COLLOQUIUM MATHEMATICUM

VOL. 137 2014 NO. 1

REPRESENTATION FUNCTIONS WITH DIFFERENT WEIGHTS

BY

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Abstract. For any given positive integer k, and any set A of nonnegative integers, let $r_{1,k}(A,n)$ denote the number of solutions of the equation $n = a_1 + ka_2$ with $a_1, a_2 \in A$. We prove that if k, l are multiplicatively independent integers, i.e., $\log k/\log l$ is irrational, then there does not exist any set $A \subseteq \mathbb{N}$ such that both $r_{1,k}(A,n) = r_{1,k}(\mathbb{N}\setminus A, n)$ and $r_{1,l}(A,n) = r_{1,l}(\mathbb{N}\setminus A, n)$ hold for all $n \geq n_0$. We also pose a conjecture and two problems for further research.

1. Introduction. Let $\mathbb{N}$ be the set of nonnegative integers. For a set $A \subseteq \mathbb{N}$, let $R_1(A,n)$, $R_2(A,n)$, $R_3(A,n)$ denote the number of solutions of $a + a' = n$, $a, a' \in A$; $a + a' = n, a, a' \in A$, $a < a'$ and $a + a' = n$, $a, a' \in A$, $a \leq a'$ respectively. These functions have been extensively studied by many authors (see [D2]–[ESS3], [N1]–[N3]) and are still a fruitful area of research in additive number theory. For $i \in \{1,2,3\}$, Sárközy asked whether there are sets A and B with infinite symmetric difference such that $R_i(A,n) = R_i(B,n)$ for all sufficiently large integers n. It is known that the answer is negative for $i = 1$ (see Dombi [D1]) and positive for $i = 2, 3$ (see Dombi [D1], Chen and Wang [CW]). In fact, Dombi [D1] for $i = 2$ and Chen and Wang [CW] for $i = 3$ proved that there exists a set $A \subseteq \mathbb{N}$ such that $R_i(A,n) = R_i(\mathbb{N}\setminus A, n)$ for all $n \geq n_0$. Lev [L] gave a simple common proof of these results. Later, Sándor [S] gave a complete answer by using generating functions, and Tang [T] gave an elementary proof. For related results, one can refer to [C, CT].

For two positive integers k_1, k_2 and any set A of nonnegative integers, let $r_{k_1,k_2}(A,n)$ denote the number of solutions of the equation $n = k_1a_1 + k_2a_2$ with $a_1, a_2 \in A$. Recently, the author and Chen [YC2] determined all pairs k_1, k_2 of positive integers for which there exists a set $A \subseteq \mathbb{N}$ such that $r_{k_1,k_2}(A,n) = r_{k_1,k_2}(\mathbb{N}\setminus A, n)$ for all $n \geq n_0$.

THEOREM A. *If k_1 and k_2 are two integers with $k_2 > k_1 \geq 2$ and $(k_1,k_2) = 1$, then there does not exist any set $A \subseteq \mathbb{N}$ such that*

$$r_{k_1,k_2}(A,n) = r_{k_1,k_2}(\mathbb{N}\setminus A, n)$$

for all sufficiently large integers n.

2010 *Mathematics Subject Classification*: Primary 11B34; Secondary 05A17.

Key words and phrases: partitions, representation functions, Sárközy problem.

DOI: 10.4064/cm137-1-1

THEOREM B. *If k is an integer with $k > 1$, then there exists a set $A \subseteq \mathbb{N}$ such that $r_{1,k}(A, n) = r_{1,k}(\mathbb{N} \setminus A, n)$ for all integers $n \geq 1$.*

For the modular version, we may refer to [YC1] and [YC3].

Given two natural numbers a, b, we call a and b *multiplicatively independent* if $\log a/\log b$ is irrational. Clearly, if k and l are multiplicatively independent, then $k, l \geq 2$ and $k \neq l$.

Professor Yong-Gao Chen asks me the following problem:

PROBLEM 1.1. *Are there two distinct integers $k \geq 2$ and $l \geq 2$ and a set $A \subseteq \mathbb{N}$ such that both $r_{1,k}(A, n) = r_{1,k}(\mathbb{N}\setminus A, n)$ and $r_{1,l}(A, n) = r_{1,l}(\mathbb{N}\setminus A, n)$ hold for all sufficiently large integers n?*

In this paper, we consider Chen's problem and prove the following theorem.

