{Tor}^R_i(R/I, M)$ for all i. The purpose of this paper is to study the I-cofiniteness and the I-weak cofiniteness of the R-modules $\mathrm{Tor}^R_i(N, M)$ for all i and for certain R-modules N and M.

In Section 2, we study the I-cofiniteness of the R-modules $\mathrm{Tor}^R_i(N, M)$. As the main result in that section, we show that $\mathrm{Tor}^R_i(N, M)$ is an I-cofinite R-module for all i if M is I-cofinite and N is finitely generated and we have $\dim \mathrm{Supp}(M) \leq 1$ or $\dim \mathrm{Supp}(N) \leq 2$. More precisely, we prove the following:

THEOREM 1.1. *Let I denote an ideal of a Noetherian ring R. Let M be an I-cofinite and N be a finitely generated R-module. Then the R-modules $\mathrm{Tor}^R_i(N, M)$ are I-cofinite for all $i \geq 0$ whenever*

$$\dim \mathrm{Supp}(M) \leq 1 \quad \textit{or} \quad \dim \mathrm{Supp}(N) \leq 2.$$

As an application, we derive the following result:

COROLLARY 1.2. *Let I denote an ideal of a Noetherian ring R and M a finitely generated R-module such that $\dim \mathrm{Supp}(M/IM) \leq 1$ (e.g., $\dim R/I \leq 1$). Then for each finitely generated R-module N, the R-modules $\mathrm{Tor}^R_i(N, H^j_I(M))$ are I-cofinite for all $i, j \geq 0$.*

An R-module M is said to be *skinny* or *weakly Laskerian* if the set of associated primes of any quotient module of M is finite (see [6] and [17]). Also, if I is an ideal of R, then an R-module T is said to be *I-weakly cofinite* if $\mathrm{Supp}(T) \subseteq V(I)$ and $\mathrm{Ext}^i_R(R/I, T)$ is weakly Laskerian for all $i \geq 0$ (see [7]).

In Section 3, we study the I-weak cofiniteness of $\mathrm{Tor}^R_i(N, M)$ over a local ring R. As the main results in that section, by using the observations from Section 2, we derive the following results:

THEOREM 1.3. *Let I denote an ideal of a Noetherian local ring R. Let M be an I-cofinite and N be a finitely generated R-module. Then the R-modules*

$\operatorname{Tor}_i^R(N, M)$ *are* I*-weakly cofinite for all* $i \geq 0$ *whenever*

$$\dim \operatorname{Supp}(M) \leq 2 \quad \textit{or} \quad \dim \operatorname{Supp}(N) \leq 3.$$

COROLLARY 1.4. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring,* I *an ideal of* R*, and* M *a finitely generated* R*-module such that* $\dim \operatorname{Supp}(M/IM) \leq 2$ (*e.g.,* $\dim R/I \leq 2$). *Then for each finitely generated* R*-module* N*, the* R*-modules* $\operatorname{Tor}_i^R(N, H_I^j(M))$ *are* I*-weakly cofinite for all* $i, j \geq 0$.

Finally, in Section 4 we construct two examples which show that Corollaries 1.2 and 1.4 are not true for ideals of dimensions 3 and 5.

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity and I will be an ideal of R. We shall use $\operatorname{Max}(R)$ to denote the set of all maximal ideals of R. An R-module T is said to be a *minimax module* if there is a finitely generated submodule N of T such that T/N is Artinian. It is clear that the class of minimax modules includes all finitely generated and all Artinian modules. For any unexplained notation and terminology we refer the reader to [3] and [14].

2. Cofiniteness. In this section, we study the I-cofiniteness of R-modules $\operatorname{Tor}_i^R(N, M)$. The main result of this section is Theorem 2.4. Proposition 2.3 will serve to shorten the proof of that theorem. In order to prove Proposition 2.3, we need the following fundamental two lemmas.

