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RELATIVELY WEAK* CLOSED IDEALS OF $A(G)$, SETS OF SYNTHESIS AND SETS OF UNIQUENESS

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Abstract. Let G be a locally compact amenable group, and $A(G)$ and $B(G)$ the Fourier and Fourier–Stieltjes algebras of G. For a closed subset E of G, let $J(E)$ and $k(E)$ be the smallest and largest closed ideals of $A(G)$ with hull E, respectively. We study sets E for which the ideals $J(E)$ or/and $k(E)$ are $\sigma(A(G), C^*(G))$-closed in $A(G)$. Moreover, we present, in terms of the uniform topology of $C_0(G)$ and the weak* topology of $B(G)$, a series of characterizations of sets obeying synthesis. Finally, closely related to the above issues, we present a series of results about closed sets of uniqueness (i.e. closed sets E for which $\overline{J(E)}^{w^*} = B(G)$).

Introduction. Let G be a locally compact amenable group and, as defined by Eymard in his seminal paper [Ey], $A(G)$ and $B(G)$ be the Fourier algebra and the Fourier–Stieltjes algebra of G, respectively. These are commutative semisimple function algebras on G, and $A(G)$ is a closed ideal of $B(G)$. The algebra $B(G)$ is the dual space of the group C^*-algebra $C^*(G)$ of G. For a subset X of $A(G)$, we denote by $\overline{X}^{\sigma}$ and $\overline{X}^{w^*}$, respectively, the $\sigma(A(G), C^*(G))$-closure of X in $A(G)$ and the weak* closure of X in $B(G)$, so that $\overline{X}^{\sigma} = A(G) \cap \overline{X}^{w^*}$. For any Banach space X, we denote by X_1 the closed unit ball of X and, for any subspace Y of X, we let $Y^{\perp}$ be the annihilator of Y in X^*.

As usual, to any closed subset E of G, the following two ideals are associated:

$$k(E) = \{a \in A(G) : a = 0 \text{ on } E\},$$
$$j(E) = \{a \in A(G) : \text{the support of } a \text{ is compact and disjoint from } E\}.$$

The ideals $J(E) = \overline{j(E)}$ and $k(E)$ are, respectively, the smallest and the largest closed ideals in $A(G)$ with hull E. When these two closed ideals coincide, the set E is said to be a *set of synthesis*. Following the terminology of [Gr-McG] and [Ke-Lo], we call a closed subset E of G a *set of uniqueness* if

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$\overline{J(E)}^{w^*} = B(G)$. Finally, for any $f \in A(G)^*$, we denote by $\sigma(f)$ the spectrum of the functional f (all the notations and terms used in this section will be defined or referenced in the next section).

We can now summarize the main results of the paper. Below, E is a closed subset of G, and $\Re_c(G)$ is the closed coset ring of G.

A. To any closed subset E of G we associate the following two closed sets:

$$E_J = \overline{\bigcup_{\varphi \in J(E)^{\perp} \cap C^*(G)} \sigma(\varphi)} \quad \text{and} \quad E_k = \overline{\bigcup_{\varphi \in k(E)^{\perp} \cap C^*(G)} \sigma(\varphi)}.$$

If $E_J = E$ (resp. $E_k = E$) we say that E is a J-set (resp. k-set).

Concerning $\sigma(A(G), C^*(G))$-closedness of the ideals $J(E)$ and $k(E)$ in $A(G)$, we present the following results:

1. E_J is the hull of the ideal $\overline{J(E)}^{\sigma}$ and E_k is the hull of the ideal $\overline{k(E)}^{\sigma}$.
2. E is a J-set iff $\overline{J(E)}^{\sigma} \subseteq k(E)$. And, E is a k-set iff $\overline{k(E)}^{\sigma} = k(E)$.
3. Any closed regular set E (i.e. $E = \overline{E^{\circ}}$) is a k-set, so a J-set.
4. The union of two J-sets (resp. k-sets) is a J-set (resp. k-set).
5. If E is a J-set (resp. k-set) then, for any closed subset F of G, the set $\overline{E \setminus F}$ is a J-set (resp. k-set).

B. We recall that $A(G)$ is contained in $C_0(G)$, the space of complex valued continuous functions on G that vanish at infinity. The space $C_0(G)$ is, as usual, equipped with the uniform convergence norm.

Concerning sets of synthesis, we present the following results:

1. E is a set of synthesis iff the uniform closures of $J(E)_1$ and $k(E)_1$ in $C_0(G)$ are the same.
2. E is a set of synthesis iff the weak* closures of $J(E)_1$ and $k(E)_1$ in $B(G)$ are the same.
3. E is a set of synthesis iff, given any $a \in k(E)_1$, there is a sequence $(b_n)_{n\geq 0}$ in $J(E)_1$ that converges pointwise on G to a.
4. E is a set of synthesis iff there is some set $F \in \Re_c(G)$ such that $E \cap F$ and $E \cup F$ are sets of synthesis.
5. Let E be a J-set. Then E is a set of synthesis iff $\overline{J(E)}^{\sigma} = J(E)$ and $J(E)^{\perp} \cap C^*(G) = k(E)^{\perp} \cap C^*(G)$.
6. Suppose that E is a J-set and its boundary is a set of uniqueness. Then E is a set of synthesis iff the ideal $J(E)$ is $\sigma(A(G), C^*(G))$-closed in $A(G)$.
7. Suppose that G is metrizable with metric d, and $\delta(E)$ denotes the diameter of the set E. Then E is a set of synthesis iff there is a sequence $(F_n)_{n\geq 0}$ of sets of synthesis contained in E such that $\delta(E \setminus F_n) \to 0$ as $n \to \infty$.

C. Concerning sets of uniqueness, we present the following results:

1. For any subset U of G the following two assertions are equivalent:
   (i) Every closed subset of U is a set of uniqueness.
   (ii) For any J-set E, $\overline{E \setminus U} = E$.
2. Every closed subset E of G decomposes in a unique way as the disjoint union of two sets: a J-set and one like U in (i) above.
3. The union of countably many sets of uniqueness does not contain any nonempty J-set.

The exposition also contains several corollaries of these results. The paper is essentially self-contained and the proofs are functional-analytic. After a preliminary section, the results are presented in three sections in the above order. The main ingredient in Section 3 concerning sets of synthesis is the result that on the unit sphere of $B(G)$, weak* convergence is equivalent to multiplier convergence [Gr-Le] (Theorem 3.0 below). As far as we know, the results listed above are new.

1. Notation and preliminary results. In this section we gather the notation that we use and the preliminary results that we need in order to logically explain the results that we aim to present.

For any Banach space X we denote by X^* the dual space of X. We always consider X as naturally embedded into its second dual. For $x \in X$ and $f \in X^*$, we denote by $\langle x, f\rangle$ (and also by $\langle f, x\rangle$) the natural (X, X^*)-duality. If A is a Banach algebra then, for $a \in A$ and $f \in A^*$, we write $a.f$ for the element of A^* defined by

$$\langle b, a.f\rangle = \langle ba, f\rangle.$$

Fourier and Fourier–Stieltjes algebras. Let now G be a locally compact group equipped with its left Haar measure. Eymard [Ey] has associated to G two important commutative Banach function algebras: the Fourier algebra $A(G)$ and the Fourier–Stieltjes algebra $B(G)$.

The Fourier–Stieltjes algebra $B(G)$ is the linear span of the set of continuous positive definite complex valued functions on G. This is also the space of coefficient functions of the unitary representations of the group G. More precisely, given $u \in B(G)$, there exists a unitary representation π of G and two vectors ξ and η in the representation space $H(\pi)$ of π such that, for $x \in G$,

$$u(x) = \langle \pi(x)\xi, \eta\rangle.$$

Equipped with the norm $\|u\| = \inf_{\xi,\eta} \|\xi\| \cdot \|\eta\|$, where the infimum is taken over all ξ and η satisfying the preceding equality, and with pointwise multiplication, $B(G)$ is a commutative Banach algebra. The Banach algebra $B(G)$ is also the dual space of the group C^*-algebra $C^*(G)$.

The Fourier algebra $A(G)$ is the closed ideal of $B(G)$ generated by the elements of $B(G)$ with compact supports. The algebra $A(G)$ can also be defined as the set of coordinate functions of the left regular representation of G in $L^2(G)$. The dual space of $A(G)$ is the group von Neumann algebra $\mathrm{VN}(G)$ of G.

We shall denote the elements of $A(G)$ by letters such as a, b, and those of $\mathrm{VN}(G)$ by f, g. The elements of $C^*(G)$ will be denoted by φ, ϕ, and those of $B(G)$ by u, v. Eymard's paper [Ey] and Pier's book [Pi] are our main references for these algebras.

When G is abelian, by way of the Fourier transform,

$$A(G) = L^1(\widehat{G}), \quad B(G) = M(\widehat{G}) \quad \text{and} \quad C^*(G) = C_0(\widehat{G}).$$

Since their introduction in the 1960's by Eymard, these algebras have been constant objects of study and they are among the most important Banach algebras of harmonic analysis. Both $A(G)$ and $B(G)$ are semisimple. Moreover $A(G)$ is regular and Tauberian. The Gelfand spectrum of $A(G)$ can be identified with the group G. Each element of G acts on $A(G)$ as point evaluation. For each $x \in G$, the corresponding multiplicative functional (i.e. the point evaluation at x) will be denoted as ρ_x. The algebra $A(G)$ has a BAI iff the group G is amenable [Lep]. As is well-known, abelian groups and compact groups are amenable. On the negative side, noncompact semisimple Lie groups, as well as any group containing a closed free subgroup generated by more than one generator, are not amenable. The reader can find ample information on this notion in [Gre] and [Pat].

