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HOCHSCHILD COHOMOLOGY OF GENERALIZED MULTICOIL ALGEBRAS

BY

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Dedicated to Ibrahim Assem on the occasion of his 65th birthday

Abstract. We determine the Hochschild cohomology of all finite-dimensional generalized multicoil algebras over an algebraically closed field, which are the algebras for which the Auslander–Reiten quiver admits a separating family of almost cyclic coherent components. In particular, the analytically rigid generalized multicoil algebras are described.

1. Introduction and the main results. Throughout the paper by an *algebra* we mean a basic, finite-dimensional k-algebra over a fixed algebraically closed field k. For an algebra A, we denote by $\operatorname{mod} A$ the category of finitely generated right A-modules, and by $\operatorname{ind} A$ a full subcategory of $\operatorname{mod} A$ consisting of a complete set of representatives of the isomorphism classes of indecomposable modules. We shall denote by $\operatorname{rad}(\operatorname{mod} A)$ the Jacobson radical of $\operatorname{mod} A$, and by $\operatorname{rad}^{\infty}(\operatorname{mod} A)$ the intersection of all powers $\operatorname{rad}^{i}(\operatorname{mod} A)$, $i \geq 1$, of $\operatorname{rad}(\operatorname{mod} A)$. Moreover, we denote by Γ_A the Auslander–Reiten quiver of A, and by τ_A and τ_A^{-} the Auslander–Reiten translations $D\operatorname{Tr}$ and $\operatorname{Tr} D$, respectively. We will not distinguish between a module in $\operatorname{ind} A$ and the corresponding vertex of Γ_A. Following [42] a family $\mathcal{C}$ of components in Γ_A is said to be *generalized standard* if $\operatorname{rad}^{\infty}(X,Y) = 0$ for all modules X and Y in $\mathcal{C}$. We note that different components of a generalized standard family $\mathcal{C}$ of components in Γ_A are orthogonal. Recall also that a family $\mathcal{C}$ of components in Γ_A is called *sincere* if any simple A-module occurs as a composition factor of a module in $\mathcal{C}$.

The Auslander–Reiten quiver is an important combinatorial and homological invariant of the module category $\operatorname{mod} A$ of an algebra A. Frequently, we may recover A from the behaviour of distinguished components of Γ_A in the category $\operatorname{mod} A$. For example, this is the case for tilted algebras [14, 19, 41], or more generally, double tilted [34] and generalized double

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tilted algebras [35, 46] whose Auslander–Reiten quiver admits a faithful component with a finite section (respectively, double section, multisection) satisfying a vanishing hom-condition.

In the representation theory of algebras a prominent role is played by algebras with a separating family of stable tubes (in the sense of Ringel [36]). This class of algebras contains the tame hereditary algebras, the tame concealed algebras, the tubular algebras, the canonical algebras and, more generally, the concealed canonical algebras. It has been proved in [17] that the class of algebras with a separating family of stable tubes coincides with the class of concealed canonical algebras. This was deepened in [33, 45], where a characterization of concealed canonical algebras in terms of external short paths (cycles) was established. Moreover, in order to deal with wider classes of algebras, a slightly more general concept of a separating family of components is natural. Namely, a family $\mathcal{C} = (\mathcal{C}_i)_{i\in I}$ of components of Γ_A is said to be *separating* in $\operatorname{mod} A$ if the modules in $\operatorname{ind} A$ split into three disjoint classes $\mathcal{P}_A$, $\mathcal{C}_A = \mathcal{C}$ and $\mathcal{Q}_A$ such that:

(S1) $\mathcal{C}_A$ is a sincere generalized standard family of components;
(S2) $\operatorname{Hom}_A(\mathcal{Q}_A, \mathcal{P}_A) = 0$, $\operatorname{Hom}_A(\mathcal{Q}_A, \mathcal{C}_A) = 0$ and $\operatorname{Hom}_A(\mathcal{C}_A, \mathcal{P}_A) = 0$;
(S3) any morphism from $\mathcal{P}_A$ to $\mathcal{Q}_A$ factors through $\operatorname{add} \mathcal{C}_A$.

We then say that $\mathcal{C}_A$ *separates* $\mathcal{P}_A$ *from* $\mathcal{Q}_A$ and write $\operatorname{ind} A=\mathcal{P}_A \vee \mathcal{C}_A \vee \mathcal{Q}_A$. We note that $\mathcal{P}_A$ and $\mathcal{Q}_A$ are then uniquely determined by $\mathcal{C}_A$ (see [5]). We also refer to the survey article [29] for the structure of arbitrary algebras with separating families of Auslander–Reiten components.

In [2, 3, 4] Assem and Skowroński introduced a natural generalization of the concept of tube, called a coil, and then the class of coil algebras, which are the tame algebras with a separating family of coils. The coil algebras have played a fundamental role in the study of tame strongly simply connected algebras [6, 43] as well as in describing the geometric and homological properties of indecomposable modules over strongly simply connected algebras of polynomial growth (see [30, 31, 32, 47] for some results).

Motivated by the importance of coils and coil algebras, the present authors introduced in [25] the concept of a generalized multicoil, and then of a generalized multicoil algebra [26]. A *generalized multicoil* is a translation quiver obtained from a finite family of stable tubes by iterated application of a sequence of admissible operations of types (ad 1)–(ad 5) and their duals (ad 1*)–(ad 5*). We recall that a *coil* in the sense of [3] is a translation quiver obtained from one stable tube by iterated application of admissible operations of types (ad 1)–(ad 3) and their duals (ad 1*)–(ad 3*). It has been proved in [25] that a component $\mathcal{C}$ of the Auslander–Reiten quiver Γ_A of an algebra A is a generalized multicoil if and only if $\mathcal{C}$ is almost cyclic and coherent. Recall that a component Γ of Γ_A is called *almost cyclic* if all but finitely many modules in Γ lie on oriented cycles contained entirely in Γ. Further, a component Γ of Γ_A is called *coherent* if the following two conditions are satisfied:

(C1) For each projective module P in Γ there is an infinite sectional path $P = X_1 \to X_2 \to \cdots \to X_i \to X_{i+1} \to X_{i+2} \to \cdots$ (that is, $X_i \neq \tau_A X_{i+2}$ for any $i \geq 1$) in Γ.

(C2) For each injective module I in Γ there is an infinite sectional path $\cdots \to Y_{j+2} \to Y_{j+1} \to Y_j \to \cdots \to Y_2 \to Y_1 = I$ (that is, $Y_{j+2} \neq \tau_A Y_j$ for any $j \geq 1$) in Γ.

The main result of [26], Theorem A, asserts that the Auslander–Reiten quiver Γ_A of an algebra A admits a separating family of almost cyclic components if and only if A is a generalized multicoil enlargement of a finite product of concealed canonical algebras by iterated application of admissible algebra operations of types (ad 1)–(ad 5) and their duals (corresponding to the translation quiver operations (ad 1)–(ad 5) and their duals, leading from a finite family of stable tubes to generalized multicoils). These algebras are called *generalized multicoil algebras* and play a prominent role in recent investigations (see [8, 21, 22, 23, 26, 27, 28] for some results concerning the geometric and homological properties of their module categories).

We note that, by [26, Theorem E], each generalized multicoil algebra A is of global dimension at most three and every module in $\operatorname{ind} A$ has projective or injective dimension at most two. We also mention that the quasitilted algebras of canonical type [18, 44] form a distinguished special class of generalized multicoil algebras. Moreover, recently generalized multicoil algebras have proved to be important in describing the support algebras of infinite cyclic components of Auslander–Reiten quivers of algebras consisting of cycle-finite indecomposable modules (see [24]).

Here, we are interested in the Hochschild cohomology spaces $H^n(A) = H^n(A, A)$, $n \geq 0$, and the deformation theory of generalized multicoil algebras A. In Section 4 we introduce numerical invariants d_A, f_A, p_A, r_A of a generalized multicoil algebra A, depending on the types of admissible operations (ad 1)–(ad 5) and their duals, leading from a product C of concealed canonical algebras to A.

The following theorem is the main result of the paper.

THEOREM 1.1. *Let A be a connected generalized multicoil algebra. Then:*

(i) $H^n(A) = 0$ *for* $n \geq 3$.
(ii) $\dim_k H^2(A) = p_A + r_A$.
(iii) $\dim_k H^1(A) = d_A + f_A$.
(iv) $\dim_k H^0(A) = 1$.

The numbers p_A, r_A, d_A, f_A will be defined at the end of Section 4.

We note that in the very special case of quasitilted algebras the above statements follow from results proved by Happel (see Section 3).

Following Gerstenhaber [11], a *one-parameter deformation* of an algebra A is a $k[[t]]$-algebra structure on $k[[t]] \otimes_k A$ given by $f : A \otimes_k A \to k[[t]] \otimes_k A$ where $f(a \otimes b) = ab + t \otimes f_1(a \otimes b) + t^2 \otimes f_2(a \otimes b) + \cdots$ for k-bilinear morphisms $f_i : A \times A \to A$. Then the algebra A is said to be *analytically rigid* if any one-parameter deformation of A is isomorphic to the trivial one given by $f_i = 0$ for $i \geq 1$. It was shown in [11] that one-parameter deformations of A are related to low Hochschild cohomology spaces, $H^i(A)$, $1 \leq i \leq 3$. In fact, for a one-parameter deformation of A given by $f : A \otimes_k A \to k[[t]] \otimes_k A$, the first f_i different from zero defines an element of $H^2(A)$. Hence, if $H^2(A) = 0$, then A is analytically rigid. Moreover, if $H^3(A) = 0$ and A is analytically rigid, then $H^2(A) = 0$.