LEMMA 2.1. *Let* R *be a Noetherian ring and* I *an ideal of* R*. Then for an* R*-module* T*, the following conditions are equivalent:*

(i) $\operatorname{Tor}_n^R(R/I, T)$ *is a finitely generated* R*-module for all* $n \geq 0$.
(ii) $\operatorname{Ext}_R^n(N, T)$ *is a finitely generated* R*-module for all* $n \geq 0$.
(iii) $\operatorname{Ext}_R^n(R/I, T)$ *is a finitely generated* R*-module for any finitely generated* R*-module* N *with support in* $V(I)$ *and for all* $n \geq 0$.
(iv) $\operatorname{Tor}_n^R(N, T)$ *is a finitely generated* R*-module for any finitely generated* R*-module* N *with support in* $V(I)$ *and for all* $n \geq 0$.

Proof. The equivalence of (i)–(iii) has been proved in [16, Theorem 2.1] and [10, Lemma 4.2]. Also, the implication (iv)⇒(i) is clear, because $\operatorname{Supp}(R/I) = V(I)$. Thus, we only need to show that (i)⇒(iv). Using a prime filtration of N (see [14, Theorem 6.4]) and induction on the length of this filtration, it is enough to show that $\operatorname{Tor}_n^R(R/\mathfrak{p}, T)$ is a finitely generated R-module for all $n \geq 0$, where $\mathfrak{p} \in \operatorname{Supp}(N)$. To this end, we use [16, Theorem 2.1] and the proof of [10, Lemma 4.2]. ∎

LEMMA 2.2. *Let* R *be a Noetherian ring,* I *an ideal of* R*, and* M *an* I*-cofinite* R*-module. Then for each non-zero* R*-module* N *of finite length, the* R*-modules* $\operatorname{Tor}_i^R(N, M)$ *are of finite length for all* $i \geq 0$.

Proof. Since N is a non-zero R-module of finite length, the set $\operatorname{Supp}(N)$ is a finite non-empty subset of $\operatorname{Max}(R)$. Let $\operatorname{Supp}(N) := \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$ and $J := \mathfrak{m}_1 \cdots \mathfrak{m}_n$. Since $\operatorname{Supp}(N) = V(J)$, in view of Lemma 2.1, it is enough to show that the R-modules $\operatorname{Tor}_i^R(R/J, M)$ are of finite length for all $i \geq 0$. To do this, since $R/J \cong \bigoplus_{j=1}^n R/\mathfrak{m}_j$, we may assume $n=1$, and hence $J=\mathfrak{m}_1$. Now, let $i \geq 0$ be an integer such that $\operatorname{Tor}_i^R(R/\mathfrak{m}_1, M) \neq 0$. Then it is easy to see that $\mathfrak{m}_1 \in \operatorname{Supp}(M) \subseteq V(I)$. Therefore, in view of Lemma 2.1, the R-module $\operatorname{Tor}_i^R(R/\mathfrak{m}_1, M)$ is finitely generated of zero dimension, and hence is of finite length. ∎

We are now ready to prove the next result, to be used in the proof of the main theorem in this section.

PROPOSITION 2.3. *Let R be a Noetherian ring, I an ideal of R, and M an I-cofinite R-module. Let N be a finitely generated R-module such that* $\dim \operatorname{Supp}(N) = 1$. *Then the R-modules* $\operatorname{Tor}_i^R(N, M)$ *are I-cofinite and minimax for all $i \geq 0$.*

Proof. Set $T := \Gamma_I(N)$. Then as $\operatorname{Supp}(T) \subseteq V(I)$, it follows from Lemma 2.1 that the R-module $\operatorname{Tor}_i^R(T, M)$ is finitely generated for all $i \geq 0$. Hence, using the exact sequence

$$0 \to T \to N \to N/T \to 0$$

and Lemma 2.2, it is enough to show that the R-module $\operatorname{Tor}_i^R(N/T, M)$ is finitely generated for all $i \geq 0$. Since N/T is I-torsion-free, we can additionally assume that N is I-torsion-free. Then, by [3, Lemma 2.1.1],

$$I \nsubseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R N} \mathfrak{p}.$$

