A NOTE ON ARC-DISJOINT CYCLES IN TOURNAMENTS

by

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Abstract. We prove that every vertex $v$ of a tournament $T$ belongs to at least

$$\max\{\min\{\delta^+(T), 2\delta^+(T) - d^+_T(v) + 1\}, \min\{\delta^-(T), 2\delta^-(T) - d^-_T(v) + 1\}\}$$

arc-disjoint cycles, where $\delta^+(T)$ (or $\delta^-(T)$) is the minimum out-degree (resp. minimum in-degree) of $T$, and $d^+_T(v)$ (or $d^-_T(v)$) is the out-degree (resp. in-degree) of $v$.

1. Introduction. Notation used in this paper is consistent with Bang-Jensen and Gutin [1]. Cycles are always directed. A tournament is an orientation of a complete graph. The out-degree (resp. in-degree) $d^+_T(v)$ (resp. $d^-_T(v)$) of a vertex $v$ of a tournament $T$ is the number of arcs with tail at $v$ (resp. with head at $v$). We denote by $\delta^+(T)$ (resp. $\Delta^+(T)$) the minimum out-degree (resp. maximum out-degree) of $T$. Moreover, we denote by $\delta^-(T)$ (resp. $\Delta^-(T)$) the minimum in-degree (resp. maximum in-degree) of $T$.

Landau [2] proved that in every tournament $T$, if a vertex $v$ has the minimum out-degree, then it belongs to $\delta^+(T)$ different 3-cycles. In this article, we prove that in every tournament $T$, every vertex $v$ belongs to at least $C_T(v)$ arc-disjoint cycles, where $C_T(v)$ is equal to

$$\max\{\min\{\delta^+(T), 2\delta^+(T) - d^+_T(v) + 1\}, \min\{\delta^-(T), 2\delta^-(T) - d^-_T(v) + 1\}\}.$$ 

This implies that $v$ belongs to at least $C_T(v)$ different 3-cycles. Moreover, if either $\Delta^+(T) \leq 2\delta^+(T)$, or $\Delta^-(T) \leq 2\delta^-(T)$, then every vertex of $T \neq K_1$ belongs to a 3-cycle.

Note that for every tournament $T$ which has a vertex $v$ such that the tournament $T - v$ is regular, the lower bound $C_T(v)$ is the best possible. Indeed, $d^+_T(v) + d^-_T(v) = 2\delta^+(T - v) + 1$. Thus, if $d^+_T(v) \leq \delta^+(T - v)$, then

$$\min\{d^+_T(v), d^-_T(v)\} = d^+_T(v) = \delta^+(T) \leq 2\delta^+(T) - d^+_T(v) + 1.$$ 

If $d^+_T(v) > \delta^+(T - v)$, then

$$\min\{d^+_T(v), d^-_T(v)\} = d^-_T(v) = 2\delta^+(T) - d^+_T(v) + 1 \leq \delta^+(T).$$ 

Hence,

$$\min\{d^+_T(v), d^-_T(v)\} = \min\{\delta^+(T), 2\delta^+(T) - d^+_T(v) + 1\}.$$
Similarly, from $d_T^+(v) + d_T^-(v) = 2\delta^-(T - v) + 1$, it follows that 
$$\min\{d_T^+(v), d_T^-(v)\} = \min\{\delta^-(T), 2\delta^-(T) - d_T^-(v) + 1\}.$$ 

2. Arc-disjoint cycles through a vertex in a tournament. Let 
$T = (V, A)$ be a tournament with vertex set $V$ and arc set $A$. For an arc 
$xy \in A$ the first vertex $x$ is its tail and the second vertex $y$ is its head. For 
a vertex $v$ in $T$ we use the following notation: 
$$N^+(v) = \{u \in V \setminus \{v\} : vu \in A\}, \quad N^-(v) = \{u \in V \setminus \{v\} : uv \in A\}.$$ 
For a pair $X, Y$ of vertex sets in $T$ we define 
$$(X, Y) = \{xy \in A : x \in X, y \in Y\}.$$ 

**Theorem 2.1.** Every vertex $v$ of a tournament $T$ belongs to at least 
$$\max\{\min\{\delta^+(T), 2\delta^+(T) - d_T^+(v) + 1\}, \min\{\delta^-(T), 2\delta^-(T) - d_T^-(v) + 1\}\}$$ 
arcs disjoint cycles.

**Proof.** For a vertex $v$ of a tournament $T$, let $\Gamma = \{\gamma^1, \ldots, \gamma^m\}$ be a 
maximum family of arc-disjoint cycles through $v$. Let $\gamma^i = v v^i_1 \ldots v^i_{n(i)} v$ for 
i = 1, \ldots, m. By Menger’s theorem (see [1]) there exists a set $\Omega$ of $m$ arcs 
covering all cycles containing the vertex $v$. Suppose that $k$ is the number of 
arcs in $\Omega$ with head $v$. If $k > 0$, we can assume that the arc $v^i_{n(i)} v$ is in $\Omega$ 
for $1 \leq i \leq k$. Let us denote $K = \{v^i_1 : 1 \leq i \leq k\}$, $L = \{v^i_1 : k < i \leq m\}$, 
$M = N^+(v) \setminus K \cup L$, $X = \{v^i_{n(i)} : 1 \leq i \leq k\}$ (if $k = 0$ we set $K = X = \emptyset$), 
and $Y = N^-(v) \setminus X$.