In order to describe analytically rigid generalized multicoil algebras we need two numerical invariants. Let C be a connected concealed canonical algebra of tubular type $p_C = (p_1, \ldots, p_t)$ (see Section 3 for details). Then we set $t_C = t$ and define the number e_C as follows:

$$e_C = \begin{cases} 0 & \text{if } t_C \geq 3, \\ 1 & \text{if } p_C = (p_1, p_2) \text{ with } p_1, p_2 \geq 2, \\ 2 & \text{if } p_C = (p_1, p_2) \text{ with } p_1 = 1,\ p_2 \geq 2, \\ 3 & \text{if } p_C = (p_1, p_2) \text{ with } p_1 = p_2 = 1. \end{cases}$$

As a consequence of Theorem 1.1 and Gerstenhaber's results, described above, we obtain the following fact.

COROLLARY 1.2. *Let A be a connected generalized multicoil algebra. The following statements are equivalent:*

(i) *A is analytically rigid.*
(ii) $H^2(A) = 0$.
(iii) *A is a generalized multicoil enlargement of a family $C_1, \ldots, C_m$ of connected concealed canonical algebras such that, for any $i \in \{1, \ldots, m\}$, $t_{C_i} \leq 3$ and the number of operations applied to modules from the mouth of stable tubes of rank one in Γ_{C_i} is at most e_{C_i}.*

We note that analytically rigid coil algebras, or more generally strongly simply connected algebras of polynomial growth, have been characterized in [10, 40].

For basic background on the representation theory of algebras we refer to the books [1, 36, 37, 38], and for Hochschild cohomology and deformations of algebras to the articles [10, 11, 12].

2. Hochschild cohomology of algebras. Let A be an algebra. Denote by $C^\bullet A$ the Hochschild complex $C^\bullet = (C^i, d^i)_{i\in\mathbb{Z}}$ defined as follows: $C^i = 0$, $d^i = 0$ for $i < 0$, $C^0 = {}_AA_A$, $C^i = \mathrm{Hom}_k(A^{\otimes i}, A)$ for $i > 0$, where $A^{\otimes i}$ denotes the i-fold tensor product over k of A itself, $d^0 : A \to \mathrm{Hom}_k(A, A)$ with $(d^0x)(a) = ax - xa$ for $x, a \in A$, $d^i : C^i \to C^{i+1}$ with

$$\begin{aligned}(d^i f)(a_1 \otimes \cdots \otimes a_{i+1}) = a_1 f(a_2 \otimes \cdots \otimes a_{i+1})&\\ + \sum_{j=1}^{i}(-1)^j f(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{i+1})&\\ + (-1)^{i+1} f(a_1 \otimes \cdots \otimes a_i)a_{i+1}&\end{aligned}$$

for $f \in C^i$ and $a_1, \ldots, a_{i+1} \in A$. Then $H^i(A) = H^i(C^\bullet A)$ is called the ith *Hochschild cohomology space* of A (see [7, Chapter IX]). Recall that the first Hochschild cohomology space $H^1(A)$ of an algebra A is isomorphic to the space $\mathrm{Der}(A, A)/\mathrm{Der}^0(A, A)$ of outer derivations of A, where $\mathrm{Der}(A, A) = \{\delta \in \mathrm{Hom}_k(A, A) \mid \delta(ab) = a\delta(b) + \delta(a)b \text{ for } a, b \in A\}$ is the space of k-linear derivations of A, and $\mathrm{Der}^0(A, A)$ is the subspace $\{\delta_x \in \mathrm{Hom}_k(A, A) \mid \delta_x(a) = ax - xa \text{ for } a \in A\}$ of inner derivations of A.

We recall the following classical result (see [12, Proposition 1.6 and Corollary 1.7])

Proposition 2.1. *Let A be the path algebra KQ of a connected quiver Q. Then:*

(i) $H^0(A) \cong k$ *and* $H^n(A) = 0$ *for* $n \geq 2$.
(ii) $H^1(A) = 0$ *if and only if* Q *is a tree.*

The following tilting invariance of Hochschild cohomology spaces of algebras has been established by Happel [12, Theorem 4.2].

Proposition 2.2. *Let B be an algebra, T a tilting B-module and $A = \mathrm{End}_B(T)$. Then $H^n(A) \cong H^n(B)$ for any $n \geq 0$.*

Frequently an algebra A can be obtained from another algebra B by a sequence of one-point extensions and one-point coextensions. Recall that the *one-point extension* of an algebra B by a B-module M is the matrix algebra

$$B[M] = \begin{bmatrix} B & 0 \\ M & k \end{bmatrix}$$

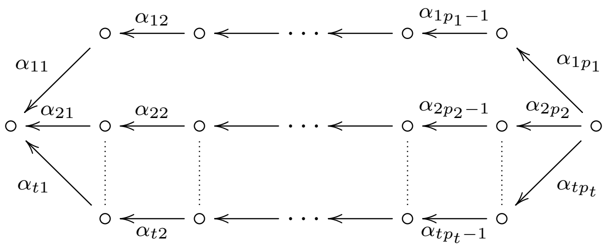
with the usual addition and multiplication of matrices. The quiver of $B[M]$ contains Q_B as a convex subquiver and there is an additional (extension) point which is a source. $B[M]$-modules are usually identified with triples (V, X, φ), where V is a k-vector space, X a B-module and $\varphi : V \to \mathrm{Hom}_B(M, X)$ a k-linear map. A $B[M]$-linear map $(V, X, \varphi) \to (V', X', \varphi')$ is then identified with a pair (f, g), where $f : V \to V'$ is k-linear, $g : X \to X'$ is B-linear and $\varphi' f = \mathrm{Hom}_B(M, g)\varphi$. One defines dually the *one-point co-extension* $[M]B$ of B by M (see [36]).

The following theorem proved by Happel [12, Theorem 5.3] provides an important tool for calculation of the Hochschild cohomology of algebras.

THEOREM 2.3. *Let A be the one-point extension $B[M]$ of an algebra B by a B-module M. Then there exists the following long exact sequence connecting the Hochschild cohomology spaces of A and B:*

$$\begin{aligned}0 \to H^0(A) \to H^0(B) \to \mathrm{Hom}_B(M, M)/k \to H^1(A) \to H^1(B)\\ \to \mathrm{Ext}^1_B(M, M) \to \cdots \to \mathrm{Ext}^i_B(M, M) \to H^{i+1}(A) \to H^{i+1}(B)\\ \to \mathrm{Ext}^{i+1}_B(M, M) \to \cdots .\end{aligned}$$

3. Concealed canonical algebras. An important role in our considerations will be played by certain tilts of canonical algebras introduced by Ringel [36]. Let $p_1, \dots, p_t$ be a sequence of positive integers with $t \geq 2$, $1 \leq p_1 \leq \dots \leq p_t$, and $p_1 \geq 2$ if $t \geq 3$. Denote by $\Delta(p_1, \dots, p_t)$ the quiver



For $t \geq 3$, consider a $(t+1)$-tuple of pairwise different elements of $\mathbb{P}_1(k) = k \cup \{\infty\}$, normalized so that $\lambda_1 = \infty$, $\lambda_2 = 0$, $\lambda_3 = 1$, and the admissible ideal $I(\lambda_1, \dots, \lambda_t)$ in the path algebra $k\Delta(p_1, \dots, p_t)$ of $\Delta(p_1, \dots, p_t)$ generated by the elements

$$\alpha_{ip_i} \dots \alpha_{i2}\alpha_{i1} + \alpha_{2p_2} \dots \alpha_{22}\alpha_{21} + \lambda_i \alpha_{1p_1} \dots \alpha_{12}\alpha_{11}, \qquad 3 \leq i \leq t.$$

Then the bound quiver algebra $\Lambda(p, \lambda) = k\Delta(p_1, \dots, p_t)/I(\lambda_1, \dots, \lambda_t)$ is said to be the *canonical algebra of type* $p = (p_1, \dots, p_t)$. Moreover, for $t = 2$, the path algebra $\Lambda(p) = k\Delta(p_1, p_2)$ is said to be the *canonical algebra of type* $p = (p_1, p_2)$. It has been proved in [36, Theorem 3.7] that if Λ is a canonical algebra of type $(p_1, \dots, p_t)$ then $\operatorname{ind}\Lambda = \mathcal{P}_\Lambda \vee \mathcal{T}_\Lambda \vee \mathcal{Q}_\Lambda$ for a $\mathbb{P}_1(k)$-family $\mathcal{T}_\Lambda$ of stable tubes of tubular type $(p_1, \dots, p_t)$, separating $\mathcal{P}_\Lambda$ from $\mathcal{Q}_\Lambda$.