Closed coset ring $\Re_c(G)$. We denote by $\Re(G)$ the Boolean ring generated by all cosets of subgroups of the algebraic group G. We denote by $\Re_c(G)$ the *closed coset ring of* G. This is the subring of $\Re(G)$ consisting of the closed elements of $\Re(G)$. For abelian groups, the structure of the elements of $\Re_c(G)$ has been described, independently, by Gilbert [Gi] and Schreiber [Schr]. Forrest [Fo] verified that the same description is also valid in the nonabelian case.

Spectrum of a functional f. For $f \in \mathrm{VN}(G)$, the spectrum $\sigma(f)$ of f can be defined in several ways, each of them having its own usage. The set $\sigma(f)$ is a closed subset of the Gelfand spectrum of $A(G)$, that we identify with G, defined in any of the following equivalent ways. For more on this notion, see [Ey, Proposition 4.4], [Ka, Chapter 5] and [Re-St, Chapter 7].

(i) For $x \in G$, $x \in \sigma(f)$ iff, for any $a \in A(G)$, $a.f = 0$ implies $a(x) = 0$.
(ii) For $x \in G$, $x \in \sigma(f)$ iff there is a net $(a_i)_{i\in I}$ in $A(G)$ such that $a_i.f \to \rho_x$ in the weak* topology of $\mathrm{VN}(G) = A(G)^*$. Here ρ_x is the point evaluation at x.

(iii) $\sigma(f)$ is the support of f as defined in [Ey, Proposition 4.4].
(iv) Let $J_f = \{a \in A(G) : a.f = 0\}$. This is a closed ideal of $A(G)$. The spectrum of f is the hull of the ideal J_f.

The properties of the spectrum that we need are:

(I) $\sigma(f) = \emptyset$ iff $f = 0$.
(II) For $a \in A(G)$, $\sigma(a.f) \subseteq \sigma(f) \cap \mathrm{supp}(a)$.
(III) For any closed subset E of G, $\sigma(f) \subseteq E$ iff $f \in J(E)^\perp$.
(IV) If E is a closed subset of G and if $(f_i)_{i\in I}$ is a weak* convergent net in $\mathrm{VN}(G)$ converging to some f, then the inclusions $\sigma(f_i) \subseteq E$ for all $i \in I$ imply that $\sigma(f) \subseteq E$ too.

Sets of uniqueness (= U-sets). As in [Gr-McG] and [Ke-Lo], we call a closed subset E of G a *set of uniqueness* if $\overline{J(E)}^{w^*} = B(G)$. Since the group G is supposed to be amenable, we have $C^*(G) \subseteq \mathrm{VN}(G)$, and the equality $\overline{J(E)}^{w^*} = B(G)$ is equivalent to $J(E)^\perp \cap C^*(G) = \{0\}$. If $\overline{k(E)}^{w^*} = B(G)$ then E is said to be a *U_1-set*. In [Bo-Py, p. 275], what is called here a *U_1-set* is called a *weak uniqueness set*.

The notion of set of uniqueness goes back to Cantor's work in the 1870's on the uniqueness of trigonometric representation of functions on the interval $[0, 2\pi]$. For the history of this subject and its development we refer the reader to [Gr-McG] and [Ke-Lo]. In these books the definitions and results related to sets of uniqueness are given for the unit circle group. For this group, our definitions coincide with the corresponding ones in [Gr-McG] and [Ke-Lo]. In general a set of uniqueness need not be closed, but since in this paper our approach is functional-analytic, we work with closed sets of uniqueness only.

Product of two closed ideals. Let I and J be two closed ideals of $A(G)$. The closed ideal generated by all products ab ($a \in I$ and $b \in J$) is denoted by IJ. Thus $IJ = \overline{\{\sum_{i=1}^n a_i b_i : a_i \in I,\, b_i \in J,\, n \in \mathbb{N}\}}$. If $J = I$, we write I^2 instead of II. We note that $J(E)^2 = J(E)$ for any closed subset E of G. Indeed, the inclusion $J(E)^2 \subseteq J(E)$ is trivial, and the other one follows from the facts that the hull of the closed ideal $J(E)^2$ is E and that $J(E)$ is the smallest closed ideal with hull E.

2. $\sigma(A(G), C^*(G))$-closedness of the ideals $J(E)$ and $k(E)$. In this section, G is a fixed locally compact amenable group and E a closed subset of it. Our aims in this section are to determine the hulls of the ideals $\overline{J(E)}^{\sigma}$ and $\overline{k(E)}^{\sigma}$, to study the problem of when the ideals $J(E)$ and/or $k(E)$ are $\sigma(A(G), C^*(G))$-closed in $A(G)$, and to study the stability properties of sets that we have called J-sets and k-sets.

For any closed ideal I of $A(G)$, we denote by $\overline{I}^{\sigma}$ its $\sigma(A(G), C^*(G))$-closure in $A(G)$. Since for any $u \in B(G)$, the multiplication operator $L_u : B(G) \to B(G)$, $L_u(v) = uv$, is weak*-weak* continuous, the space $\overline{I}^{\sigma}$ is also a closed ideal of $A(G)$. As one can check easily, the annihilator of the ideal $\overline{I}^{\sigma}$ in $A(G)^* = \mathrm{VN}(G)$ is the weak* closure of $I^\perp \cap C^*(G)$:

$$(\overline{I}^{\sigma})^\perp = \overline{I^\perp \cap C^*(G)}^{w^*}.$$

If E is the hull of I, we denote by E_I the following closed subset of G:

$$E_I = \overline{\bigcup_{\varphi \in I^\perp \cap C^*(G)} \sigma(\varphi)}.$$

If $I = J(E)$ or $I = k(E)$, instead of E_I we write E_J and E_k, respectively. We recall that if $E = E_J$ (resp. $E = E_k$) we say that E is a *J-set* (resp. *k-set*).

The next result explains the meaning of the set E_I.

THEOREM 2.1. *Let I be a closed ideal of $A(G)$ with hull E. Then E_I is the hull of the ideal $\overline{I}^{\sigma}$.*

Proof. It is enough to prove that

$$J(E_I) \subseteq \overline{I}^{\sigma} \subseteq k(E_I).$$

The left inclusion is equivalent to

$$I^\perp \cap C^*(G) \subseteq J(E_I)^\perp. \tag{1}$$

Since, for a functional f in $\mathrm{VN}(G)$,

$$f \in J(E_I)^\perp \quad \text{iff} \quad \sigma(f) \subseteq E_I,$$

inclusion (1) follows from the definition of E_I.

To prove the inclusion $\overline{I}^{\sigma} \subseteq k(E_I)$, since $k(E_I)^\perp = \overline{\mathrm{span}(E_I)}^{w^*}$, it is enough to show that $E_I \subseteq \overline{I^\perp \cap C^*(G)}^{w^*}$. Since

$$E_I = \overline{\bigcup_{\varphi \in I^\perp \cap C^*(G)} \sigma(\varphi)},$$

it is enough to prove that $\sigma(\varphi) \subseteq \overline{I^\perp \cap C^*(G)}^{w^*}$ for each $\varphi \in I^\perp \cap C^*(G)$. So let $x \in \sigma(\varphi)$. Then, by the definition (ii) of the spectrum in Section 1, $\rho_x = w^*\text{-}\lim_i a_i.\varphi$ for some net $(a_i)_{i \in I}$ in $A(G)$. Since $a_i.\varphi \in I^\perp \cap C^*(G)$ for all $i \in I$, it follows that $\sigma(\varphi) \subseteq \overline{I^\perp \cap C^*(G)}^{w^*}$. Hence $E_I \subseteq \overline{I^\perp \cap C^*(G)}^{w^*}$, and $\overline{I}^{\sigma} \subseteq k(E_I)$. ■

As immediate consequences of this theorem and the definitions of the sets E_J and E_k, we record the following results, valid for any closed subset E of G.

(2) Since E_J is the hull of the ideal $\overline{J(E)}^{\sigma}$, we have

$$J(E_J) \subseteq \overline{J(E)}^{\sigma} \subseteq k(E_J).$$

Similarly, since E_k is the hull of $\overline{k(E)}^{\sigma}$, we have

$$J(E_k) \subseteq \overline{k(E)}^{\sigma} \subseteq k(E_k).$$

The set E_J is empty iff E is a set of uniqueness. The set E_k is empty iff E is a U_1-set.

(3) Since $E_k \subseteq E_J \subseteq E$, if $E = E_k$ then $E = E_J$ too. That is, every k-set is a J-set.

(4) Since $E_J \subseteq E$, so $J(E) \subseteq J(E_J)$, we see by (2) that, $\overline{J(E_J)}^{\sigma} = \overline{J(E)}^{\sigma}$.

(5) Since $E_k \subseteq E_J \subseteq E$, we have

$$J(E_J) \subseteq \overline{J(E)}^{\sigma} \subseteq \overline{k(E)}^{\sigma} \subseteq k(E_k).$$

In the next result we present a characterization of J-sets and k-sets.

THEOREM 2.2.

(a) *E is a J-set iff $\overline{J(E)}^{\sigma} \subseteq k(E)$.*
(b) *E is a k-set iff $\overline{k(E)}^{\sigma} = k(E)$.*

Proof. (a) If E is a J-set then, from (2), the inclusion $\overline{J(E)}^{\sigma} \subseteq k(E)$ is clear. Conversely, suppose that this inclusion holds. We have to prove that

$$E = \overline{\bigcup_{\varphi \in J(E)^{\perp} \cap C^*(G)} \sigma(\varphi)}. \tag{6}$$

By hypothesis,

$$k(E)^{\perp} \subseteq \overline{J(E)^{\perp} \cap C^*(G)}^{w^*}. \tag{7}$$

Since $k(E)^{\perp} = \overline{\mathrm{span}(E)}^{w^*}$, we have

$$E \subseteq \overline{J(E)^{\perp} \cap C^*(G)}^{w^*}.$$

In other words, for each $x \in E$, we have $\rho_x = w^*\text{-}\lim_i \varphi_i$ for some net $(\varphi_i)_{i \in I}$ in $J(E)^{\perp} \cap C^*(G)$. Since $\sigma(\varphi_i) \subseteq E_J$ for all $i \in I$, we see that $x \in E_J$, so $E \subseteq E_J$. Hence $E = E_J$ so that E is a J-set.