First we prove that 
$$|(K \cup X \cup M, Y)| \leq |(L, K \cup X \cup M)|.$$ 
Assume that an arc $wv$ belongs to $(K \cup X \cup M, Y)$. Notice that $vy \notin \Omega$. If 
w $\in K \cup M$, then the arc $wv$ of the cycle $wvyv$ belongs to $\Omega \setminus \{v^i_{n(i)} v : i \leq k\}$. 
If $w \in X$, then $w = v^i_{n(i)}$ for some $i \leq k$. Hence, the arc $wv = v^i_{n(i)}y$ of the 
cycle $v v^i_1 \ldots v^i_{n(i)} v y$ belongs to $\Omega \setminus \{v^i_{n(i)} v : i \leq k\}$. Thus, $wy$ is an arc 
of the cycle $\gamma^i$, for some $i > k$. Suppose that $v^i_1$ is the first vertex of the 
cycle $\gamma^i$ which does not belong to $L$. Notice that $wy$ is the only arc of $\gamma^i$ 
which belongs to $\Omega$, because $\Omega$ and $\Gamma$ have the same number of elements. 
Hence, the vertex $v^i_1$ does not belong to $Y$. Otherwise, the cycle $v v^i_1 \ldots v^i_{l-1} v^i_l v$ 
would not be covered by $\Omega$. Thus the edge $v^i_{l-1} v^i_l$ of the cycle $\gamma^i$ belongs to 
$(L, K \cup X \cup M)$. Accordingly, to every arc in $(K \cup X \cup M, Y)$ we can assign 
an arc in $(L, K \cup X \cup M)$ such that the two arcs belong to the same cycle 
$\gamma^i$, for some $i > k$. The above assignment is injective, because $\Omega$ and $\Gamma$ have 
the same number of elements, and $\Gamma$ is a family of arc-disjoint cycles. Hence, 
(1) holds.
By (1) we obtain
\[
|K \cup X \cup M|(|V| - 1) = |(V \setminus L, K \cup X \cup M)| + |(L, K \cup X \cup M)| \\
+ |(K \cup X \cup M, V)| \\
\geq |(V \setminus L, K \cup X \cup M)| + |(K \cup X \cup M, Y)| \\
+ |(K \cup X \cup M, V)| \\
= |(K \cup X \cup M, K \cup X \cup M)| \\
+ |(\{v\}, K \cup X \cup M)| + |(Y, K \cup X \cup M)| \\
+ |(K \cup X \cup M, Y)| + |(K \cup X \cup M, V)| \\
\geq |K \cup X \cup M| \cdot \frac{|K \cup X \cup M| - 1}{2} + |K| + |M| \\
+ |K \cup X \cup M||Y| + (|K| + |X| + M))\delta^+(T).
\]
Since \(|V| - 1 = d_T^+(v) + |X| + |Y|\) and \(|K| = |X|\), we have
\[
(2|K| + |M|)d_T^+(v) \geq (2|K| + |M|)\frac{|M|}{2} + \frac{|M|}{2} + (2|K| + |M|)\delta^+(T).
\]
Thus, either \(|M| = 0\), or \(d_T^+(v) > |M|/2 + \delta^+(T)\). Hence, either \(|M| = 0\), or
\[
d_T^+(v) - |M| > 2\delta^+(T) - d_T^+(v).
\]
Accordingly, the vertex \(v\) belongs to at least
\[
\min\{\delta^+(T), 2\delta^+(T) - d_T^+(v) + 1\}
\]
arc-disjoint cycles. By considering the tournament obtained from \(T\) by reversing the directions of the arcs of \(A\), we conclude in a similar fashion that the vertex \(v\) belongs to at least \(\min\{\delta^-(T), 2\delta^-(T) - d_T^-(v) + 1\}\) arc-disjoint cycles.

**Remark 1.** There exists a regular tournament \(R\) with a vertex which does not belong to \(\delta^+(R)\) arc-disjoint 3-cycles. For example, let \(R\) be the

![Diagram](image.png)

Fig. 1. A regular tournament \(R\). \((\{h, i, j\}, \{a, b, c\}) \cup (\{k\}, \{a, b, c, d, e\}) \cup \{ca, gd, jh\}\) is the set of all backward arcs with respect to the ordering \(a, b, c, d, e, f, g, h, i, j, k\) of vertices in \(R\).
tournament in Fig. 1. Let $kv_1v_2$ be a 3-cycle through the vertex $k$. Notice that, if $v_1 \in \{a, b, c\}$, then $v_2 \in \{f, g\}$. Hence, the vertex $k$ does not belong to $\delta^+(R)$ arc-disjoint 3-cycles.

REFERENCES


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