Therefore, there exists an $x \in I$ such that $x \notin \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R N} \mathfrak{p}$. Now, the exact sequence

$$0 \to N \xrightarrow{x} N \to N/xN \to 0$$

induces an exact sequence

$$\operatorname{Tor}_{j+1}^R(N/xN, M) \to (0 :_{\operatorname{Tor}_j^R(N,M)} x) \to 0.$$

Since the R-module N/xN is of finite length, it follows from the above exact sequence and Lemma 2.2 that the R-module $(0 :_{\operatorname{Tor}_j^R(N,M)} x)$ is of finite length for all $j \geq 0$. Hence the R-module $(0 :_{\operatorname{Tor}_j^R(N,M)} I)$ has the same property (note that $x \in I$). On the other hand, as

$$\operatorname{Supp}(\operatorname{Tor}_j^R(N, M)) \subseteq \operatorname{Supp}(M) \subseteq V(I),$$

it follows that the R-module $\operatorname{Tor}_j^R(N, M)$ is I-torsion. Hence, Melkerssons' theorem [15, Theorem 1.3] implies that the R-module $\operatorname{Tor}_j^R(N, M)$ is Artinian and hence minimax. Now it follows from [16, Proposition 4.3] that $\operatorname{Tor}_j^R(N, M)$ is also I-cofinite for all $j \geq 0$, as required. ∎

We are now ready to state and prove the main theorem of this section.

THEOREM 2.4. *Let R be a Noetherian ring, I an ideal of R, and M an I-cofinite R-module. Let N be a finitely generated R-module such that* $\dim \operatorname{Supp}(N) = 2$. *Then the R-modules $\operatorname{Tor}_i^R(N, M)$ are I-cofinite for all $i \geq 0$.*

Proof. In view of the proof of Proposition 2.3, we may assume that N is an I-torsion-free R-module and $\dim \operatorname{Supp}(N) = 2$. Then, by [3, Lemma 2.1.1], $I \not\subseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R N} \mathfrak{p}$. Therefore, there exists $x \in I$ such that $x \notin \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R N} \mathfrak{p}$. Now, the exact sequence

$$0 \to N \xrightarrow{x} N \to N/xN \to 0$$

induces an exact sequence

$$\operatorname{Tor}_{j+1}^R(N/xN, M) \to \operatorname{Tor}_j^R(N, M) \xrightarrow{x} \operatorname{Tor}_j^R(N, M) \to \operatorname{Tor}_j^R(N/xN, M)$$

for all $j \geq 0$. Consequently, using Proposition 2.3, Lemma 2.2 and [16, Corollary 4.4], we deduce that the R-modules

$$(0 :_{\operatorname{Tor}_j^R(N,M)} x) \quad \text{and} \quad \operatorname{Tor}_j^R(N, M)/x \operatorname{Tor}_j^R(N, M)$$

are I-cofinite for all $j \geq 0$. Therefore it follows from [16, Corollary 3.4] that $\operatorname{Tor}_j^R(N, M)$ is I-cofinite for all $j \geq 0$. ∎

3. Weak cofiniteness. In this section, we use the results of Section 2 to study the I-weak cofiniteness of $\operatorname{Tor}_i^R(N, M)$ over a local ring R. Recall that an R-module M is said to be *weakly Laskerian* or *skinny* if the set of associated primes of any quotient module of M is finite. Also, if I is an ideal of R, then an R-module T is said to be *I-weakly cofinite* if $\operatorname{Supp}(T) \subseteq V(I)$ and $\operatorname{Ext}_R^i(R/I, T)$ is weakly Laskerian for all $i \geq 0$. The following lemma is needed in the proof of the main results of this section.