(b) If E is a k-set then, from (2), the inclusion $\overline{k(E)}^{\sigma} \subseteq k(E)$, hence the equality $\overline{k(E)}^{\sigma} = k(E)$, is clear. Conversely, suppose that $\overline{k(E)}^{\sigma} = k(E)$ so that

$$k(E)^{\perp} = \overline{k(E)^{\perp} \cap C^*(G)}^{w^*}. \tag{8}$$

Then $E \subseteq \overline{k(E)^{\perp} \cap C^*(G)}^{w^*}$. From this inclusion and the definition of E_k, as in the proof of (a), we see that $E = E_k$. ∎

Here we note that the preceding theorem, compared with [Ke-Lo, p. 227, Proposition 8], shows that what we have called here "J-set" coincides with "set of pure multiplicity" introduced, by a completely different definition, in the setting of the Fourier algebra of the unit circle group T by Piatetski-Shapiro [Pi-Sh].

(9) Since $J(E_J) \subseteq \overline{J(E)}^{\sigma} \subseteq k(E_J)$, from the above theorem it is clear that, for any closed subset E of G, the set E_J is a J-set. Actually, by its very definition, E_J is the largest J-set contained in E. In this paper, J-sets play a more important role than k-sets.

(10) As Theorem 2.2(b) shows, for any k-set E, the equality $k(E) = \overline{k(E)}^{\sigma}$ holds whereas in general $J(E_J) \neq \overline{J(E_J)}^{\sigma}$. However, as we shall see below (Theorem 3.6), if E_J is a set of synthesis then $J(E_J) = \overline{J(E_J)}^{\sigma}$. These results show that the properties of the sets E_J and E_k are quite different.

We can now exhibit some classes of concrete J-sets and k-sets. The next result and the examples following it present a very large class of k-sets, hence J-sets.

Proposition 2.3. *Any regular closed subset E of G (i.e. E is the closure of its interior) is a k-set.*

Proof. Let E be a regular closed subset of G. We want to show that $\overline{k(E)}^{\sigma} = k(E)$. Let $(a_i)_{i\in I}$ be a net in $k(E)$ that $\sigma(A(G), C^*(G))$-converges to some a in $A(G)$. Let $x \in E^\circ$. Since the set $G \setminus E^\circ$ is closed and x is not in this set, by regularity of $A(G)$, there is some $b \in A(G)$ such that $b(x) = 1$ and $b = 0$ on $G \setminus E^\circ$. Hence, the algebra $A(G)$ being semisimple, $ba_i = 0$ for all $i \in I$. Since multiplication in $B(G)$ is weak* continuous when one of the factors is kept fixed, passing to the weak* limit we get $ba = 0$. As $b(x) = 1$, we conclude that $a(x) = 0$. This being true for each $x \in E^\circ$, $a = 0$ on E°. Since a is continuous on G, $a = 0$ on the closure of the set E°. Since E is supposed to be regular, $a = 0$ on E. Hence $a \in k(E)$, and $\overline{k(E)}^{\sigma} = k(E)$. ∎

We note that, since the closure of any open set is a regular set, the above proposition implies that $\overline{E^\circ} \subseteq E_k$ for any closed subset E of G. Actually, for any closed set F, the inclusion $F \subseteq E$ implies $F_k \subseteq E_k$. As an illustration of the above proposition we present the following examples.

Examples 2.4. (a) Since, for any open subset O of G, the set $E = \overline{O}$ is regular, the support of any $u \in B(G)$ is a regular set. Hence, if $E = \operatorname{supp}(u)$, the ideal $k(E)$ is $\sigma(A(G), C^*(G))$-closed in $A(G)$. This example shows that $\sigma(A(G), C^*(G))$-closed ideals in $A(G)$ abundantly exist to justify studying them. Associate to $u \in B(G)$ the ideal $I(u) = \{a \in A(G) : au = 0\}$. It is easy to see directly, without using the preceding proposition, that $I(u)$ is $\sigma(A(G), C^*(G))$-closed in $A(G)$ and $I(u) = k(E)$, where $E = \operatorname{supp}(u)$.

(b) Let $\varphi \in C^*(G)$ and $E = \sigma(\varphi)$. Since $\varphi \in J(E)^\perp \cap C^*(G)$, we have $E_J = E$ so E is a J-set. Since E is the hull of the closed ideal $J_\varphi = \{a \in A(G) : a.\varphi = 0\}$, we have $J(E) \subseteq J_\varphi \subseteq k(E)$. The ideal J_φ is obviously $\sigma(A(G), C^*(G))$-closed in $A(G)$ so that $\overline{J(E)}^\sigma \subseteq J_\varphi \subseteq k(E)$, in conformity with Theorem 2.2.

(c) If $\varphi \in C^*(G)$ and $F = \sigma(\varphi)$ is a set of synthesis then F is a k-set. The class of k-sets is much richer than the class of regular sets. In this connection, see Proposition 2.6 and Theorem 2.7 below.

(d) For $E \in \Re_c(G)$ we can explicitly determine the sets E_k and E_J. Actually $E_J = E_k = E^\circ$. Indeed, since E is a set of synthesis [FKLS], $E_J = E_k$. Again by [FKLS], the closed ideal $k(E)$ has a bounded approximate identity $(e_i)_{i\in I}$. Taking a subnet of it, we can assume that (e_i) converges weak* to some idempotent element u of $B(G)$. Since $au = a$ for $a \in k(E)$, one can easily see that u is the characteristic function of $G \setminus E^\circ$. It follows that E° is also closed and the sets E° and $\partial E = E \setminus E^\circ$ are also in the ring $\Re_c(G)$. As ∂E is in $\Re_c(G)$, the ideal $k(\partial E)$ has a bounded approximate identity so that $k(E) = k(E^\circ)k(\partial E)$. As the ideal $k(\partial E)$ has a bounded approximate identity and the set ∂E is nowhere dense,

$$\overline{k(\partial E)}^{w^*} = B(G).$$

Since E° is closed, hence regular, we have $\overline{k(E^\circ)}^\sigma = k(E^\circ)$. From this and the equality $k(E) = k(E^\circ)k(\partial E)$, using the fact that $\overline{k(\partial E)}^{w^*} = B(G)$, we obtain $\overline{k(E)}^\sigma = k(E^\circ)$. Since E is a set of synthesis, by Theorem 2.2, $\overline{k(E)}^\sigma = k(E_k)$ too, and we conclude that $E_k = E^\circ$.

(e) The preceding example shows that the $\sigma(A(G), C^*(G))$-closed ideals with bounded approximate identities of $A(G)$ are exactly the ideals of the form $k(E) = \theta A(G)$, where E is a closed and open element of the ring $\Re(G)$, and θ is the idempotent element of $B(G)$ whose support is $G \setminus E$. (Compare with the main result of [Ül].)

In the next example we give examples of "thin J-sets" that are not k-sets.

Example 2.5. Extending a famous theorem of Körner [Kö], originally proved for the unit circle group, Saeki [Sa] showed that every locally compact metrizable abelian infinite group Γ contains a closed set E such that $k(E)^\perp \cap C_0(\Gamma) = \{0\}$ but $J(E)^\perp \cap C_0(\Gamma) \neq \{0\}$ (i.e. E is a U_1-set but not a set of uniqueness). Since, by a theorem of Zelmanov [Ze], every infinite compact group contains an infinite compact abelian group, every locally compact group containing an infinite compact metrizable group contains a closed set E which is not a set of uniqueness but is a U_1-set. Let $E \subseteq G$ be such a set. For any $\varphi \in J(E)^\perp \cap C^*(G)$, the set $F = \sigma(\varphi)$ is a J-set that is not a k-set.

Next we present a stability property of J-sets and k-sets.

PROPOSITION 2.6. *The union of two J-sets* (*resp. k-sets*) *is also a J-set* (*resp. k-set*).

Proof. Let E and F be two J-sets in G. To show that the set $E \cup F$ is a J-set, by Theorem 2.2, it will be enough to show that $\overline{J(E \cup F)}^{\sigma} \subseteq k(E \cup F)$. To see this, let $a \in \overline{J(E \cup F)}^{\sigma}$. Then there is a net $(a_i)_{i \in I}$ in $J(E \cup F)$ that $\sigma(A(G), C^*(G))$-converges to a. As

$$J(E \cup F) \subseteq J(E) \cap J(F),$$

we see that $a \in \overline{J(E)}^{\sigma}$ and $a \in \overline{J(F)}^{\sigma}$. As E and F are J-sets, $\overline{J(E)}^{\sigma} \subseteq k(E)$ and $\overline{J(F)}^{\sigma} \subseteq k(E)$ so that $a \in k(E) \cap k(F)$. It follows that $a \in k(E \cup F)$ so that $\overline{J(E \cup F)}^{\sigma} \subseteq k(E \cup F)$, and $E \cup F$ is a J-set.