Following [16] a connected algebra C is called a *concealed canonical algebra of type* $(p_1, \dots, p_t)$ if C is the endomorphism algebra $\mathrm{End}_\Lambda(T)$ for some canonical algebra Λ of type $(p_1, \dots, p_t)$ and a tilting Λ-module T whose indecomposable direct summands belong to $\mathcal{P}_\Lambda$. Then the images of modules from $\mathcal{T}_\Lambda$ via the functor $\mathrm{Hom}_\Lambda(T, -)$ form a separating family $\mathcal{T}_C$ of stable tubes of Γ_C, and in particular we have a decomposition ind $C = \mathcal{P}_C \vee \mathcal{T}_C \vee \mathcal{Q}_C$. It has been proved by Lenzing and de la Peña [17, Theorem 1.1] that the class of (connected) concealed canonical algebras coincides with the class of all connected algebras with a separating family of stable tubes. It is also known that the class of concealed canonical algebras of type (p_1, p_2) coincides with the class of hereditary algebras Euclidean types $\widetilde{\mathbb{A}}_m$, $m \geq 1$ (see [15]).

We need the following facts on the Hochschild cohomology of concealed canonical algebras, proved in [12, Proposition 1.6] and [13, Theorem 2.4].

Proposition 3.1. *Let C be a connected concealed canonical algebra of type (p_1, p_2). Then $H^0(C) \cong k$, $H^n(C) = 0$ for $n \geq 2$, and*

(i) $H^1(C) \cong k$ *if* $p_1, p_2 \geq 2$,
(ii) $H^1(C) \cong k^2$ *if* $p_1 = 1$, $p_2 \geq 2$,
(iii) $H^1(C) \cong k^3$ *if* $p_1 = p_2 = 1$.

Proposition 3.2. *Let C be a connected concealed canonical algebra of type $(p_1, \ldots, p_t)$ with $t \geq 3$. Then $H^0(C) \cong k$, $H^2(C) \cong k^{t-3}$, and $H^n(C) = 0$ for $n \neq 0, 2$.*

4. Admissible operations. Recall from [9, 36] that a translation quiver Γ is called a *tube* if it contains a cyclic path, and if its underlying topological space is homeomorphic to $S^1 \times \mathbb{R}^+$ (where S^1 is the unit circle, and $\mathbb{R}^+$ the nonnegative real half-line). A tube has only two types of arrows: arrows pointing to infinity and arrows pointing to the mouth. Tubes containing neither projective vertices nor injective vertices are called *stable*. Recall that if $\mathbb{A}_\infty$ is the quiver $0 \to 1 \to 2 \to \cdots$, then $\mathbb{Z}\mathbb{A}_\infty$ is the translation quiver

$$\begin{tikzcd}[column sep=small]
 & (i-1,0) \arrow[dr] & & (i,0) \arrow[dr] & & (i+1,0) \arrow[dr] & & (i+2,0) \arrow[dr] & \\
\iddots \arrow[ur] & & (i-1,1) \arrow[ur] \arrow[dr] & & (i,1) \arrow[ur] \arrow[dr] & & (i+1,1) \arrow[ur] \arrow[dr] & & \ddots \\
 & \iddots \arrow[ur] & & (i-1,2) \arrow[ur] \arrow[dr] & & (i,2) \arrow[ur] \arrow[dr] & & \ddots & \\
 & & \iddots \arrow[ur] & & \vdots \arrow[ur] & & \ddots & &
\end{tikzcd}$$

with $\tau(i, j) = (i-1, j)$ for $i \in \mathbb{Z}$, $j \in \mathbb{N}$. For $r \geq 1$, denote by $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ the translation quiver Γ obtained from $\mathbb{Z}\mathbb{A}_\infty$ by identifying each vertex (i, j) of $\mathbb{Z}\mathbb{A}_\infty$ with the vertex $\tau^r(i, j)$, and each arrow $x \to y$ in $\mathbb{Z}\mathbb{A}_\infty$ with the arrow $\tau^r x \to \tau^r y$. Translation quivers of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, $r \geq 1$, are called *stable tubes of rank* r. The *rank* of a stable tube Γ is the least positive integer r such that $\tau^r x = x$ for all x in Γ. A stable tube of rank 1 is said to be *homogeneous*. The τ-orbit of a stable tube Γ formed by all vertices having exactly one direct predecessor is said to be the *mouth* of Γ.

We also note that the generalized canonical algebras (introduced in [45]) provide a wide class of algebras whose Auslander–Reiten quivers admit generalized standard stable tubes.

It has been proved in [25, Theorem A] that a connected component Γ of Γ_A is almost cyclic and coherent if and only if Γ is a generalized multicoil obtained from a family of stable tubes by a sequence of operations called admissible. Our task in this section is to recall the latter and simultaneously define the corresponding enlargements of algebras.

For $r \geq 1$, we denote by $T_r(k)$ the $r \times r$ lower triangular matrix algebra

$$\begin{bmatrix} k & 0 & 0 & \dots & 0 & 0 \\ k & k & 0 & \dots & 0 & 0 \\ k & k & k & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k & k & k & \dots & k & 0 \\ k & k & k & \dots & k & k \end{bmatrix}$$

Given a generalized standard component Γ of Γ_A, and an indecomposable module X in Γ, the *support* $\mathcal{S}(X)$ of the functor $\mathrm{Hom}_A(X,-)|_\Gamma$ is the k-linear category defined as follows [4]. Let $\mathcal{H}_X$ denote the full subcategory of Γ consisting of all indecomposable modules M in Γ such that $\mathrm{Hom}_A(X,M) \neq 0$, and $\mathcal{I}_X$ denote the ideal of $\mathcal{H}_X$ consisting of all morphisms $f: M \to N$ (with M, N in $\mathcal{H}_X$) such that $\mathrm{Hom}_A(X,f) = 0$. We define $\mathcal{S}(X)$ to be the quotient category $\mathcal{H}_X/\mathcal{I}_X$. Following the above convention, we usually identify the k-linear category $\mathcal{S}(X)$ with its quiver.

From now on let A be an algebra and Γ be a family of generalized standard infinite components of Γ_A. For an indecomposable module X in Γ, called the *pivot*, one defines five admissible operations (ad 1)–(ad 5) and their duals (ad 1*)–(ad 5*) modifying the translation quiver $\Gamma = (\Gamma, \tau)$ to a new translation quiver (Γ', τ') and the algebra A to a new algebra A', depending on the shape of the support $\mathcal{S}(X)$ (see [25, Section 2] for figures illustrating the modified translation quivers Γ').

(ad 1) Let $t \in \mathbb{N}$ and assume $\mathcal{S}(X)$ consists of an infinite sectional path starting at X:

$$X = X_0 \to X_1 \to X_2 \to \cdots .$$

If $t \geq 1$ then $D = T_t(k)$ and $Y_1, \dots, Y_t$ denote indecomposable injective D-modules with $Y = Y_1$ the unique indecomposable projective-injective D-module. We define the *modified algebra* A' of A to be the one-point extension

$$A' = (A \times D)[X \oplus Y],$$

and the *modified translation quiver* Γ' of Γ to be obtained by inserting in Γ the rectangle consisting of the modules $Z_{ij} = \big(k, X_i \oplus Y_j, \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\big)$ for $i \geq 0$, $1 \leq j \leq t$, and $X'_i = (k, X_i, 1)$ for $i \geq 0$. The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 1$, $j \geq 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau' Z_{0j} = Y_{j-1}$ if $j \geq 2$, Z_{01} is projective, $\tau' X'_0 = Y_t$, $\tau' X'_i = Z_{i-1,t}$ if $i \geq 1$, $\tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not an injective A-module, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ, or Γ_D, respectively.

Finally, if $t = 0$ we define the modified algebra A' to be the one-point extension $A' = A[X]$, and the modified translation quiver Γ' to be the translation quiver obtained from Γ by inserting only the sectional path consisting of the vertices X'_i, $i \geq 0$.

The nonnegative integer t is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangle equals $t + 1$. We call t the *parameter* of the operation.

In case Γ is a stable tube, it is clear that any module on the mouth of Γ satisfies the condition for being a pivot for the above operation. Actually, the above operation is, in this case, the tube insertion as considered in [9].

(ad 2) Suppose that $\mathcal{S}(X)$ admits two sectional paths starting at X, one infinite and the other finite with at least one arrow:

$$Y_t \leftarrow \cdots \leftarrow Y_2 \leftarrow Y_1 \leftarrow X = X_0 \to X_1 \to X_2 \to \cdots$$

where $t \geq 1$. In particular, X is necessarily injective. We define the *modified algebra* A' of A to be the one-point extension $A' = A[X]$, and the *modified translation quiver* Γ' of Γ to be obtained by inserting in Γ the rectangle consisting of the modules $Z_{ij} = \big(k, X_i \oplus Y_j, \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\big)$ for $i \geq 1$, $1 \leq j \leq t$, and $X'_i = (k, X_i, 1)$ for $i \geq 1$. The translation τ' of Γ' is defined as follows: X'_0 is projective-injective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2$, $j \geq 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau' Z_{1j} = Y_{j-1}$ if $j \geq 2$, $\tau' X'_i = Z_{i-1,t}$ if $i \geq 2$, $\tau' X'_1 = Y_t$, $\tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not an injective A-module, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation τ of Γ.

The integer $t \geq 1$ is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangle equals $t + 1$. We call t the *parameter* of the operation.