LEMMA 3.1. *Let R be a Noetherian ring, I an ideal of R, and M an R-module. Then the following statements are equivalent:*

(i) $\operatorname{Tor}_n^R(R/I, M)$ *is a weakly Laskerian R-module for all $n \geq 0$.*
(ii) $\operatorname{Ext}_R^n(N, M)$ *is a weakly Laskerian R-module for all $n \geq 0$.*

Proof. The proof is the same as that of [16, Theorem 2.1]. ∎

THEOREM 3.2. *Let $(R, \mathfrak{m})$ be a local (Noetherian) ring, I an ideal of R and M an I-cofinite R-module. Let N be a finitely generated R-module such that* $\dim \operatorname{Supp}(N) = 3$. *Then the R-modules $\operatorname{Tor}_i^R(N, M)$ are I-weakly cofinite for all $i \geq 0$.*

Proof. In view of Lemma 3.1, it is enough to show that, for all $i, j \geq 0$, the R-modules $\operatorname{Tor}_j^R(R/I, \operatorname{Tor}_i^R(N, M))$ are weakly Laskerian. To this end, let Φ denote the set of all modules $\operatorname{Tor}_j^R(R/I, \operatorname{Tor}_i^R(N, M))$ where $i, j \geq 0$.

Let $L \in \Phi$ and let L' be a submodule of L. In view of the definition, it is enough to show that $\mathrm{Ass}_R L/L'$ is finite. To this end, according to [14, Ex. 7.7] and [12, Lemma 2.1], without loss of generality, we may assume that R is complete. Now, suppose that $\mathrm{Ass}_R L/L'$ is infinite. Then it has a countably infinite subset $\{\mathfrak{p}_k\}_{k=1}^{\infty}$ none of whose elements is equal to $\mathfrak{m}$. Then, by [13, Lemma 3.2], $\mathfrak{m} \nsubseteq \bigcup_{k=1}^{\infty} \mathfrak{p}_k$. Let S be the multiplicatively closed subset $R \setminus \bigcup_{k=1}^{\infty} \mathfrak{p}_k$. Since $S^{-1}N$ has dimension at most 2, it follows from Lemma 2.2, Proposition 2.3 and Theorem 2.4 that $S^{-1}L/S^{-1}L'$ is a finitely generated $S^{-1}R$-module, and so $\mathrm{Ass}_{S^{-1}R} S^{-1}L/S^{-1}L'$ is a finite set. But

$$S^{-1}\mathfrak{p}_k \in \mathrm{Ass}_{S^{-1}R} S^{-1}L/S^{-1}L' \quad \text{for all } k = 1, 2, \dots,$$

which is a contradiction. ∎

LEMMA 3.3. *Let R be a Noetherian ring, I an ideal of R, and M an I-cofinite R-module such that* $\dim \mathrm{Supp}(M) \le 1$. *Then for each non-zero finitely generated R-module N, the R-modules* $\mathrm{Tor}_i^R(N, M)$ *are I-cofinite for all $i \ge 0$.*

Proof. As R is Noetherian and N is finitely generated, it follows that N has a free resolution

$$\mathbb{F}_\bullet\colon \cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to 0,$$

where all the free modules F_i have finite ranks. Thus $\mathrm{Tor}_i^R(N, M) = H_i(\mathbb{F}_\bullet \otimes_R N))$ is a subquotient of a direct sum of finitely many copies of M. Now, the assertion follows easily from [2, Theorem 2.7]. ∎

COROLLARY 3.4. *Let I be an ideal of R and M a non-zero finitely generated R-module such that* $\dim \mathrm{Supp}(M/IM) \le 1$ (*e.g.*, $\dim R/I \le 1$). *Then for each finitely generated R-module N, the R-modules* $\mathrm{Tor}_i^R(N, H_I^j(M))$ *are I-cofinite for all $i, j \ge 0$.*

Proof. As $\mathrm{Supp}(H_I^i(M)) \subseteq \mathrm{Supp}(M/IM)$ and $\dim \mathrm{Supp}(M/IM) \le 1$, we see that

$$\dim \mathrm{Supp}(H_I^i(M)) \le 1.$$

Now the assertion follows from [1, Corollary 2.7] and Lemma 3.3. ∎

COROLLARY 3.5. *Let* $(R, \mathfrak{m})$ *be a local* (*Noetherian*) *ring, I an ideal of R and M a finitely generated R-module such that* $\dim \mathrm{Supp}(M/IM) \le 2$ (*e.g.*, $\dim R/I \le 2$). *Then for each finitely generated R-module N, the R-modules* $\mathrm{Tor}_i^R(N, H_I^j(M))$ *are I-weakly cofinite for all $i, j \ge 0$.*