THEOREM 1.2. *If two integers k, l are multiplicatively independent, then there does not exist any set $A \subseteq \mathbb{N}$ such that both $r_{1,k}(A, n) = r_{1,k}(\mathbb{N} \setminus A, n)$ and $r_{1,l}(A, n) = r_{1,l}(\mathbb{N} \setminus A, n)$ hold for all sufficiently large integers n.*

By a famous result of Senge and Straus [SS], the condition in Theorem 1.2 is equivalent to there being at most finitely many positive integers with the property that the sum of the digits in each of the two bases k and l lies below any prescribed bound.

If $\log k/\log l$ is rational, we suppose that $\log k/\log l = a/b$ with $(a, b) = 1$. From the proof, we know that Theorem 1.2 also holds when a, b have different parities. However, the case that both a and b are odd seems to be difficult. We believe that the following conclusion is correct:

CONJECTURE 1.3. *Let $\log k/\log l = a/b$ with $(a, b) = 1$. If a, b are both odd, then there does not exist any set $A \subseteq \mathbb{N}$ such that both $r_{1,k}(A, n) = r_{1,k}(\mathbb{N} \setminus A, n)$ and $r_{1,l}(A, n) = r_{1,l}(\mathbb{N} \setminus A, n)$ hold for all sufficiently large integers n.*

REMARK 1.4. We can only prove that there does not exist such a set A when one of the equalities holds for all $n \geq 1$.

Now we pose two problems which we cannot settle.

PROBLEM 1.5. *Let k_1 and k_2 be two integers with $k_2 > k_1 \geq 2$ and $(k_1, k_2) = 1$. Do there exist two sets A and B with infinite symmetric difference such that $r_{k_1,k_2}(A, n) = r_{k_1,k_2}(B, n)$ for all sufficiently large integers n?*

PROBLEM 1.6. *Given two positive integers k, l with $k > l \geq 2$, do there exist two sets A and B with infinite symmetric difference such that both $r_{1,k}(A, n) = r_{1,k}(B, n)$ and $r_{1,l}(A, n) = r_{1,l}(B, n)$ hold for all sufficiently large integers n?*

2. Proofs. We first give a definition. The *Farey series* $\mathfrak{F}_n$ of order n is the ascending series of irreducible fractions between 0 and 1 whose denominators do not exceed n. To prove Theorem 1.2, we need some lemmas.

LEMMA 2.1. *Let $k > 1$ and $n_0 \geq 1$ be two integers. If $A \subseteq \mathbb{N}$ is such that $r_{1,k}(A, n) = r_{1,k}(\mathbb{N} \setminus A, n)$ for all $n \geq n_0$, then*

$$n \in A \;\Leftrightarrow\; \lfloor n/k \rfloor \notin A,\, n \geq n_0 + k.$$

Lemma 2.1 follows from Corollary 2 in [YC2].

LEMMA 2.2 (see [HW, Theorem 28]). *If $h_1/h_2, h_3/h_4$ are two successive terms of $\mathfrak{F}_n$, then $h_2h_3 - h_1h_4 = 1$.*

LEMMA 2.3 (see [HW, Theorem 30]). *If $h_1/h_2, h_3/h_4$ are two successive terms of $\mathfrak{F}_n$, then $h_2 + h_4 > n$.*

Now we shall use Lemmas 2.2 and 2.3 to prove Lemma 2.4, which is similar to Dirichlet's theorem on Diophantine approximation.

LEMMA 2.4. *If ξ is any irrational number with $0 < \xi < 1$, and n_0 a positive integer, then there is an integer $n \geq n_0$ and an irreducible fraction h_1/h_2 such that*

$$0 < h_2 \leq n, \qquad 2 \nmid h_1 + h_2, \qquad \left|\frac{h_1}{h_2} - \xi\right| \leq \frac{1}{h_2(n+1)}.$$

Proof. Since ξ is irrational, there exist two successive terms $n_1/n_1', n_2/n_2'$ of $\mathfrak{F}_{n_0}$ such that $n_1/n_1' < \xi < n_2/n_2'$, and so ξ falls in one of the intervals

$$\left(\frac{n_1}{n_1'}, \frac{n_1 + n_2}{n_1' + n_2'}\right), \qquad \left(\frac{n_1 + n_2}{n_1' + n_2'}, \frac{n_2}{n_2'}\right).$$

Without loss of generality, we may assume that ξ belongs to the latter interval. By Lemmas 2.2 and 2.3, it follows that $n_1'n_2 - n_1n_2' = 1$ and

$$(2.1) \qquad 0 < \frac{n_2}{n_2'} - \xi \leq \frac{n_2}{n_2'} - \frac{n_1 + n_2}{n_1' + n_2'} = \frac{1}{n_2'(n_1' + n_2')} \leq \frac{1}{n_2'(n_0 + 1)}.$$