(ad 3) Assume $\mathcal{S}(X)$ is the mesh-category of two parallel sectional paths:

$$\begin{array}{ccccccccccc} & & Y_1 & \to & Y_2 & \to & \cdots & \to & Y_t & & \\ & & \uparrow & & \uparrow & & & & \uparrow & & \\ X = & & X_0 & \to & X_1 & \to & \cdots & \to & X_{t-1} & \to & X_t \to \cdots \end{array}$$

where $t \geq 2$. In particular, X_{t-1} is necessarily injective. Moreover, we consider the translation quiver $\overline{\Gamma}$ of Γ obtained by deleting the arrows

$Y_i \to \tau_A^{-1} Y_{i-1}$. We assume that the union $\widehat{\Gamma}$ of the connected components of $\overline{\Gamma}$ containing the vertices $\tau_A^{-1} Y_{i-1}$, $2 \le i \le t$, is a finite translation quiver. Then $\overline{\Gamma}$ is the disjoint union of $\widehat{\Gamma}$ and a cofinite full translation subquiver Γ^*, containing the pivot X. We define the *modified algebra* A' of A to be the one-point extension $A' = A[X]$, and the *modified translation quiver* Γ' of Γ to be obtained from Γ^* by inserting the rectangle consisting of the modules $Z_{ij} = \left(k, X_i \oplus Y_j, \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right)$ for $i \ge 1$, $1 \le j \le t$, $j \le i$, and $X'_i = (k, X_i, 1)$ for $i \ge 1$. The translation τ' of Γ' is defined as follows: X'_0 is projective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \ge 2$, $2 \le j \le t$, $\tau' Z_{i1} = X_{i-1}$ if $i \ge 1$, $\tau' X'_i = Y_i$ if $1 \le i \le t$, $\tau' X'_i = Z_{i-1,t}$ if $i \ge t+1$, $\tau' Y_j = X'_{j-2}$ if $2 \le j \le t$, $\tau'(\tau^{-1} X_i) = X'_i$ if $i \ge t$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation τ of Γ^*. We note that X'_{t-1} is injective.

The integer $t \ge 2$ is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangle equals $t+1$. We call t the *parameter* of the operation.

(ad 4) Suppose that $\mathcal{S}(X)$ consists of an infinite sectional path starting at X,

$$X = X_0 \to X_1 \to X_2 \to \cdots,$$

and let

$$Y = Y_1 \to Y_2 \to \cdots \to Y_t$$

with $t \ge 1$ be a finite sectional path in Γ_A. Let $r \in \mathbb{N}$. Moreover, we consider the translation quiver $\overline{\Gamma}$ of Γ obtained by deleting the arrows $Y_i \to \tau_A^{-1} Y_{i-1}$. We assume that the union $\widehat{\Gamma}$ of the connected components of $\overline{\Gamma}$ containing the vertices $\tau_A^{-1} Y_{i-1}$, $2 \le i \le t$, is a finite translation quiver. Then $\overline{\Gamma}$ is the disjoint union of $\widehat{\Gamma}$ and a cofinite full translation subquiver Γ^*, containing the pivot X. For $r = 0$ we define the *modified algebra* A' of A to be the one-point extension $A' = A[X \oplus Y]$, and the *modified translation quiver* Γ' of Γ to be obtained from Γ^* by inserting the rectangle consisting of the modules $Z_{ij} = \left(k, X_i \oplus Y_j, \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right)$ for $i \ge 0$, $1 \le j \le t$, and $X'_i = (k, X_i, 1)$ for $i \ge 1$. The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \ge 1$, $j \ge 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \ge 1$, $\tau' Z_{0j} = Y_{j-1}$ if $j \ge 2$, Z_{01} is projective, $\tau' X'_0 = Y_t$, $\tau' X'_i = Z_{i-1,t}$ if $i \ge 1$, $\tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ^*.

For $r \ge 1$, let $G = T_r(k)$, let $U_{1,t+1}, U_{2,t+1}, \ldots, U_{r,t+1}$ denote the indecomposable projective G-modules, and $U_{r,t+1}, U_{r,t+2}, \ldots, U_{r,t+r}$ denote the indecomposable injective G-modules, with $U_{r,t+1}$ the unique indecomposable projective-injective G-module. We define the *modified algebra* A' of A

to be the triangular matrix algebra of the form

$$A' = \begin{bmatrix} A & 0 & 0 & \dots & 0 & 0 \\ Y & k & 0 & \dots & 0 & 0 \\ Y & k & k & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Y & k & k & \dots & k & 0 \\ X \oplus Y & k & k & \dots & k & k \end{bmatrix}$$

with $r+2$ columns and rows, and the *modified translation quiver* Γ' of Γ to be obtained from Γ^* by inserting the rectangles consisting of the modules $U_{sl} = \big(k, Y_l \oplus U_{s,t+1}, \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\big)$ for $1 \le s \le r$, $1 \le l \le t$, and $Z_{ij} = \big(k, X_i \oplus U_{rj}, \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\big)$ for $i \ge 0$, $1 \le j \le t+r$, and $X'_i = (k, X_i, 1)$ for $i \ge 0$. The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \ge 1$, $j \ge 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \ge 1$, $\tau' Z_{0j} = U_{r,j-1}$ if $2 \le j \le t+r$, Z_{01}, U_{k1}, $1 \le k \le r$ are projective, $\tau' U_{kl} = U_{k-1,l-1}$ if $2 \le k \le r$, $2 \le l \le t+r$, $\tau' U_{1l} = Y_{l-1}$ if $2 \le l \le t+1$, $\tau' X'_0 = U_{r,t+r}$, $\tau' X'_i = Z_{i-1,t+r}$ if $i \ge 1$, $\tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ^*, or Γ_G, respectively.

We note that the quiver $Q_{A'}$ of A' is obtained from the quiver of the double one-point extension $A[X][Y]$ by adding a path of length $r+1$ with source at the extension vertex of $A[X]$ and sink at the extension vertex of $A[Y]$.

The integers $t \ge 1$ and $r \ge 0$ are such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangles equals $t+r+1$. We call $t+r$ the *parameter* of the operation.

To define of the next admissible operation we also need finite versions of the admissible operations (ad 1)–(ad 4), which we denote by (fad 1)–(fad 4), respectively. In order to obtain these operations we replace all infinite sectional paths of the form $X_0 \to X_1 \to X_2 \to \cdots$ (in the definitions of (ad 1)–(ad 4)) by finite sectional paths of the form $X_0 \to X_1 \to X_2 \to \cdots \to X_s$. For the operation (fad 1) we have $s \ge 0$, for (fad 2) and (fad 4) we have $s \ge 1$, and for (fad 3) we have $s \ge t-1$. In all the above operations, X_s is injective (see [25] or [26] for the details).

(ad 5) We define the *modified algebra* A' of A to be the iteration of the extensions described in the definitions of the admissible operations (ad 1)–(ad 4), and their finite versions corresponding to the operations (fad 1)–(fad 4). The *modified translation quiver* Γ' of Γ is obtained in the following three steps: first we perform on Γ one of the operations (fad 1)–(fad 3), next a finite number (possibly empty) of times the operation (fad 4) and finally the operation (ad 4), all in such a way that the sectional paths starting from all the new projective vertices have a common cofinite (infinite) sectional subpath.

Finally, together with the admissible operations (ad 1)–(ad 5), we consider their duals, denoted by (ad 1*)–(ad 5*). These ten operations are now called the *admissible operations*. Following [25] a connected translation quiver Γ is said to be a *generalized multicoil* if Γ can be obtained from a finite family $\mathcal{T}_1, \dots, \mathcal{T}_s$ of stable tubes by iterated application of admissible operations (ad 1), (ad 1*), (ad 2), (ad 2*), (ad 3), (ad 3*), (ad 4), (ad 4*), (ad 5) or (ad 5*). If $s = 1$, such a translation quiver Γ is said to be a *generalized coil*. The admissible operations of types (ad 1)–(ad 3), (ad 1*)–(ad 3*) have been introduced in [2, 4, 5], and the admissible operations (ad 4) and (ad 4*) for $r = 0$ in [20].

Observe that any stable tube is trivially a generalized coil. A *tube* (in the sense of [9]) is a generalized coil having the property that each admissible operation in the sequence defining it is of the form (ad 1) or (ad 1*). If we apply only operations of type (ad 1) (respectively, of type (ad 1*)) then such a generalized coil is called a *ray tube* (respectively, a *coray tube*). Observe that a generalized coil without injective (respectively, projective) vertices is a ray tube (respectively, a coray tube). A *quasi-tube* (in the sense of [39]) is a generalized coil with the property that each of the admissible operations in the sequence defining it is of type (ad 1), (ad 1*), (ad 2) or (ad 2*). Finally, following [3] a *coil* is a generalized coil having the property that each of the admissible operations in the sequence defining it is of one of the forms (ad 1), (ad 1*), (ad 2), (ad 2*), (ad 3) or (ad 3*). We note that any generalized multicoil Γ is a coherent translation quiver with trivial valuations, and its *cyclic part* ${}_c\Gamma$ (the translation subquiver of Γ obtained by removing from Γ all acyclic vertices and the arrows attached to them) is infinite, connected and cofinite in Γ, and so Γ is almost cyclic.