Proof. In view of Lemma 3.1, it is enough to show that, for all $i, j \ge 0$, the R-modules $\mathrm{Tor}_j^R(R/I, \mathrm{Tor}_i^R(N, H_I^j(M)))$ are weakly Laskerian. To this end, let Φ denote the set of all R-modules $\mathrm{Tor}_k^R(R/I, \mathrm{Tor}_i^R(N, H_I^j(M)))$, where $i, j, k \ge 0$. Let $L \in \Phi$ and let L' be a submodule of L. In view of the definition, it is enough to show that the set $\mathrm{Ass}_R L/L'$ is finite. To this end, according to [3, Theorem 4.3.2], [14, Ex. 7.7] and [12, Lemma 2.1], without loss of generality, we may assume that R is complete. Now, suppose that $\mathrm{Ass}_R L/L'$ is infinite. Then it has a countably infinite subset $\{\mathfrak{p}_t\}_{t=1}^{\infty}$ none of whose elements is equal to $\mathfrak{m}$. Then, by [13, Lemma 3.2], $\mathfrak{m} \not\subseteq \bigcup_{t=1}^{\infty} \mathfrak{p}_t$. Let S be the multiplicatively closed subset $R \setminus \bigcup_{t=1}^{\infty} \mathfrak{p}_t$. Then it follows from Corollary 3.4 that $S^{-1}L/S^{-1}L'$ is a finitely generated $S^{-1}R$-module, and so $\mathrm{Ass}_{S^{-1}R} S^{-1}L/S^{-1}L'$ is a finite set. But $S^{-1}\mathfrak{p}_t \in \mathrm{Ass}_{S^{-1}R} S^{-1}L/S^{-1}L'$ for all $t = 1, 2, \ldots$, which is a contradiction. ∎

Finally, the following result is a generalization of Corollary 3.4 over local rings.

THEOREM 3.6. *Let $(R, \mathfrak{m})$ be a local (Noetherian) ring, I an ideal of R, and M an I-cofinite R-module such that* $\dim \mathrm{Supp}(M) \le 2$. *Then for each non-zero finitely generated R-module N, the R-modules* $\mathrm{Tor}_i^R(N, M)$ *are I-weakly cofinite for all $i \ge 0$.*

Proof. The proof is analogous to that of Theorem 3.2; it uses Lemma 3.3. ∎

4. Two examples. In this section we construct two examples of a Noetherian local ring R, an ideal I of R, and a finitely generated R-module N, with the property that $H_I^2(R)$ is I-cofinite, but the R-modules $\mathrm{Ext}_R^0(N, H_I^2(R))$ and $\mathrm{Tor}_0^R(N, H_I^2(R))$ are not I-weakly cofinite. The following lemma, which is a consequence of the definition, will be used several times in the proof of Examples 4.2 and 4.3.

LEMMA 4.1. *Let R be a Noetherian ring and I an ideal of R. If*

$$0 \to M' \to M \to M'' \to 0$$

is exact and two of the modules in the sequence are I-cofinite or I-weakly cofinite, then so is the third one.

Proof. The assertion follows immediately from the definition of I-cofiniteness and I-weakly cofiniteness. ∎

EXAMPLE 4.2. *Let k be a field and let $R = k[[x, y, z, u]]/(xy - zu)$. Set $I = (x, u)$ and $f = xy - zu$. Then:*

(i) *The R-module $H_I^2(R)$ is I-cofinite.*
(ii) *The R-module $R/Rf \otimes_R H_I^2(R)$ is not I-cofinite.*
(iii) *The R-module* $\mathrm{Hom}_R(R/Rf, H_I^2(R))$ *is not I-cofinite.*