Suppose now that E and F are k-sets, so $\overline{k(E)}^{\sigma} = k(E)$ and $\overline{k(F)}^{\sigma} = k(F)$. As $k(E \cup F) = k(E) \cap k(F)$, the inclusions

$$\overline{k(E \cup F)}^{\sigma} \subseteq \overline{k(E)}^{\sigma} \cap \overline{k(F)}^{\sigma} \subseteq k(E) \cap k(F) \subseteq k(E \cup F)$$

are clear. The inclusion $k(E \cup F) \subseteq \overline{k(E \cup F)}^{\sigma}$ being always true, we see that $k(E \cup F) = \overline{k(E \cup F)}^{\sigma}$ so that $E \cup F$ is a k-set. ∎

The next result was a kind of surprise for us. It plays an important role in the study of sets of uniqueness. Its proof uses the following simple observation: Let (X, τ) be a Hausdorff topological space, H an arbitrary subset of X and F a closed subset of X. Then $\overline{\overline{H} \setminus F} = \overline{H \setminus F}$.

THEOREM 2.7.

(a) *Let E be a J-set. Then, for any closed subset F of G, the set $\overline{E \setminus F}$ is also a J-set.*

(b) *Let E be a k-set. Then, for any closed subset F of G, the set $\overline{E \setminus F}$ is also a k-set.*

Proof. (a) Since E is a J-set, by the very definition of J-sets, we have

$$E = \overline{\bigcup_{\varphi \in J(E)^{\perp} \cap C^*(G)} \sigma(\varphi)}.$$

We want to prove that

$$\overline{E \setminus F} = \overline{\bigcup_{\varphi \in J(\overline{E \setminus F})^{\perp} \cap C^*(G)} \sigma(\varphi)}.$$

It is enough to show

$$E \setminus F \subseteq \overline{\bigcup_{\varphi \in J(\overline{E \setminus F})^{\perp} \cap C^*(G)} \sigma(\varphi)}. \tag{11}$$

To see this, let

$$H = \bigcup_{\varphi \in J(E)^{\perp} \cap C^*(G)} \sigma(\varphi),$$

so that $H \setminus F$ is a union of sets of the form $\sigma(\varphi) \setminus F$ for $\varphi \in J(E)^{\perp} \cap C^*(G)$. To prove (11), by the observation preceding the statement of the theorem, it is enough to show that, for each $\varphi \in J(E)^{\perp} \cap C^*(G)$, the set $\sigma(\varphi) \setminus F$ is contained in the right hand side of (11). So let $\varphi \in J(E)^{\perp} \cap C^*(G)$ and $x \in \sigma(\varphi) \setminus F$. Then, since F is closed and $x \notin F$, there is a relatively compact neighborhood V of x such that $V \cap F = \emptyset$. Choose $a \in A(G)$ such that $a(x) = 1$ and $\operatorname{supp}(a) \subseteq V$. Then the functional $a.\varphi$ is in $C^*(G)$ and

$$\sigma(a.\varphi) \subseteq E \cap \operatorname{supp}(a) \subseteq E \setminus F.$$

Hence $a.\varphi \in J(\overline{E \setminus F})^{\perp} \cap C^*(G)$. As $x \in \sigma(a.\varphi)$, we see that x indeed belongs to the right hand side of (11).

(b) Let E be a k-set, so

$$E = \overline{\bigcup_{\varphi \in k(E)^{\perp} \cap C^*(G)} \sigma(\varphi)}. \tag{12}$$

Let F be any closed subset of G. To see that $\overline{E \setminus F}$ is also a k-set, it is enough to show that

$$E \setminus F \subseteq \overline{\bigcup_{\varphi \in k(\overline{E \setminus F})^{\perp} \cap C^*(G)} \sigma(\varphi)}. \tag{13}$$

As before, it is enough to prove that, for each $\varphi \in k(E)^{\perp} \cap C^*(G)$, the set $\sigma(\varphi) \setminus F$ is contained in the right hand side of (13). Let $\varphi \in k(E)^{\perp} \cap C^*(G)$ and $x \in \sigma(\varphi) \setminus F$. As above, choose an $a \in A(G)$ such that $a(x) = 1$ and $\operatorname{supp}(a) \cap F = \emptyset$. We claim that $a.\varphi \in k(\overline{E \setminus F})^{\perp}$. If this were not the case, there would be a $b \in k(\overline{E \setminus F})$ such that $\langle b, a.\varphi \rangle = \langle ba, \varphi \rangle \neq 0$. Since $ab = 0$ on E and $\varphi \in k(E)^{\perp}$, this is not possible. Hence $a.\varphi \in k(\overline{E \setminus F})^{\perp} \cap C^*(G)$. Since $x \in \sigma(a.\varphi)$, we conclude that x belongs to the right hand side of (13). ∎

The next result will be needed in the subsequent sections. It suggests that every closed set F included in $E \setminus E_J$ is a set of uniqueness, a result that will be justified in Section 4.

Proposition 2.8. *For any closed subset E of G, we have*

$$J(E_J)^{\perp} \cap C^*(G) = J(E)^{\perp} \cap C^*(G).$$

Proof. The inclusion $\subseteq$ being clear, we prove the reverse inclusion. By the definition of E_J, for $\varphi \in J(E)^{\perp} \cap C^*(G)$, we have $\sigma(\varphi) \subseteq E_J$. Hence $\varphi \in J(E_J)^{\perp} \cap C^*(G)$ so that $J(E_J)^{\perp} \cap C^*(G) = J(E)^{\perp} \cap C^*(G)$. ∎

If E is a set of synthesis then $J(E)^{\perp}\cap C^*(G)=k(E)^{\perp}\cap C^*(G)$. However, this equality may hold even if E is not a set of synthesis (see Remark 3.7 and Example 3.14 below). Actually, as the next proposition shows, this equality holds iff $\overline{J(E)}^{\sigma}=\overline{k(E)}^{\sigma}$. Compare this result with Theorem 3.1(c) below.

Proposition 2.9. *For a closed subset E of G, we have*

$$J(E)^{\perp}\cap C^*(G)=k(E)^{\perp}\cap C^*(G) \quad \textit{iff} \quad \overline{J(E)}^{\sigma}=\overline{k(E)}^{\sigma}.$$

Proof. Since, for any closed ideal I of $A(G)$,

$$(\overline{I}^{\sigma})^{\perp}=\overline{I^{\perp}\cap C^*(G)}^{w^*},$$

the direct implication is clear. Conversely, suppose that $\overline{J(E)}^{\sigma}=\overline{k(E)}^{\sigma}$. Then

$$\overline{J(E)^{\perp}\cap C^*(G)}^{w^*}=\overline{k(E)^{\perp}\cap C^*(G)}^{w^*}.$$

Since $J(E)^{\perp}\cap C^*(G)\subseteq\overline{J(E)^{\perp}\cap C^*(G)}^{w^*}$ and $\overline{k(E)^{\perp}\cap C^*(G)}^{w^*}\subseteq k(E)^{\perp}$, we see that

$$J(E)^{\perp}\cap C^*(G)\subseteq k(E)^{\perp}.$$

Hence $J(E)^{\perp}\cap C^*(G)\subseteq k(E)^{\perp}\cap C^*(G)$; the reverse inclusion is obvious. ∎

As seen above (Examples 2.4), closed sets E for which the ideal $k(E)$ is $\sigma(A(G),C^*(G))$-closed in $A(G)$ abundantly exist. If such an E is also a set of synthesis then $\overline{J(E)}^{\sigma}=J(E)$. Except the k-sets that are also sets of synthesis we do not know of any closed set E for which $\overline{J(E)}^{\sigma}=J(E)$. Actually, as we shall see below, for a J-set E to be a set of synthesis, the equality $\overline{J(E)}^{\sigma}=J(E)$ is a necessary condition.

3. Characterizations of sets of synthesis. Since Malliavin's celebrated theorem [Ma], it has been known that every nondiscrete locally compact abelian group G contains a closed set E which is not a set of synthesis for the algebra $A(G)$. This is also true for nonabelian locally compact groups [Ka-La]. Two major unsolved problems in this area are (a) the *union problem*, which asks whether the union of two sets of synthesis is also a set of synthesis, and (b) the so-called *Ditkin set vs. set of synthesis problem*, which asks whether these two classes of sets are the same. We recall that a closed subset E of G is said to be a *Ditkin set* if $a\in\overline{aj(E)}$ for each $a\in k(E)$. This notion is stronger than that of set of synthesis.

In this section we do not attack either of these two unsolved problems. Our aim is much more modest: we want to better understand sets of synthesis. We think that to achieve this, one should find as many characterizations of them as one can, in terms of better known notions. That is what we are going to do in this section, in the hope that some day one of these characterizations might be helpful in the solution of the above mentioned problems.

In the first part of this section we do not assume that the group G is amenable. Instead, we assume that

(14) $a \in \overline{aA(G)}$ for any $a \in A(G)$.

We do not know of any locally compact group for which (14) does not hold. Our main ingredient in this section is the following result, which is the fruit of the work of several mathematicians. For a proof and short history, we refer the reader to [Gr-Le]. This theorem is valid for any locally compact group G.

Theorem 3.0. *Let $u \in B(G)$ and suppose that $(u_i)_{i\in I}$ is a net in $B(G)$ such that $\lim_i \|u_i\| = \|u\|$ and $u_i \to u$ in the weak* topology of $B(G)$. Then $\|u_i a - ua\| \to 0$ for any $a \in A(G)$.* ∎

For any subset X of $A(G)$, we denote by $\overline{X}^{\infty}$ the uniform closure of X in $C_0(G)$, and by $\overline{X}^{w^*}$ the weak* closure of X in $B(G)$. We also recall that, for any Banach space X, we denote by X_1 the closed unit ball of X. One of the main results of this section is the following theorem.