Finally, let C be a (not necessarily connected) concealed canonical algebra and $\mathcal{T}_C$ a separating family of stable tubes of Γ_C. Following [26] we say that an algebra A is a *generalized multicoil enlargement* of C using modules from $\mathcal{T}_C$ if there exists a sequence of algebras

$$C = A_0, A_1, \dots, A_n = A$$

such that A_{i+1} is obtained from A_i by an admissible operation of one of the types (ad 1)–(ad 5), (ad 1*)–(ad 5*) performed either on stable tubes of $\mathcal{T}_{A_i}$, or on generalized multicoils obtained from stable tubes of $\mathcal{T}_{A_i}$ by means of operations done so far. Observe that this definition extends the concept of a coil enlargement of a concealed canonical algebra introduced in [5]. We note that a generalized multicoil enlargement A of C invoking only admissible operations of type (ad 1) (respectively, of type (ad 1*)) is a tubular extension (respectively, tubular coextension) of C in the sense of [36]. An algebra A is said to be a *generalized multicoil algebra* if A is a connected generalized multicoil enlargement of a product C of connected concealed canonical algebras.

In order to formulate our main result, we define some numerical invariants of A. Let A be a generalized multicoil enlargement of a concealed canonical algebra C. Let $C = C_1 \times \cdots \times C_l \times C_{l+1} \times \cdots \times C_m$ be a decomposition of C into a product of connected algebras such that $C_1, \ldots, C_l$ are of type (p_1, p_2) and $C_{l+1}, \ldots, C_m$ are of type $(p_1, \ldots, p_t)$ with $t \geq 3$. By h_i we denote the number of all homogeneous tubes from Γ_{C_i} with $1 \leq i \leq l$ used in the whole process of creating A from C, and $h_i = 0$ if $l+1 \leq i \leq m$. Moreover, let

$$e_i = \begin{cases} 0 & \text{if } C_i \text{ is of type } (p_1, \ldots, p_t) \text{ with } t \geq 3, \\ 1 & \text{if } C_i \text{ is of type } (p_1, p_2) \text{ with } p_1, p_2 \geq 2, \\ 2 & \text{if } C_i \text{ is of type } (p_1, p_2) \text{ with } p_1 = 1,\ p_2 \geq 2, \\ 3 & \text{if } C_i \text{ is of type } (p_1, p_2) \text{ with } p_1 = p_2 = 1, \end{cases}$$

for $i \in \{1, \ldots, m\}$. We also define $f_{C_i} = \max(e_i - h_i, 0)$ for $i \in \{1, \ldots, m\}$ and set $f_A = \sum_{i=1}^{m} f_{C_i} = \sum_{i=1}^{l} f_{C_i}$. Note that we can apply (ad 4), (fad 4), (ad 4*), (fad 4*) in two ways. The first way is when the sectional paths occurring in the definitions of these operations come from a component or two components of the same connected algebra. The second one is when these sectional paths come from two components of two connected algebras. We denote by d_A the number of operations of type (ad 4), (fad 4), (ad 4*) and (fad 4*) used in the whole process of creating A from C which are of the first type. Moreover, we denote by v_i the number of all admissible operations of type (ad 1), (ad 1*), (ad 4), (ad 4*), (ad 5) and (ad 5*) applied to an indecomposable pivot or copivot A_i-module X lying on the mouth of a stable tube of rank one in the whole process of creating of A from C. Next, we define the nonnegative integer numbers $r_i = \max(v_i - e_i, 0)$ and $r_A = \sum_{i=1}^{m} r_i$. Finally, we set $p_i = \max(t_i - 3, 0)$, $1 \leq i \leq m$, where t_i is the number of arms of the algebra C_i, and define the invariant p_A of A by $p_A = \sum_{i=1}^{m} p_i = \sum_{i=l+1}^{m} p_i$.

Proposition 4.1 ([26, Proposition 3.7]). *Let C be a concealed canonical algebra, $\mathcal{T}_C$ a separating family of stable tubes of Γ_C, and A a generalized multicoil enlargement of C using modules from $\mathcal{T}_C$. Then Γ_A admits a generalized standard family $\mathcal{C}_A$ of generalized multicoils obtained from the family $\mathcal{T}_C$ of stable tubes by a sequence of admissible operations corresponding to the admissible operations leading from C to A.*

Let A be an algebra with a separating family $\mathcal{C}_A$ of almost cyclic coherent components in Γ_A, and $\operatorname{ind} A = \mathcal{P}_A \vee \mathcal{C}_A \vee \mathcal{Q}_A$. Then, by [26, Theorem C], there are uniquely determined quasitilted algebras A_l and A_r such that $\mathcal{P}_A = \mathcal{P}_{A_l}$ and $\mathcal{Q}_A = \mathcal{Q}_{A_r}$. Moreover, from the proof of [26, Theorem C] we obtain the following fact.

PROPOSITION 4.2. *Let A be an algebra with a separating family of almost cyclic coherent components in Γ_A. Then:*

(i) *A can be obtained from A_l by a sequence of admissible operations of types* (ad 1)–(ad 5).
(ii) *A can be obtained from A_r by a sequence of admissible operations of types* (ad 1*)–(ad 5*).

5. Proof of Theorem 1.1

(i) It follows from [26, Theorem E(iv)] that $\operatorname{gl.dim} A \leq 3$, and hence $H^n(A) = 0$ for $n \geq 4$. Moreover, we know (see details below) that A can be obtained from a concealed canonical algebra C by one-point extensions or one-point coextensions. Applying [27, Proposition 2.6], where we proved that if M is a module in $\operatorname{add} \mathcal{C}_A$ then $\operatorname{Ext}^r_A(M, M) = 0$ for $r \geq 2$, Propositions 3.1, 3.2 and Theorem 2.3 (and its dual) we obtain $H^3(A) = 0$.

(iv) It follows from [26, Corollary B] that the ordinary quiver of A has no oriented cycles. Since A is connected, the center of A is a field, so that $H^0(A) \cong k$. Hence $\dim_k H^0(A) = 1$.

(ii), (iii) From [26, Theorem A] we may assume that A is a generalized multicoil enlargement of a concealed canonical algebra C. Let $C = C_1 \times \cdots \times C_m$ be a decomposition of C into a product of connected algebras and let h_i, e_i, v_i, r_i, p_i and p_A, r_A, d_A, f_A be as above. We shall prove our claims by induction on the number n of admissible operations leading from C to the algebra A.

Assume $n = 1$. Then we can only apply an admissible operation of type (ad 1) or (ad 1*). In particular, we have $m = 1$ and $d_A = 0$. Assume that A is obtained from C by applying an operation of type (ad 1). Then $A = C[X]$ if $t = 0$, and $A = (C \times D)[X \oplus Y]$ if $t \geq 1$, where the C-module X is the pivot and Y is the unique indecomposable projective-injective D-module (see definition of (ad 1)). From Theorem 2.3 and the statements (i), (iv) established above, we have the following exact sequences:

$$0 \to k \to k \to \operatorname{Hom}_C(X, X)/k \to H^1(A) \to H^1(C) \to \operatorname{Ext}^1_C(X, X) \to H^2(A) \to H^2(C) \to \operatorname{Ext}^2_C(X, X) \to 0$$

if $t = 0$, and

$$0 \to k \to k^2 \to \operatorname{Hom}_{C\times D}(X \oplus Y, X \oplus Y)/k \to H^1(A) \to H^1(C \times D) \to \operatorname{Ext}^1_{C\times D}(X \oplus Y, X \oplus Y) \to H^2(A) \to H^2(C \times D) \to \operatorname{Ext}^2_{C\times D}(X \oplus Y, X \oplus Y) \to 0$$

if $t \geq 1$. In this case X is an indecomposable module from the mouth of a stable tube of Γ_C, and Y is either zero or a directing D-module. Therefore we have $\mathrm{Hom}_C(X,X) \cong k$, $\mathrm{Hom}_D(Y,Y) \cong k$, and hence

$$\mathrm{Hom}_{C\times D}(X\oplus Y, X\oplus Y) \cong \mathrm{Hom}_C(X,X) \oplus \mathrm{Hom}_D(Y,Y) \cong k\oplus k \cong k^2.$$

Moreover,

$$\mathrm{Ext}^i_{C\times D}(X\oplus Y, X\oplus Y) \cong \mathrm{Ext}^i_C(X,X) \oplus \mathrm{Ext}^i_D(Y,Y) \cong \mathrm{Ext}^i_C(X,X)$$

for $i \geq 0$. Always $H^i(C\times D) \cong H^i(C)$ for $i = 1,2$. We have the following four cases:

(1) $l = 1$, $h_1 = 1$. Then $\mathrm{Ext}^1_C(X,X) \cong k$ and $\mathrm{Ext}^2_C(X,X) = 0$, because $\mathrm{pd}_C X = 1$. Moreover, by our assumption on C and by Proposition 2.1, we have $H^2(C) = 0$. Hence, from the above exact sequences, we deduce that $\dim_k H^1(A) = \dim_k H^1(C) - 1 = d_A + f_A$ and $\dim_k H^2(A) = \dim_k H^2(C) = 0 = p_A + r_A$.

(2) $l = 1$, $h_1 = 0$. Then $\mathrm{Ext}^1_C(X,X) = \mathrm{Ext}^2_C(X,X) = 0$. Moreover, by our assumption on C, we have $H^2(C) = 0$. Then from the above exact sequences we obtain $\dim_k H^1(A) = \dim_k H^1(C) = e_1 = f_{C_1} = d_A + f_A$, $\dim_k H^2(A) = \dim_k H^2(C) = 0 = p_A + r_A$.