Proof. Applying the functor $H_I^0(-)$ to the exact sequence

$$0 \to R \xrightarrow{f} R \to R/Rf \to 0,$$

we obtain the exact sequence

$$\cdots \to H_I^2(R) \xrightarrow{f} H_I^2(R) \to H_I^2(R/Rf) \to 0.$$

Since $H_I^i(R) = 0$ for all $i \neq 2$, one can show that

$$\operatorname{Ext}_R^i(R/I, H_I^2(R)) \cong \operatorname{Ext}_R^{i+2}(R/I, R)$$

for all i. Thus, $H_I^2(R)$ is I-cofinite. However,

$$\operatorname{coker} f = R/Rf \otimes_R H_I^2(R) = H_I^2(R/Rf)$$

is not I-cofinite [9] (note that $\dim R/Rf = 3$). This proves (i) and (ii).

In order to prove (iii), suppose that the contrary is true. Then it follows from Lemma 4.1 and the exact sequence

$$0 \to \operatorname{Hom}_R(R/Rf, H_I^2(R)) \to H_I^2(R) \to fH_I^2(R) \to 0$$

that the R-module $fH_I^2(R)$ is also I-cofinite. Again using Lemma 4.1 and the exact sequence

$$0 \to fH_I^2(R)) \to H_I^2(R) \to H_I^2(R)/fH_I^2(R) \to 0,$$

we deduce that the R-module $H_I^2(R)/fH_I^2(R)$ is I-cofinite, which is a contradiction. ∎

Example 4.3. *Let k be a field,*

$$S = k[x, y, s, t, u, v] \quad \text{and} \quad \mathfrak{m} = (x, y, s, t, u, v).$$

Set $R = S_{\mathfrak{m}}$, $I = (u, v)R$ and $f = sx^2v^2 + (t+s)xyuv + ty^2u^2 \in R$. Then:

(i) *The R-module $H_I^2(R)$ is I-cofinite.*
(ii) *The R-module $R/Rf \otimes_R H_I^2(R)$ is not I-weakly cofinite.*
(iii) *The R-module $\operatorname{Hom}_R(R/Rf, H_I^2(R))$ is not I-weakly cofinite.*

Proof. Analogously to the proof of Example 4.2 we see that $H_I^2(R)$ is I-cofinite and $H_I^2(R)/fH_I^2(R) \cong H_I^2(R/fR)$. In view of [11], the R-module $H_I^2(R/fR)$ has infinitely many associated primes, and so the R-module $R/Rf \otimes_R H_I^2(R)$ is not I-weakly cofinite (note that $\dim R/Rf = 5$). Thus (i) and (ii) hold.

In order to prove (iii), suppose that the contrary is true. Then it follows from Lemma 4.1 and the exact sequence

$$0 \to \operatorname{Hom}_R(R/Rf, H_I^2(R)) \to H_I^2(R) \to fH_I^2(R) \to 0$$

that the R-module $fH_I^2(R)$ is also I-weakly cofinite. Hence Lemma 4.1 and the exact sequence

$$0 \to fH_I^2(R)) \to H_I^2(R) \to H_I^2(R)/fH_I^2(R) \to 0$$

imply that the R-module $H_I^2(R)/fH_I^2(R)$ is I-weakly cofinite, which is a contradiction. ∎

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Reza Naghipour (corresponding author)
Department of Mathematics
University of Tabriz
Tabriz, Iran
E-mail: naghipour@tabrizu.ac.ir
and
School of Mathematics
Institute for Research
in Fundamental Sciences (IPM)
P.O. Box 19395-5746
Tehran, Iran
E-mail: naghipour@ipm.ir

Kamal Bahmanpour
Faculty of Science
University of Mohaghegh Ardabili
Ardabil, Iran
and
School of Mathematics
Institute for Research
in Fundamental Sciences (IPM)
P.O. Box 19395-5746
Tehran, Iran
E-mail: bahmanpour.k@gmail.com

Imaneh Khalili Gorji
Department of Basic Sciences
Imam Khomeini International University
P.O. Box 34149-1-6818
Qazvin, Iran
E-mail: i.khalili@ikiu.ac.ir

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