Theorem 3.1. *For any closed subset E of G the following assertions are equivalent:*

(a) *E is a set of synthesis.*
(b) $\overline{J(E)_1}^{\infty} = \overline{k(E)_1}^{\infty}$.
(c) $\overline{J(E)_1}^{w^*} = \overline{k(E)_1}^{w^*}$.

Proof. The implication (a)⇒(b) is clear. Let us prove (b)⇒(a). To this end, let $a \in k(E)$ with $\|a\| = 1$. By (b), there is a sequence $(b_n)_{n\ge 0}$ in the unit ball of $J(E)$ that converges uniformly on G to a. As this sequence is bounded in the norm of $B(G)$, hence in the supremum norm, by the Lebesgue Dominated Convergence Theorem we have, for $f \in L^1(G)$,

$$\langle b_n, f\rangle = \int_G b_n(t)f(t)\,dt \to \int_G a(t)f(t)\,dt = \langle a, f\rangle.$$

Since $L^1(G)$ is dense in $C^*(G)$ and the sequence $(b_n)_{n\ge 0}$ is bounded in the norm of $B(G)$, we conclude that $b_n \to a$ in the weak* topology of $B(G)$. Hence, as the norm of $B(G)$ is lower semicontinuous with respect to that topology, we have

$$1 = \|a\| \le \liminf \|b_n\| \le \limsup \|b_n\| \le 1.$$

Thus, $\lim_{n\to\infty} \|b_n\| = \|a\| = 1$. So, by Theorem 3.0, for each $b \in A(G)$, $\|b_n b - ab\| \to 0$ as $n \to \infty$. Since $b_n b \in J(E)$, we conclude that $ba \in \overline{J(E)}$ for all $b \in A(G)$. Now, by (14), $a \in \overline{aA(G)}$. So, for some sequence $(e_n)_{n\ge 0}$ in $A(G)$, $\|ae_n - a\| \to 0$ as $n \to \infty$. By what we have seen, $ae_n \in J(E)$ for all $n \in \mathbb{N}$, so $a \in J(E)$. Now let $a \in k(E)$ be an arbitrary nonzero element. Then, as $a/\|a\| \in J(E)$, we conclude that $k(E) = J(E)$, hence E is a set of synthesis.

The implication (a)⇒(c) is clear. That the converse is also true follows from the proof of (b)⇒(a). ∎

REMARKS 3.2. (a) Let $S = \{a \in k(E) : \|a\| = 1\}$ be the unit sphere of $k(E)$. The above proof shows in fact that E is a set of synthesis iff $S \subseteq \overline{J(E)_1}^{w^*}$.

(b) One can show that $\overline{J(E)}^{\infty} = \overline{k(E)}^{\infty}$ for any closed subset E of G. However, this equality does not imply $\overline{J(E)_1}^{\infty} = \overline{k(E)_1}^{\infty}$.

Since the set $k(E)_1$ is convex, its norm and weak closures in $C_0(G)$ are the same. Since a sequence $(f_n)_{n\geq 0}$ in $C_0(G)$ converges weakly to some $f \in C_0(G)$ iff $(f_n)_{n\geq 0}$ is uniformly bounded on G and converges pointwise to f on G, from the preceding theorem we deduce the following result.

COROLLARY 3.3. *A closed subset E of G is a set of synthesis iff, given any $a \in k(E)_1$ there is a sequence $(b_n)_{n\geq 0}$ in the unit ball of $J(E)$ that converges pointwise to a on G.* ∎

From this point on until the end of the paper we assume that the group G is amenable.

As recalled in Section 1, $\Re_c(G)$ is the closed coset ring of the group G. Any $F \in \Re_c(G)$ is a set of synthesis and the ideal $k(F)$ has a bounded approximate identity [FKLS]. The next theorem, which involves elements of $\Re_c(G)$, gives another necessary and sufficient condition for a given closed set to be a set of synthesis.

THEOREM 3.4. *Let E be a closed subset of G. Then E is a set of synthesis iff there is some $F \in \Re_c(G)$ such that $E \cap F$ and $E \cup F$ are sets of synthesis.*

Proof. If E is a set of synthesis, we can take for F either the empty set or the group G itself: both are in the ring $\Re_c(G)$, and in either case, the sets $E \cap F$ and $E \cup F$ are sets of synthesis.

Conversely, suppose that, for some $F \in \Re_c(G)$, the sets $E \cap F$ and $E \cup F$ are sets of synthesis. Since

$$k(E \cup F) = k(E) \cap k(F),$$

we have

$$k(E \cup F)^{\perp} = \overline{k(E)^{\perp} + k(F)^{\perp}}^{w^*}.$$

Since the ideal $k(E)$ has a bounded approximate identity, by a theorem of Rudin [Ru], the sum $k(E) + k(F)$ is closed in $A(G)$. Hence, $k(E)^{\perp} + k(F)^{\perp}$ is weak* closed in $\mathrm{VN}(G)$ (see e.g. [Kat, p. 221, Theorem 4.8]). Thus

$$k(E \cup F)^{\perp} = k(E)^{\perp} + k(F)^{\perp}.$$

Since, by hypothesis, $E \cup F$ is a set of synthesis, $k(E \cup F) = J(E \cup F)$ so that $J(E)^{\perp} \subseteq k(E \cup F)^{\perp}$. Let now $f \in J(E)^{\perp}$. Then $f = g + h$ for some $g \in k(E)^{\perp}$ and $h \in k(F)^{\perp}$. Since $k(E)^{\perp} \subseteq J(E)^{\perp}$, $h = f - g$ is in $J(E)^{\perp}$. So $\sigma(h) \subseteq E$. On the other hand, since $h \in k(F)^{\perp}$, also $\sigma(h) \subseteq F$. Hence

$$\sigma(h) \subseteq E \cap F.$$

Let now $a \in k(E)$. Since $a \in k(E \cap F)$ and $E \cap F$ is a set of synthesis, we have $\langle a, h\rangle = 0$. Hence $\langle a, f\rangle = \langle a, g\rangle$. As $g \in k(E)^{\perp}$, also $\langle a, g\rangle = 0$, so that $\langle a, f\rangle = 0$. This being true for each $f \in J(E)^{\perp}$ and $a \in k(E)$, we conclude that E is a set of synthesis. ∎

We note that in the preceding proof we did not use the hypothesis that $E \cup F$ is a set of synthesis; we only used the inclusion $J(E)^{\perp} \subseteq k(E \cup F)^{\perp}$, which holds if there is a set of synthesis in between E and $E \cup F$. Moreover, if E is a set of synthesis and $F \in \Re_c(G)$ then one can prove that $E \cap F$ is a set of synthesis iff $k(E \cap F) = k(E) + k(F)$. We do not present the proof of this result since we do not need it.

The following result is actually much stronger than the preceding theorem but the hypotheses here are not as elegant.

Proposition 3.5. *Let E be a closed subset of G. Then E is a set of synthesis iff there are a set F in $\Re_c(G)$ and two sets of synthesis D_1, D_2 such that $E \cap F \subseteq D_1 \subseteq E \subseteq D_2 \subseteq E \cup F$.*

Proof. If E is a set of synthesis we can take $F = G$ and $D_1 = D_2 = E$.

To prove the converse, suppose that there are sets F and D_1, D_2 as in the statement. As in the preceding proof, we have

$$k(E \cup F)^{\perp} = k(E)^{\perp} + k(F)^{\perp}.$$

Since $E \subseteq D_2 \subseteq E \cup F$ and D_2 is a set of synthesis, we obtain

$$J(E)^{\perp} \subseteq J(D_2)^{\perp} \subseteq k(D_2)^{\perp} \subseteq k(E \cup F)^{\perp}.$$

Let $f \in J(E)^{\perp}$. Then, since $J(E)^{\perp} \subseteq k(E)^{\perp} + k(F)^{\perp}$, f is of the form $f = g + h$ for some $g \in k(E)^{\perp}$ and $h \in k(F)^{\perp}$. As in the previous proof, since $k(E)^{\perp} \subseteq J(E)^{\perp}$, $h = f - g$ is in $J(E)^{\perp}$. Hence $\sigma(h) \subseteq E$. On the other hand, since $h \in k(F)^{\perp}$, also $\sigma(h) \subseteq F$. Thus $\sigma(h) \subseteq E \cap F$. Since $E \cap F \subseteq D_1 \subseteq E$ and D_1 is a set of synthesis, for $a \in k(E) \subseteq k(D_1)$ we have $\langle a, h\rangle = 0$. So $\langle a, f\rangle = \langle a, g\rangle$. Since $g \in k(E)^{\perp}$ and $a \in k(E)$, also $\langle g, a\rangle = 0$. Hence $\langle f, a\rangle = 0$ for all $a \in k(E)$, so that E is a set of synthesis. ∎

Now we return to J-sets and study the question when a J-set is a set of synthesis. The main message of the next theorem is that a J-set E, in particular a regular closed set E, cannot be a set of synthesis unless the ideal $J(E)$ is $\sigma(A(G), C^*(G))$-closed in $A(G)$. This is one of the rare necessary conditions that we know for a set to be a set of synthesis.

Theorem 3.6. *Let E be a J-subset of G. Then E is a set of synthesis iff $\overline{J(E)}^{\sigma} = J(E)$ and $J(E)^{\perp} \cap C^*(G) = k(E)^{\perp} \cap C^*(G)$.*

Proof. Suppose first that E is a set of synthesis. Then obviously $J(E)^{\perp} \cap C^*(G) = k(E)^{\perp} \cap C^*(G)$. Since E is a J-set, by Theorem 2.2,

$$J(E) \subseteq \overline{J(E)}^{\sigma} \subseteq k(E).$$

Since E is supposed to be set of synthesis, $J(E) = k(E)$ so that $J(E) = \overline{J(E)}^{\sigma}$.