(3) $l = 0$, $h_1 = 0$, $v_1 = 1$. Then we get $\mathrm{Ext}^1_C(X,X) \cong k$, $\mathrm{Ext}^2_C(X,X) = 0$. Moreover, by our assumption on C, we have $H^1(C) = 0$. So from the above exact sequences we find that $\dim_k H^1(A) = \dim_k H^1(C) = 0 = d_A + f_A$, $\dim_k H^2(A) = \dim_k H^2(C) + 1 = p_1 + 1 = p_A + 1 = p_A + r_A$.

(4) $l = 0$, $h_1 = 0$, $v_1 = 0$. Then $\mathrm{Ext}^1_C(X,X) = \mathrm{Ext}^2_C(X,X) = 0$. Moreover, by our assumption on C, we have $H^1(C) = 0$. Then from the above exact sequences we infer that $\dim_k H^1(A) = \dim_k H^1(C) = 0 = d_A + f_A$, $\dim_k H^2(A) = \dim_k H^2(C) = p_A + r_A$.

If the algebra A is obtained from C by applying (ad 1*), then the proof is dual.

Let $n > 1$ and Γ be a generalized multicoil of Γ_A which is obtained from a finite family $\mathcal{T}_1, \ldots, \mathcal{T}_s$ of stable tubes of Γ_C. Assume that the statement holds for $n-1$, so after applying $n-1$ admissible operations we have a disjoint union of a finite family of generalized multicoils $\Omega_1, \ldots, \Omega_q$, $1 \leq q \leq s$. If the nth admissible operation is of type (ad 1), (ad 1*), (ad 2), (ad 2*), (ad 3) or (ad 3*) then $q = 1$, so Γ is obtained from Ω_1. Note that we can apply an admissible operation (ad 2), (ad 3), or (ad 4) (that is, also (ad 5)) (respectively (ad 2*), (ad 3*), (ad 4*)) if the number of all successors (respectively, predecessors) of the module Y_i (which occurs in the definitions of the above admissible operations) is finite for each $1 \leq i \leq t$. Indeed, if this is not the case, then the family of generalized multicoils obtained after applying such admissible operations is not sincere, and then it is not separating. If the nth admissible operation is of type (ad 1), then let $A = B[X]$ if $t = 0$, and $A = (B \times D)[X \oplus Y]$ if $t \geq 1$, where X is an indecomposable nondirecting B-module (pivot) and Y is either zero or a directing D-module. Therefore $\mathrm{Hom}_B(X,X) \cong k$, $\mathrm{Hom}_{B\times D}(X \oplus Y, X \oplus Y) \cong \mathrm{Hom}_B(X,X) \oplus \mathrm{Hom}_D(Y,Y) \cong k \oplus k \cong k^2$, and $\mathrm{Ext}^i_{B\times D}(X \oplus Y, X \oplus Y) \cong \mathrm{Ext}^i_B(X,X) \oplus \mathrm{Ext}^i_D(Y,Y) \cong \mathrm{Ext}^i_B(X,X)$ for $i \geq 1$. Moreover, $H^1(B \times D) \cong H^1(B)$ and $H^2(B \times D) \cong H^2(B)$. Then from Theorem 2.3 and the statements (i), (iv) we have the following exact sequences:

$$0 \to k \to k \to 0 \to H^1(A) \to H^1(B) \xrightarrow{\eta} \mathrm{Ext}^1_B(X,X) \\ \to H^2(A) \to H^2(B) \to \mathrm{Ext}^2_B(X,X) \to 0$$

if $t = 0$, and

$$0 \to k \to k^2 \to k \xrightarrow{0} H^1(A) \to H^1(B) \xrightarrow{\eta} \mathrm{Ext}^1_B(X,X) \\ \to H^2(A) \to H^2(B) \to \mathrm{Ext}^2_B(X,X) \to 0$$

if $t \geq 1$.

Note that B is a connected algebra. We will show that $\mathrm{Ext}^2_B(X,X) = 0$. Clearly, we may assume that X is noninjective. Denote by Ω the generalized multicoil of Γ_B containing X. Let $0 \to X \xrightarrow{j} E$ be an injective envelope of X in $\mathrm{mod}\, B$, and V be the cokernel of j. Then $\mathrm{Ext}^2_B(X,X) \cong \mathrm{Ext}^1_B(X,V)$. In order to prove that $\mathrm{Ext}^2_B(X,X) = 0$, it is sufficient to show, by [40, Lemma 5.7], that $\mathrm{mod}\, B$ has no path $W \to \cdots \to X$ with W an indecomposable direct summand of V. Since Ω is standard (see [25, Section 3]) and convex, we infer from the proof of [40, Theorem 5.1] that $\mathrm{Ext}^2_B(X,X) = 0$. Note that the Auslander–Reiten formula gives an isomorphism $\mathrm{Ext}^1_B(X,X) \cong D\underline{\mathrm{Hom}}_B(\tau^- X, X)$. We have three cases:

(1) $f_A = f_B$, $r_A = r_B + 1$. Then $\mathrm{Ext}^1_B(X,X) \cong k$, $\eta = 0$ and from the above exact sequences we deduce that $\dim_k H^1(A) = \dim_k H^1(B) = d_B + f_B = d_A + f_A$, $\dim_k H^2(A) = \dim_k H^2(B) + 1 = p_B + r_B + 1 = p_A + r_A$.

(2) $f_B \neq 0$, $f_A = f_B - 1$, $r_A = r_B$. Then $\mathrm{Ext}^1_B(X,X) \cong k$, $\eta \neq 0$ and from the above exact sequences we find that $\dim_k H^1(A) = \dim_k H^1(B) - 1 = d_B + f_B - 1 = d_A + f_A$, $\dim_k H^2(A) = \dim_k H^2(B) = p_B + r_B = p_A + r_A$.

(3) $f_A = f_B$, $r_A = r_B$. Then $\mathrm{Ext}^1_B(X,X) = 0$ and from the above exact sequences we infer that $\dim_k H^1(A) = \dim_k H^1(B) = d_B + f_B = d_A + f_A$, $\dim_k H^2(A) = \dim_k H^2(B) = p_B + r_B = p_A + r_A$.

If the operation is of type (ad 1*), then the proof is dual.

If the nth admissible operation is of type (ad 2) then let A be the algebra obtained from B by applying this admissible operation with pivot X, so $A = B[X]$. Note that B is a connected algebra. From Theorem 2.3 and the statements (i), (iv) we have the exact sequence

$$(*) \quad 0 \to k \to k \to \mathrm{Hom}_B(X,X)/k \to H^1(A) \to H^1(B) \to \mathrm{Ext}^1_B(X,X) \\ \to H^2(A) \to H^2(B) \to \mathrm{Ext}^2_B(X,X) \to 0.$$

From the definition of the operation (ad 2) we know that X is an injective B-module, so $\mathrm{Ext}_B^1(X,X) = \mathrm{Ext}_B^2(X,X) = 0$. Since X is the pivot, we get $\mathrm{Hom}_B(X,X) \cong k$. Moreover, $d_A = d_B$, $f_A = f_B$, $p_A = p_B$, $r_A = r_B$, for arbitrary $t \geq 1$. Therefore, by the above remarks, by the exact sequence $(*)$ and by our inductive assumption we have $\dim_k H^1(A) = \dim_k H^1(B) = d_B + f_B = d_A + f_A$, $\dim_k H^2(A) = \dim_k H^2(B) = p_B + r_B = p_A + r_A$.

If the admissible operation leading from B to A is of type (ad 2*), then the proof is dual.

If the nth admissible operation is of type (ad 3) then let A be the algebra obtained from B by applying this admissible operation with pivot X, so $A = B[X]$. Note that B is a connected algebra. Again, from Theorem 2.3 and the statements (i), (iv) we have the exact sequence $(*)$. Denote by Ω the generalized multicoil of Γ_B containing X. Let $0 \to X \xrightarrow{j} E$ be an injective envelope of X in $\mathrm{mod}\, B$ and V be the cokernel of j. Then $\mathrm{Ext}_B^2(X,X) \cong \mathrm{Ext}_B^1(X,V)$. Again, in order to prove that $\mathrm{Ext}_B^2(X,X) = 0$, it is sufficient to show, by [40, Lemma 5.7], that $\mathrm{mod}\, B$ has no path $W \to \cdots \to X$ with W an indecomposable direct summand of V. Since Ω is standard (see [25, Section 3]) and convex we infer from the proof of [40, Theorem 5.1] that $\mathrm{Ext}_B^2(X,X) = 0$. Moreover, the Auslander–Reiten formula gives $\mathrm{Ext}_B^1(X,X) \cong D\underline{\mathrm{Hom}}_B(\tau^- X, X)$, and by definition of (ad 3) we have $D\underline{\mathrm{Hom}}_B(\tau^- X, X) = D\,\mathrm{Hom}_B(\tau^- X, X) = D\,\mathrm{Hom}_B(Y_2, X) = 0$, where Y_2 is a directing B-module. Again, X is the pivot, which yields $\mathrm{Hom}_B(X,X) \cong k$. We have $d_A = d_B$, $f_A = f_B$, $p_A = p_B$, $r_A = r_B$, for arbitrary $t \geq 2$. Hence, by the above remarks, by the exact sequence $(*)$ and by our inductive assumption we have $\dim_k H^1(A) = \dim_k H^1(B) = d_B + f_B = d_A + f_A$, $\dim_k H^2(A) = \dim_k H^2(B) = p_B + r_B = p_A + r_A$.

If the admissible operation leading from B to A is of type (ad 3*), then the proof is dual.

If the nth admissible operation is of type (ad 4) then let A be the algebra obtained from B by applying this admissible operation with pivot X, so for $r = 0$, $A = B[X \oplus Y]$, and for $r \geq 1$,

$$A = \begin{bmatrix} B & 0 & 0 & \dots & 0 & 0 \\ Y & k & 0 & \dots & 0 & 0 \\ Y & k & k & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Y & k & k & \dots & k & 0 \\ X \oplus Y & k & k & \dots & k & k \end{bmatrix}$$

with $r+2$ columns and rows. In this case $q = 1$ or $q = 2$, so Γ is obtained from Ω or from the disjoint union of two generalized multicoils Ω, Ω'.

If $r = 0$, then by Theorem 2.3 and condition (i) we have the exact sequence

$$0 \to H^0(A) \to H^0(B) \to \mathrm{Hom}_B(X \oplus Y, X \oplus Y)/k \to H^1(A) \to H^1(B) \\ \to \mathrm{Ext}^1_B(X \oplus Y, X \oplus Y) \to H^2(A) \to H^2(B) \to \mathrm{Ext}^2_B(X \oplus Y, X \oplus Y) \to 0.$$

If $r \geq 1$, then observe that the modified algebra A of B can be obtained by applying $r+1$ one-point extensions in the following way: $B_0 = B[Y_0]$, $B_1 = B_0[Y_1]$, $B_2 = B_1[Y_2], \dots, B_{r-1} = B_{r-2}[Y_{r-1}]$ and finally $A = B_r = B_{r-1}[X \oplus Y_r]$, where $Y_0 = Y$, and Y_j is a projective B_{j-1}-module such that $\operatorname{rad} Y_j = Y_{j-1}$, for $r \geq 1$, $1 \leq j \leq r$. Therefore, from Theorem 2.3 and the statement (i) we have the following finite sequence of exact sequences:

$$0 \to H^0(B_0) \to H^0(B) \to \mathrm{Hom}_B(Y_0, Y_0)/k \to H^1(B_0) \to H^1(B) \\ \to \mathrm{Ext}^1_B(Y_0, Y_0) \to H^2(B_0) \to H^2(B) \to \mathrm{Ext}^2_B(Y_0, Y_0) \to 0,$$

$$0 \to H^0(B_i) \to H^0(B_{i-1}) \to \mathrm{Hom}_{B_{i-1}}(Y_i, Y_i)/k \to H^1(B_i) \to H^1(B_{i-1}) \\ \to \mathrm{Ext}^1_{B_{i-1}}(Y_i, Y_i) \to H^2(B_i) \to H^2(B_{i-1}) \to \mathrm{Ext}^2_{B_{i-1}}(Y_i, Y_i) \to 0,$$

for $i = 1, \dots, r-1$, and

$$0 \to H^0(A) \to H^0(B_{r-1}) \to \mathrm{Hom}_{B_{r-1}}(X \oplus Y_r, X \oplus Y_r)/k \to H^1(A) \\ \to H^1(B_{r-1}) \to \mathrm{Ext}^1_{B_{r-1}}(X \oplus Y_r, X \oplus Y_r) \to H^2(A) \to H^2(B_{r-1}) \\ \to \mathrm{Ext}^2_{B_{r-1}}(X \oplus Y_r, X \oplus Y_r) \to 0.$$

Since the modules $Y, Y_1, \dots, Y_r$ are directing and X is a pivot, we get

$$\mathrm{Hom}_B(X \oplus Y, X \oplus Y) \cong \mathrm{Hom}_B(X, X) \oplus \mathrm{Hom}_B(Y, Y) \cong k \oplus k \cong k^2,$$
$$\mathrm{Ext}^j_B(X \oplus Y, X \oplus Y) \cong \mathrm{Ext}^j_B(X, X) \oplus \mathrm{Ext}^j_B(Y, Y) \cong \mathrm{Ext}^j_B(X, X),$$

for $j = 1, 2$,

$$\mathrm{Hom}_B(Y_0, Y_0) \cong k, \quad \mathrm{Ext}^1_B(Y_0, Y_0) = \mathrm{Ext}^2_B(Y_0, Y_0) = 0,$$
$$\mathrm{Hom}_{B_{i-1}}(Y_i, Y_i) \cong k, \quad \mathrm{Ext}^1_{B_{i-1}}(Y_i, Y_i) = \mathrm{Ext}^2_{B_{i-1}}(Y_i, Y_i) = 0,$$

for $i = 1, \dots, r-1$,

$$\mathrm{Hom}_{B_{r-1}}(X \oplus Y_r, X \oplus Y_r) \cong \mathrm{Hom}_{B_{r-1}}(X, X) \oplus \mathrm{Hom}_{B_{r-1}}(Y_r, Y_r) \\ \cong k \oplus k \cong k^2,$$
$$\mathrm{Ext}^j_{B_{r-1}}(X \oplus Y_r, X \oplus Y_r) \cong \mathrm{Ext}^j_{B_{r-1}}(X, X) \oplus \mathrm{Ext}^j_{B_{r-1}}(Y_r, Y_r) \\ \cong \mathrm{Ext}^j_{B_{r-1}}(X, X)$$

for $j = 1, 2$. Therefore, for $r \geq 1$, we obtain

$$H^0(B_0) \cong H^0(B), \qquad H^1(B_0) \cong H^1(B), \qquad H^2(B_0) \cong H^2(B),$$
$$H^0(B_i) \cong H^0(B_{i-1}), \quad H^1(B_i) \cong H^1(B_{i-1}), \quad H^2(B_i) \cong H^2(B_{i-1}),$$

where $i = 1, \dots, r-1$. These isomorphisms imply $H^j(B) \cong H^j(B_{r-1})$, $j = 0, 1, 2$.

Hence, for $r = 0$ we have the exact sequence

$$(**) \quad 0 \to H^0(A) \to H^0(B) \to k \to H^1(A) \to H^1(B) \xrightarrow{\eta} \mathrm{Ext}^1_B(X,X) \\ \to H^2(A) \to H^2(B) \to \mathrm{Ext}^2_B(X,X) \to 0,$$

and for $r \geq 1$ we have the exact sequence

$$0 \to H^0(A) \to H^0(B) \to k \to H^1(A) \to H^1(B) \to \mathrm{Ext}^1_{B_{r-1}}(X,X) \\ \to H^2(A) \to H^2(B) \to \mathrm{Ext}^2_{B_{r-1}}(X,X) \to 0.$$

In order to simplify the notation we will consider the case when $r = 0$, so X is an B-module, because the proof for $r \geq 1$ is the same (just replace B by B_{r-1}). Note that in our situation the algebra B is not necessarily connected. Without loss of generality we may assume that X belongs to the generalized multicoil Ω of Γ_B. We shall show now that $\mathrm{Ext}^2_B(X,X) = 0$. Clearly, we may assume that X is noninjective. Let $0 \to X \xrightarrow{j} E$ be an injective envelope of X in $\operatorname{mod} B$, and V be the cokernel of j. Then $\mathrm{Ext}^2_B(X,X) \cong \mathrm{Ext}^1_B(X,V)$. Again, in order to prove that $\mathrm{Ext}^2_B(X,X) = 0$, it is sufficient to show, by [40, Lemma 5.7], that $\operatorname{mod} B$ has no path $W \to \cdots \to X$ with W an indecomposable direct summand of V. In our case the full subcategory $\mathrm{Hom}_B(X,\Omega)$ of the vector space category $\mathrm{Hom}_B(X, \operatorname{mod} B)$ consisting of all objects $\mathrm{Hom}_B(X,V) \neq 0$ with V from Ω has the form

$$\mathrm{Hom}_B(X,X_0) \to \mathrm{Hom}_B(X,X_1) \to \mathrm{Hom}_B(X,X_2) \to \cdots$$

with $X = X_0$.

If all modules X_i, $i \geq 0$, are noninjective then E is a direct sum of modules which do not belong to Ω. Then, since Ω is convex, $\operatorname{mod} B$ has no path $W \to \cdots \to X$ with W an indecomposable direct summand of V.

Hence assume that one of the modules X_i is injective. Let s be the smallest index such that X_s is injective. Then Ω contains a full translation subquiver

$$\begin{array}{ccccccccccc}
 & & Z_1 & \to & Z_2 & \to & \cdots & \to & Z_s & & \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \\
X = X_0 & \to & X_1 & \to & X_2 & \to & \cdots & \to & X_s & \to X_{s+1} & \to \cdots
\end{array}$$

and there is an exact sequence (see [2, (2.2)])

$$0 \to X \to X_s \to Z_s \to 0.$$

Since X_s is indecomposable, we get $E \cong X_s$ and $V \cong Z_s$. Moreover, Ω is standard (see [25, Section 3]) and is the convex generalized multicoil of Γ_B, which implies that there is no path in $\operatorname{mod} B$ from Z_s to X, and so our claim follows.