Conversely, suppose $\overline{J(E)}^{\sigma} = J(E)$ and $J(E)^{\perp} \cap C^*(G) = k(E)^{\perp} \cap C^*(G)$. Then, since by Proposition 2.9, the equality $J(E)^{\perp} \cap C^*(G) = k(E)^{\perp} \cap C^*(G)$ is equivalent to $\overline{J(E)}^{\sigma} = \overline{k(E)}^{\sigma}$, we have

$$J(E)^{\perp} = (\overline{J(E)}^{\sigma})^{\perp} = (\overline{k(E)}^{\sigma})^{\perp} = \overline{k(E)^{\perp} \cap C^*(G)}^{w^*} \subseteq k(E)^{\perp}.$$

Hence $J(E) = k(E)$, and E is a set of synthesis. ∎

Remark 3.7. The equality $J(E)^{\perp} \cap C^*(G) = k(E)^{\perp} \cap C^*(G)$ may hold even if E is not a set of synthesis. Indeed, let G be the Euclidean group $\mathbb{R}^3$ and $E = \{x \in \mathbb{R}^3 : \|x\| = 1\}$ its unit sphere. A well-known result due to L. Schwartz [Schw] says that E is not a set of synthesis for the Fourier algebra $A(\mathbb{R}^3)$. On the other hand, as proved by Varopoulos [Va, Theorem 2], $J(E)^{\perp} \cap C_0(\mathbb{R}^3) = k(E)^{\perp} \cap C_0(\mathbb{R}^3)$. Equivalently, $\overline{J(E)}^{\sigma} = \overline{k(E)}^{\sigma}$. See also Example 3.14 below for a way of producing sets of nonsynthesis satisfying $J(E)^{\perp} \cap C^*(G) = k(E)^{\perp} \cap C^*(G)$.

We do not know of any necessary and sufficient condition on the set E for the equality $J(E)^{\perp} \cap C^*(G) = k(E)^{\perp} \cap C^*(G)$ to hold. Next we present a sufficient condition in terms of sets of uniqueness.

To every closed subset E of G we associate the set

$$E_0 = \overline{\bigcup_{a \in k(E),\, f \in J(E)^{\perp}} \sigma(a.f)}. \tag{15}$$

This is a subset of the boundary of E, so if the boundary is a set of uniqueness then so is E_0.

Proposition 3.8. *If E_0 is a set of uniqueness then $J(E)^{\perp} \cap C^*(G) = k(E)^{\perp} \cap C^*(G)$.*

Proof. Suppose that E_0 is a set of uniqueness. Let $a \in k(E)$ and $\varphi \in J(E)^{\perp} \cap C^*(G)$. Then $a.\varphi \in J(E_0)^{\perp} \cap C^*(G)$. Since E_0 is a set of uniqueness, we have $J(E_0)^{\perp} \cap C^*(G) = \{0\}$ so that $a.\varphi = 0$. Hence, since $A(G)$ has a bounded approximate identity, $\langle a, \varphi\rangle = 0$. This being true for each $a \in k(E)$, we conclude that $\varphi \in k(E)^{\perp} \cap C^*(G)$ so that $J(E)^{\perp} \cap C^*(G) = k(E)^{\perp} \cap C^*(G)$. ∎

Corollary 3.9. *Suppose that E is a J-set and its boundary is a set of uniqueness. Then E is a set of synthesis iff $J(E)$ is $\sigma(A(G),C^*(G))$-closed in $A(G)$.* ■

The next result shows that if $J(E)^\perp \cap C^*(G) \neq k(E)^\perp \cap C^*(G)$, then not only E, but also E_J cannot be a set of synthesis. Actually, as one can see easily, no closed set F in between E_J and E can be a set of synthesis in this case.

Proposition 3.10. *If E_J is a set of synthesis then $J(E)^\perp \cap C^*(G) = k(E)^\perp \cap C^*(G)$ so that $E_J = E_k$.*

Proof. Since E_J is a set of synthesis, by Theorem 3.6 we have

$$\overline{J(E_J)}^\sigma = J(E_J) \quad \text{and} \quad J(E_J)^\perp \cap C^*(G) = k(E_J)^\perp \cap C^*(G).$$

On the other hand, by (2),

$$J(E_J) \subseteq \overline{J(E)}^\sigma \subseteq \overline{k(E)}^\sigma.$$

Since $E_J \subseteq E$, and so $k(E) \subseteq k(E_J)$, we have $\overline{k(E)}^\sigma \subseteq \overline{k(E_J)}^\sigma$. As E_J is a set of synthesis, Theorem 3.6 yields $\overline{k(E_J)}^\sigma = \overline{J(E_J)}^\sigma = J(E_J)$. Hence

$$J(E_J) \subseteq \overline{J(E)}^\sigma \subseteq \overline{k(E)}^\sigma \subseteq J(E_J)$$

so that $\overline{J(E)}^\sigma = \overline{k(E)}^\sigma$. By Proposition 2.9, this implies that $J(E)^\perp \cap C^*(G) = k(E)^\perp \cap C^*(G)$. ■

Combining Theorem 3.6 and Proposition 3.10 we get

Corollary 3.11. *For any closed subset E of G, the set E_J is a set of synthesis iff $J(E_J) = \overline{k(E)}^\sigma$.* ■

For the next characterization of sets of synthesis, we need the following lemma. Here E is a closed subset of G and the set E_0 is as defined in (15).

Lemma 3.12. *For any set of synthesis $F \subseteq E$, we have $E_0 \subseteq \overline{E \setminus F}$.*

Proof. If F as above is empty then there is nothing to prove. So we suppose that F is not empty. Let $a \in k(E)$ and $f \in J(E)^\perp$. As F is a set of synthesis, $k(F) = \overline{j(F)}$. So, since $a \in k(F)$, there is a sequence $(b_n)_{n\geq 1}$ in $j(F)$ such that $\|a - b_n\| \to 0$. As $b_n \in j(F)$, we have $\operatorname{supp}(\widehat{b_n}) \cap F = \emptyset$ for each $n \geq 1$. So

$$\sigma(b_n.f) \subseteq \sigma(f) \cap \operatorname{supp}(\widehat{b_n}) \subseteq E \setminus F.$$

Hence, since $b_n.f \to a.f$ in the weak* topology of $\mathrm{VN}(G)$, we obtain

$$\sigma(a.f) \subseteq \overline{E \setminus F}.$$

It follows that

$$\bigcup_{a\in k(E),\, f\in J(E)^\perp} \sigma(a.f) \subseteq \overline{E \setminus F}.$$

Hence $E_0 \subseteq \overline{E \setminus F}$. ■

In the next theorem we assume that G is metrizable with metric d. For any subset E of G, we denote by $\delta(E) = \sup\{d(x, y) : x, y \in E\}$ (finite or not) the diameter of the set E. We note that $\delta(\overline{E}) = \delta(E)$. The theorem presents yet another characterization of sets of synthesis.

THEOREM 3.13. *A closed subset E of G is a set a synthesis iff for each $\varepsilon > 0$ there is a set of synthesis $F \subseteq E$ such that $\delta(E \setminus F) < \varepsilon$.*

Proof. If E is a set of synthesis then for F_ε we take E itself.

To prove the converse, suppose that for each $\varepsilon > 0$ there is a set of synthesis $F_\varepsilon \subseteq E$ such that $\delta(E \setminus F_\varepsilon) < \varepsilon$. Then, by the preceding lemma, $E_0 \subseteq \overline{E \setminus F_\varepsilon}$. Hence, since $\delta(\overline{E \setminus F_\varepsilon}) = \delta(E \setminus F_\varepsilon) < \varepsilon$, we see that $\delta(E_0) < \varepsilon$. As E_0 is independent of ε, we conclude that $\delta(E_0) = 0$. So either E_0 is empty or it contains only one point. But E_0 cannot be a singleton since each set $\sigma(a.f)$ is perfect; so E_0 must be empty. In this case, by definition of E_0, for any $f \in J(E)^\perp$ and $a \in k(E)$, $a.f = 0$ so that $\langle a, f \rangle = 0$, and E is a set of synthesis. ∎

As another application of Lemma 3.12 we give the following example, which shows that sets E disobeying synthesis but satisfying the equality $J(E)^\perp \cap C^*(G) = k(E)^\perp \cap C^*(G)$ abundantly exist.

EXAMPLE 3.14. Let D be a set of uniqueness disobeying synthesis (see Example 2.5 above) and F a set of synthesis. Then $E = D \cup F$ need not be a set of synthesis. Actually if D and F are disjoint, then E is not a set of synthesis (see [Ka, Theorem 5.2.5]). Since $E \setminus F = D \setminus F \subseteq D$, by Lemma 3.12, $E_0 \subseteq D$ so that E_0 is a set of uniqueness. Hence, by Proposition 3.8, $J(E)^\perp \cap C^*(G) = k(E)^\perp \cap C^*(G)$.

Let 1_G be the unit element of the algebra $B(G)$. If E_0 is a set of uniqueness then, since $\overline{J(E_0)}^{w^*} = B(G)$, we have $1_G \in \overline{J(E_0)}^{w^*}$. The preceding example shows that the condition "$1_G \in \overline{J(E_0)}^{w^*}$" is not sufficient to conclude that E is a set of synthesis. The next result shows that if we replace $J(E_0)$ by its closed unit ball then we get a necessary and sufficient condition for E to be a set of synthesis.

PROPOSITION 3.15. *A closed subset E of G is a set of synthesis iff $1_G \in \overline{J(E_0)_1}^{w^*}$.*

Proof. If E is a set of synthesis then $E_0 = \emptyset$ so that $J(E_0) = A(G)$. Since G is supposed to be amenable, the algebra $A(G)$ has a bounded approximate identity bounded in norm by 1. This bounded approximate identity converges in the weak* topology of $B(G)$ to 1_G so that $1_G \in \overline{J(E_0)_1}^{w^*}$.

Conversely, suppose that $1_G \in \overline{J(E_0)_1}^{w^*}$. Since, for each $a \in k(E)$ and $f \in J(E)^\perp$, $\sigma(a.f) \subseteq E_0$, for each $b \in J(E_0)$ we have $\langle b, a.f \rangle = \langle ba, f \rangle = 0$.

This shows that $J(E_0)k(E) \subseteq J(E)$. Now let $(e_i)_{i\in I}$ be a net in the unit ball of $J(E_0)$ that weak* converges to 1_G. Then, for $a \in k(E)_1$, the net $(ae_i)_{i\in I}$ is in the unit ball of $J(E)$ and weak* converges to a, so that $\overline{k(E)_1}^{w^*} = \overline{J(E)_1}^{w^*}$. Hence, by Theorem 3.1 above, E is a set of synthesis. ■

Perhaps it is worth noting here that the condition "$1_G \in \overline{J(E)_1}^{w^*}$" is much stronger than Proposition 3.15; it implies that every closed subset of E is a set of synthesis. To see this it is enough to apply the proposition to each closed subset of E. This demonstrates, if need be, how much the conditions "$1_G \in \overline{J(E)}^{w^*}$" and "$1_G \in \overline{J(E)_1}^{w^*}$" are far away from each other. In contrast with the proposition, we do not know whether the condition "$1_G \in \overline{k(E_0)_1}^{w^*}$" is sufficient to conclude that E is a set of synthesis.

The results of this section suggest that to study sets of synthesis, the class of J-sets and the class of sets of uniqueness should be considered separately. For instance, instead of the union problem, the problem of whether the union of two J-sets which are also sets of synthesis is a set of synthesis seems to be more tractable. Sets of uniqueness are very elusive and play a role similar to that of negligible sets in measure theory (see the results in the next section).

4. Sets of uniqueness. In this section we present a series of results about closed sets of uniqueness. Since our approach is entirely functional-analytic, we are working with closed sets of uniqueness exclusively. Sets of uniqueness and notions closely related to them are usually studied for the Fourier algebra $A(T)$ of the unit circle group T. In the books [Gr-McG] and [Ke-Lo] the reader can find ample information about this and closely related notions.

For noncommutative locally compact groups the only works about sets of uniqueness and U_1-sets that we have been able to find in the literature are the papers [Bo, Bo-Py] of Bożejko and Pytlik. In [Bo-Py] it is proved that, for certain discrete groups H, not even finite subsets of H are U_1-sets, so that the situation in noncommutative groups is quite different from that of commutative ones. On the other hand, the main result of [Bo] implies that, when H is a nondiscrete locally compact amenable group, every compact scattered subset E of H is a set of uniqueness. For amenable groups H, such sets are sets of synthesis for the algebra $A(H)$.

As above, in this section too, G is a fixed amenable locally compact group. Our aim here is to present some characterizations of sets of uniqueness in terms of J-sets and to study their properties.

Since we assume that G is amenable, sets of uniqueness abundantly exist. As mentioned above, compact scattered subsets of G are sets of uniqueness. If G is connected, every set $E \in \Re_c(G)$ is also a set of uniqueness. Indeed, in this case, the ideal $J(E) = k(E)$ has a bounded approximate identity, which necessarily converges in the weak* topology of $B(G)$ to the unit element of $B(G)$. More generally, whether G is connected or not, each $F \in \Re_c(G)$ with empty interior is a set of uniqueness. Further, every closed subset of a set of uniqueness is a set of uniqueness, as also are their finite unions.

Comparing the definition of sets of uniqueness with the definition of J-sets we see that the following lemma holds.

LEMMA 4.1. *A closed subset F of G is a set of uniqueness iff F does not contain any nonempty J-set.* ∎

To study sets of uniqueness, our main tool is the following lemma, which is of independent interest.

LEMMA 4.2.

(a) *Let E be a closed subset of G. Then, for any two closed subsets D and F of E with $D \cup F = E$, we have $J(E) = J(D)J(F)$.*

(b) *Let E be a closed subset of G that is a set of synthesis. Then, for any two closed subsets D and F of E with $D \cup F = E$, we have $k(E) = k(D)k(F)$.*

Proof. (a) Let D and F be two closed subsets of E with $D \cup F = E$. Then since $J(E) \subseteq J(D)$, $J(E) \subseteq J(F)$ and $J(E)^2 = J(E)$, the inclusion $J(E) \subseteq J(D)J(F)$ is clear. For the reverse inclusion, let $a \in J(D)$ and $b \in J(F)$. Since $J(D) = \overline{j(D)}$ and $J(F) = \overline{j(F)}$, there exist a sequence $(a_n)_{n\geq 0}$ in $j(D)$ and a sequence $(b_n)_{n\geq 0}$ in $j(F)$ such that $\|a - a_n\| \to 0$ and $\|b - b_n\| \to 0$ as $n \to \infty$. Since $\operatorname{supp}(a_n b_n)$ is compact and disjoint from $D \cup F = E$, we have $a_n b_n \in j(E)$ so that $ab \in J(E)$. Since $J(D)J(F)$ is the smallest closed ideal of $A(G)$ containing all the products ab ($a \in J(D)$ and $b \in J(F)$), we conclude that $J(D)J(F) \subseteq J(E)$ and hence $J(E) = J(D)J(F)$.

(b) Since E is a set of synthesis, by (a), $k(E) = J(E) = J(D)J(F)$. Hence $k(E) \subseteq k(D)k(F)$. On the other hand, since $k(D)k(F) \subseteq k(D) \cap k(F) = k(E)$, we conclude that $k(E) = k(D)k(F)$. ∎

Another result that we shall use repeatedly is the following lemma.

LEMMA 4.3. *For any two closed subsets D and F of G we have:*

(a) $\overline{J(D)J(F)}^{\sigma} \supseteq \overline{J(D)}^{\sigma}\,\overline{J(F)}^{\sigma}$.

(b) *If* $\overline{J(F)}^{w^*} = B(G)$ *then* $\overline{J(D)J(F)}^{\sigma} = \overline{J(D)}^{\sigma}$.

Proof. (a) Let $a \in \overline{J(D)}^{\sigma}$ and $b \in \overline{J(F)}^{\sigma}$. Then there exist two nets $(a_i)_{i\in I}$ and $(b_j)_{j\in J}$ in $J(D)$ and $J(F)$, respectively, that $\sigma(A(G), C^*(G))$-converge to a and b, respectively. All the products $a_i b_j$ are in $J(D)J(F)$. Hence the iterated limit

$$ab = \sigma(A(G), C^*(G))\text{-}\lim_i \lim_j a_i b_j = \sigma(A(G), C^*(G))\text{-}\lim_i a_i b$$

is in the ideal $\overline{J(D)J(F)}^{\sigma}$. It follows that the latter ideal contains the closed ideal generated by the products ab with $a \in \overline{J(D)}^{\sigma}$ and $b \in \overline{J(F)}^{\sigma}$. Hence $\overline{J(D)J(F)}^{\sigma} \supseteq \overline{J(D)}^{\sigma}\,\overline{J(F)}^{\sigma}$.

(b) If $\overline{J(F)}^{w^*} = B(G)$ then there is a net $(e_i)_{i\in I}$ in $J(F)$ that converges weak* to the unit element of $B(G)$. Hence, for any $a \in \overline{J(D)}^{\sigma}$, $ae_i \to a$ in the topology $\sigma(A(G), C^*(G))$ of $A(G)$, so that $\overline{J(D)J(F)}^{\sigma} \supseteq \overline{J(D)}^{\sigma}$. Since $J(D)J(F) \subseteq J(D)$, also $\overline{J(D)J(F)}^{\sigma} \subseteq \overline{J(D)}^{\sigma}$, and we get $\overline{J(D)J(F)}^{\sigma} = \overline{J(D)}^{\sigma}$. ∎

We need yet another lemma, which is of purely topological nature.

Lemma 4.4. *Let X be a regular topological space, E a closed subset of X and U an arbitrary subset of E. If, for every closed set $F \subset X$ contained in U, we have $\overline{E \setminus F} = E$ then $\overline{E \setminus U} = E$ too.*

Proof. Suppose that, for every closed subset F of X contained in U, we have $\overline{E \setminus F} = E$. To see that $\overline{E \setminus U} = E$, let, if possible, x be a point in E which is not in $\overline{E \setminus U}$. So, there exists an open neighborhood V of x such that $\overline{V} \cap (E \setminus U) = \emptyset$. That is, $\overline{V} \cap E \subseteq U$. Let $F = \overline{V} \cap E$. Since F is a closed subset of U, by hypothesis, $\overline{E \setminus F} = E$. So there is a net (x_i) in $E \setminus F$ that converges to x. Since $x_i \notin V$ for all i, V is open and $x \in V$, the net (x_i) cannot converge to x. This proves that $\overline{E \setminus U} = E$. ∎

A subset U of G is said to be a *set of interior uniqueness* [Ke-Lo, p. 47] if every closed subset F of U is a set of uniqueness. The next result characterizes sets of interior uniqueness in terms of J-sets.