Assume first that B is a connected. Then by (iv) we get $H^0(B) \cong H^0(A) \cong k$, and by our inductive assumption we have $\dim_k H^1(B) = d_B + f_B$, $\dim_k H^2(B) = p_B + r_B$. Moreover, $d_A = d_B + 1$. Hence, by the above remarks and from the exact sequence $(**)$ we get three cases:

(1) $f_A = f_B$, $r_A = r_B$, $p_A = p_B$. Then $\mathrm{Ext}^1_B(X, X) = 0$ and we deduce that $\dim_k H^1(A) = \dim_k H^1(B) + 1 = d_B + f_B + 1 = d_A + f_A$, $\dim_k H^2(A) = \dim_k H^2(B) = p_B + r_B = p_A + r_A$.

(2) $f_A = f_B$, $r_A = r_B + 1$, $p_A = p_B$. Then $\mathrm{Ext}^1_B(X, X) \cong k$, $\eta = 0$ and we infer that $\dim_k H^1(A) = \dim_k H^1(B) + 1 = d_B + f_B + 1 = d_A + f_A$, $\dim_k H^2(A) = \dim_k H^2(B) + 1 = p_B + r_B + 1 = p_A + r_A$.

(3) $f_B \neq 0$, $f_A = f_B - 1$, $r_A = r_B$, $p_A = p_B$. Then $\mathrm{Ext}^1_B(X, X) \cong k$, $\eta \neq 0$ and we have the factorization

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & k \arrow[r] & k \arrow[r, "0"] & k \arrow[r] & H^1(A) \arrow[rr] \arrow[dr] & & H^1(B) \arrow[r, "\eta"] & k \arrow[r, "0"] & H^2(A) \arrow[r] & H^2(B) \arrow[r] & 0 \\ & & & & & k^m \arrow[ur] & & & & & \end{tikzcd}$$

where $m = \dim_k H^1(B) - 1$. Note that $f_B \neq 0$, so $\dim_k H^1(B) \geq 1$. Hence, $\dim_k H^1(A) = \dim_k H^1(B) = d_B + f_B = d_A - 1 + f_A + 1 = d_A + f_A$, $\dim_k H^2(A) = \dim_k H^2(B) = p_B + r_B = p_A + r_A$.

Assume now that B is not connected. Then by (iv) we see that $H^0(B) \cong k^2$, $H^0(A) \cong k$, and by our inductive assumption we have $\dim_k H^1(B) = d_B + f_B$, $\dim_k H^2(B) = p_B + r_B$. Moreover, $d_A = d_B$. Hence, by the above remarks and from the exact sequence $(**)$ we again get three cases:

(1) $f_A = f_B$, $r_A = r_B$, $p_A = p_B$. Then $\mathrm{Ext}^1_B(X, X) = 0$ and we find that $\dim_k H^1(A) = \dim_k H^1(B) = d_B + f_B = d_A + f_A$, $\dim_k H^2(A) = \dim_k H^2(B) = p_B + r_B = p_A + r_A$.

(2) $f_A = f_B$, $r_A = r_B + 1$, $p_A = p_B$. Then $\mathrm{Ext}^1_B(X, X) \cong k$, $\eta = 0$ and we deduce that $\dim_k H^1(A) = \dim_k H^1(B) = d_B + f_B = d_A + f_A$, $\dim_k H^2(A) = \dim_k H^2(B) + 1 = p_B + r_B + 1 = p_A + r_A$.

(3) $f_B \neq 0$, $f_A = f_B - 1$, $r_A = r_B$, $p_A = p_B$. Then $\mathrm{Ext}^1_B(X, X) \cong k$, $\eta \neq 0$ and we have $\dim_k H^1(A) = \dim_k H^1(B) - 1 = d_B + f_B - 1 = d_A + f_A$, $\dim_k H^2(A) = \dim_k H^2(B) = p_B + r_B = p_A + r_A$.

If the admissible operation is of type (ad 4*), then the proof is dual.

If the nth admissible operation is of type (ad 5) then Γ is obtained from the disjoint union of the finite family of generalized multicoils $\Omega_1, \ldots, \Omega_q$, $1 \leq q \leq s$. Since in the definition of (ad 5) we use the finite versions (fad 1)–(fad 4) of the admissible operations (ad 1), (ad 2), (ad 3), (ad 4) and the admissible operation (ad 4), we conclude that the required statement follows from the above considerations. For type (ad 5*) the proof is dual.

This finishes the proof of Theorem 1.1.

6. Examples. We present two examples illustrating Theorem 1.1 and Corollary 1.2.

EXAMPLE 6.1. Consider the algebra A given by the quiver

$$\begin{tikzcd}
 & 1 & & 4 \arrow[dl, "\alpha"'] \\
 & & 3 \arrow[ul, "\beta"'] \arrow[dll, "\lambda"'] \arrow[dl, "\delta"] & \\
6 & 2 & 9 \arrow[u, "\rho"'] \arrow[dl, "\sigma"] & 5 \arrow[ul, "\gamma"'] \\
 & 7 \arrow[ul, "\mu"] & & 8 \arrow[ll, "\nu"]
\end{tikzcd}$$

bound by $\alpha\lambda = 0$, $\gamma\lambda = 0$, $\rho\lambda = 0$, $\rho\beta = 0$, $\rho\delta = 0$, $\sigma\mu = 0$. We first show that A is a generalized multicoil enlargement of a concealed canonical algebra C. Indeed, let C be the hereditary algebra of Euclidean type $\widetilde{\mathbb{D}}_4$ given by the vertices $1, 2, 3, 4, 5$. Consider the dimension-vectors

$$\mathbf{a}_1 = \begin{smallmatrix}0&&0\\&1&\\0&&0\end{smallmatrix}, \quad \mathbf{a}_2 = \begin{smallmatrix}&0&&0\\&&1&\\00&&&0\\&0&&0\end{smallmatrix}, \quad \mathbf{b}_1 = \begin{smallmatrix}&0&&0\\&&0&\\00&&&0\\&1&&0\end{smallmatrix}, \quad \mathbf{b}_2 = \begin{smallmatrix}&0&&0\\&&0&\\00&&&0\\&1&&1\end{smallmatrix}.$$

We apply (ad 1^*) to C with pivot the simple regular C-module with dimension-vector $\mathbf{a}_1$, and with parameter $t = 2$. The modified algebra B is given by the quiver with the vertices $1, \ldots, 8$ bound by $\alpha\lambda = 0$, $\gamma\lambda = 0$. Finally, we apply the admissible operation (ad 4) to B with pivot the simple B-module X with dimension-vector $\mathbf{a}_2$ and with the finite sectional path $Y_1 \to Y_2$ in Γ_B consisting of all indecomposable B-modules Y_1 and Y_2 with dimension-vectors $\mathbf{b}_1$ and $\mathbf{b}_2$, respectively, and with parameter $r = 0$. The modified algebra is then equal to A.

We now compute the Hochschild cohomology spaces of A. We have $d_A = 1$, $f_A = 0$, $p_A = 0$, $r_A = 0$. Hence, applying Theorem 1.1, we get $\dim_k H^0(A) = 1$, $\dim_k H^1(A) = d_A + f_A = 1$, $\dim_k H^2(A) = p_A + r_A = 0$, and $H^n(A) = 0$ for $n \geq 3$. In particular, A is an analytically rigid algebra.

EXAMPLE 6.2. Consider the algebra D given by the quiver

$$\begin{tikzcd}
 & 1 & & 4 \arrow[dl, "\alpha"'] & \\
 & & 3 \arrow[ul, "\beta"'] \arrow[dll, "\lambda"'] \arrow[dl, "\delta"] & & 10 \arrow[ul, "\varphi"'] \arrow[dl, "\psi"] \\
6 & 2 & 9 \arrow[u, "\rho"'] \arrow[dl, "\sigma"] & 5 \arrow[ul, "\gamma"'] & \\
 & 7 \arrow[ul, "\mu"] & & 8 \arrow[ll, "\nu"] &
\end{tikzcd}$$

bound by $\alpha\lambda = 0$, $\gamma\lambda = 0$, $\rho\lambda = 0$, $\rho\beta = 0$, $\rho\delta = 0$, $\sigma\mu = 0$, $\varphi\alpha\beta = \psi\gamma\beta$, $\varphi\alpha\delta = a\psi\gamma\delta$, where $a \in k \setminus \{0, 1\}$. Then D is a generalized multicoil enlargement of the concealed canonical algebra C from Example 6.1. Indeed, we apply the admissible operation (ad 1) with parameter $t = 0$ to the algebra A from Example 6.1 with pivot the regular C-module X corresponding to the indecomposable representation of the form

$$\begin{tikzcd}
k & & k \arrow[dl, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] \\
& k^2 \arrow[ul, "{[1\ 1]}"'] \arrow[dl, "{[a\ 1]}"] & \\
k & & k \arrow[ul, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"']
\end{tikzcd}$$

lying in a stable tube of rank 1 in Γ_C (see [37, XIII.2.6(d)]). The modified algebra is then equal to D.

We now compute the Hochschild cohomology spaces of D. We have $d_D = 1$, $f_D = 0$, $p_D = 0$, $r_D = 1$. Hence, applying Theorem 1.1, we get $\dim_k H^0(D) = 1$, $\dim_k H^1(D) = d_D + f_D = 1$, $\dim_k H^2(D) = p_D + r_D = 1$, and $H^n(D) = 0$ for $n \geq 3$. In particular, D is not an analytically rigid algebra.

We also mention that there exist connected generalized multicoil algebras A with arbitrarily large $\dim_k H^1(A)$ and $\dim_k H^2(A)$.

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