Theorem 4.5. *Let U be an arbitrary subset of G. Then U is a set of interior uniqueness iff $\overline{E \setminus U} = E$ for any J-set E.*

Proof. Suppose first that U is a set of interior uniqueness and E is a J-set. Let F be a closed subset of U. By hypothesis F is a set of uniqueness so that the ideal $J(F)$ is weak* dense in $B(G)$. Since $E = \overline{E \setminus F} \cup F$, by Lemma 4.2(a), we have

$$J(E) = J(\overline{E \setminus F} \cup F) = J(\overline{E \setminus F})J(F). \tag{16}$$

Now, since $\overline{J(F)}^{w^*} = B(G)$, by Lemma 4.3(b) we get

$$\overline{J(E)}^{\sigma} = \overline{J(\overline{E \setminus F})}^{\sigma}.$$

Hence, since E is a J-set, by Theorem 2.2 we have $\overline{J(E)}^{\sigma} \subseteq k(E)$, so that

$$J(\overline{E \setminus F}) \subseteq \overline{J(\overline{E \setminus F})}^{\sigma} = \overline{J(E)}^{\sigma} \subseteq k(E).$$

Thus

$$J(\overline{E \setminus F}) \subseteq k(E).$$

The algebra $A(G)$ being regular, this last inclusion implies that $\overline{E \setminus F} = E$. This being true for each closed subset F of U, by the preceding lemma, $\overline{E \setminus U} = E$.

Conversely, suppose that, for any J-set E, we have $\overline{E \setminus U} = E$. If U is not a set of interior uniqueness then, for some closed subset F of U, we should have $J(F)^{\perp} \cap C^*(G) \neq \{0\}$. Let $\varphi \in J(F)^{\perp} \cap C^*(G)$, $\varphi \neq 0$. Then $E = \sigma(\varphi)$ is a J-set and $E \subseteq F$. As $E \setminus F = \emptyset$, the set $E \setminus F$ cannot be dense in E. So neither can be $E \setminus U$. This proves that U is a set of interior uniqueness. ∎

As two immediate corollaries of this theorem, we present the following results.

Corollary 4.6. *A closed subset F of G is a set of uniqueness iff $\overline{E \setminus F} = E$ for any J-set E.* ∎

Since one-point sets are sets of uniqueness, we have the following corollary.

Corollary 4.7. *Every nonempty J-set E is perfect. More generally, no nonempty set U of interior uniqueness can be open and closed in any nonempty J-set E.* ∎

A classical result due to N. Bary [Ba, p. 78, Théorème IV] says that the union of countably many closed sets of uniqueness does not contain any set of nonuniqueness. Below, as an application of Theorem 4.5, we present a very short functional-analytic proof of a weaker version of this result. It is weaker because it only shows that the union does not contain any closed set of nonuniqueness.

Theorem 4.8. *Let $(F_n)_{n\geq 1}$ be a sequence of closed sets of uniqueness ($F_n \subseteq G$). Then $\bigcup_{n\geq 1} F_n$ does not contain any nonempty J-set.*

Proof. For a contradiction, suppose that this union contains a nonempty J-set E. Let $O_n = E \setminus F_n = E \setminus (E \cap F_n)$. The set O_n is open in E. The set E, being a closed subset of the locally compact space G, is locally compact under its relative topology, hence it is a Baire space. Since, by Corollary 4.6, the open set O_n is dense in E, by the Baire Theorem, so is the intersection $\bigcap_{n\geq 1} O_n$. Since $E \subseteq \bigcup_{n\geq 1} F_n$, this is not possible unless $E = \emptyset$. Hence $\bigcup_{n\geq 1} F_n$ does not contain any nonempty J-set. ∎

The next theorem shows that closed subsets of G can be decomposed into a disjoint union of a J-set and a set of interior uniqueness.

Theorem 4.9. *Every closed subset of G decomposes in a unique way as the union of a J-set and a set of interior uniqueness. If G is metrizable then the second component is an F_σ-set.*

Proof. Let E be a closed subset of G. If E is a set of uniqueness then there is nothing to prove. So assume that E is not a set of uniqueness. Then

$$E_J = \overline{\bigcup_{\varphi \in J(E)^{\perp} \cap C^*(G)} \sigma(\varphi)}$$

is a nonempty J-set, and it is the largest J-set contained in E. It follows from Proposition 2.6 that $U = E \setminus E_J$ does not contain any nonempty J-set, so that every closed subset of U is a set of uniqueness. Hence U is a set of interior uniqueness. Thus $E = E_J \cup U$, and the sets E_J and U are disjoint.

To prove the uniqueness of such a decomposition, let $E = F \cup H$ be another disjoint decomposition of E into a J-set F and a set H of interior uniqueness. Since E_J is the largest J-set contained in E, we have $F \subseteq E_J$ so that $U \subseteq H$. From the equalities

$$E = E_J \cup U = F \cup H$$

we deduce that, since $U \subseteq H$,

$$F = E \setminus H = (E_J \setminus H) \cup (U \setminus H) = E_J \setminus H.$$

As H is a set of interior uniqueness, Theorem 4.5 yields $\overline{E_J \setminus H} = E_J$ so that $E_J = F$ and $H = U$. This proves the desired uniqueness.

In the case where G is metrizable, every open subset of G is an F_σ-set. So, if $O = G \setminus E_J = \bigcup_{n\geq 0} F_n$, where each F_n is closed, then

$$U = E \setminus E_J = E \cap \bigcup F_{n\geq 0} = \bigcup_{n\geq 0} (E \cap F_n)$$

is an F_σ-set. ∎

The next result is a partial analogue of Corollary 4.6 for U_1-sets.

PROPOSITION 4.10. *If E is a k-set then for any U_1-set F we have $\overline{E \setminus F} = E$.*

Proof. Let F be a U_1-set and E a k-set. Then $k(F)^{\perp} \cap C^*(G) = \{0\}$ and, by Theorem 2.2(b), we have $\overline{k(E)}^{\sigma} = k(E)$. Recall that, by Theorem 2.7, $\overline{E \setminus F}$ is also a k-set. As

$$k(E) \supseteq k(\overline{E \setminus F})k(F)$$

and F is a U_1-set, so that $k(F)$ is weak* dense in $B(G)$, we have $\overline{k(E)}^{\sigma} \supseteq \overline{k(\overline{E \setminus F})}^{\sigma}$. Since both E and $\overline{E \setminus F}$ are k-sets, we have $\overline{k(E)}^{\sigma} = k(E)$ and $\overline{k(\overline{E \setminus F})}^{\sigma} = k(\overline{E \setminus F})$. This implies that $k(\overline{E \setminus F}) \subseteq k(E)$, which is only possible if $\overline{E \setminus F} = E$. ∎

The following is the analogue of Theorem 4.8 for U_1-sets.

COROLLARY 4.11. *Let $(F_n)_{n\geq 1}$ be a sequence of U_1-sets. Then $\bigcup_{n\geq 1} F_n$ does not contain any nonempty k-set.*

Proof. Suppose that $\bigcup_{n\geq 1} F_n$ contains a k-set E. Then, by the preceding proposition, each $O_n = E\setminus F_n$ is open and dense in E. Hence, as in the proof of Theorem 4.8, by the Baire Theorem, $\bigcap_{n\geq 1} O_n$ is dense in E, which is not possible since $E \subseteq \bigcup_{n\geq 1} F_n$. From this we conclude that E is empty. ∎

We finish this paper with a series of questions.

QUESTION 1. Let $\varphi \in C^*(G)$ and $E = \sigma(\varphi)$. Then $J(E) \subseteq J_\varphi$, where $J_\varphi = \{a \in A(G) : a.\varphi = 0\}$. For which φ's in $C^*(G)$, do we have $J(E) = J_\varphi$?

QUESTION 2. Let E_0 be as in (15) and F be a closed set with $E_0 \subseteq F \subseteq E$. If F is a Ditkin set then, as one can see easily, E is a set of synthesis. Is E a set of synthesis if F is a set of synthesis?

QUESTION 3. Let E be a closed subset of G. What is a necessary condition for the equality $\overline{J(E)}^\sigma = \overline{k(E)}^\sigma$ to hold?

QUESTION 4. How are the synthesis properties of the sets E, E_J and E_k related to each other?

QUESTION 5. If E is a k-set, $E^\circ \neq \emptyset$ and E is a set of synthesis, do we have $E_k = \overline{E^\circ}$?

QUESTION 6. For any closed set E, as seen in Section 2, $J(E_J) \subseteq \overline{J(E)}^\sigma$. When does equality hold?

QUESTION 7. For any closed set E, we know that $\overline{k(E)}^\sigma \subseteq k(E_k)$. When does equality hold?

QUESTION 8. Let E_0 be as in (15). We always have $J(E_0)k(E) = J(E)$ (see the proof of Proposition 3.15). For which sets E, do we have $k(E_0)k(E) = J(E)$?

QUESTION 9. If $\overline{J(E)}^\sigma = J(E)$ and E_0 is a set of uniqueness then E is a set of synthesis (Theorem 3.6 and Proposition 3.8). Is the condition $\overline{J(E)}^\sigma = J(E)$ alone sufficient for E to be a set of synthesis?

Finally, in connection with Question 2, we make the following conjecture.

CONJECTURE. If there is a set F of synthesis with $E_0 \subseteq F \subseteq E$ then E is a set of synthesis.

If this conjecture turns out to be true then the answer to the union problem is also positive.

If this conjecture turns out to be wrong then the answer to the Ditkin set vs. set of synthesis problem is also negative.

REMARK 4.12. As suggested by the referee, working, instead of the spaces $B(G)$, $C^*(G)$, with the reduced Fourier–Stieltjes algebra $B_r(G)$ and its predual $C_r^*(G)$, one could drop the amenability hypothesis on G. Given that the problems studied in this paper (understanding sets of synthesis and sets of uniqueness) are not really fully understood even in the case where G is the unit circle group, for uniformity of hypotheses, we preferred to work with amenable groups. For the algebras $B_r(G)$, $C_r^*(G)$ and some results related to sets of synthesis in this context, we refer the reader to the paper [La-Lo], and in particular to Lemma 7.3 and Proposition 7.4 of that paper.

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