If $2 \nmid n_2 + n_2'$, then we take $n = n_0$, $h_1 = n_2$, $h_2 = n_2'$. By $n_2' \leq n_0$ and the inequality (2.1), Lemma 2.4 is true. Now we suppose that both n_2 and n_2' are odd. Let $n_3 = n_1 + n_2$, $n_3' = n_1' + n_2'$, and let

$$n_s = n_2 + n_{s-1}, \qquad n_s' = n_2' + n_{s-1}'$$

for $s \geq 4$. Then $n_s/n_s', n_2/n_2'$ are two successive terms of $\mathfrak{F}_{n_s'}$ for all $s \geq 3$. From Lemma 2.2 it follows that $2 \nmid n_s + n_s'$ for all $s \geq 3$. Since

$$\lim_{s\to\infty} \frac{n_s}{n_s'} = \lim_{s\to\infty} \frac{n_1 + (s-2)n_2}{n_1' + (s-2)n_2'} = \frac{n_2}{n_2'}$$

and n_s/n_s' is strictly increasing, there exists an integer $t \geq 3$ such that

$$\frac{n_t}{n_t'} < \xi < \frac{n_{t+1}}{n_{t+1}'} < \frac{n_2}{n_2'}.$$

Hence, by Lemma 2.2 we have

$$0 < \xi - \frac{n_t}{n'_t} < \frac{n_{t+1}}{n'_{t+1}} - \frac{n_t}{n'_t} = \frac{1}{n'_t n'_{t+1}} = \frac{1}{n'_t(n'_2 + n'_t)} \le \frac{1}{n'_t(n'_t + 1)}.$$

Therefore, if we take $h_1 = n_t$ and $h_2 = n = n'_t$, then the conclusion is true. ∎

Proof of Theorem 1.2. Suppose that k, l are multiplicatively independent and, for some integer $n_0 \ge 1$, both $r_{1,k}(A, n) = r_{1,k}(\mathbb{N} \setminus A, n)$ and $r_{1,l}(A, n) = r_{1,l}(\mathbb{N} \setminus A, n)$ for all $n \ge n_0$. By Lemma 2.1, there exists a positive integer n_1 such that for all $n \ge n_1$, the following two conditions hold:

(i) $n \in A \Leftrightarrow kn + s \notin A$ for all s with $0 \le s < k - 1$;
(ii) $n \in A \Leftrightarrow ln + s \notin A$ for all s with $0 \le s < l - 1$.

Without loss of generality, we may assume $k < l$. Let

$$n_2 = \frac{\log l}{\log(1 + 1/n_1)}.$$

By Lemma 2.4, there exists an integer $n_3 \ge n_2$ and an irreducible fraction β/α such that

$$0 < \alpha \le n_3, \qquad 2 \nmid \alpha + \beta, \qquad \left| \frac{\beta}{\alpha} - \frac{\log k}{\log l} \right| \le \frac{1}{\alpha(n_3 + 1)} < \frac{1}{\alpha n_2}.$$

Without loss of generality, we assume that α is odd and β is even.

If $\beta/\alpha - \log k/\log l > 0$, then we take a real number t such that

$$\frac{\beta}{\alpha} - \frac{\log k}{\log l} = \frac{\log(1 + t/n_1)}{\alpha \log l}.$$

It follows that $0 < t < 1$ and

$$n_1 l^\beta = n_1 k^\alpha + s, \quad \text{where} \quad s = k^\alpha t < k^\alpha.$$

Now we express the integer s in the scale of k as

$$s = b_{\alpha-1}k^{\alpha-1} + b_{\alpha-2}k^{\alpha-2} + \cdots + b_1 k + b_0, \text{ where } 0 \le b_0, b_1, \ldots, b_{\alpha-1} \le k - 1.$$

By (i), since β is even, we have

$$n_1 l^\beta \in A \;\Leftrightarrow\; n_1 l^{\beta-1} \notin A \;\Leftrightarrow\; \cdots \;\Leftrightarrow\; n_1 \in A.$$

On the other hand, by (ii), since α is odd, we have

$$n_1 k^\alpha + s \in A \;\Leftrightarrow\; n_1 k^{\alpha-1} + b_{\alpha-1}k^{\alpha-2} + \cdots + b_1 \notin A \;\Leftrightarrow\; \cdots \;\Leftrightarrow\; n_1 \notin A.$$

This is a contradiction.

If $\beta/\alpha - \log k/\log l < 0$, then we take a number t such that

$$\frac{\log k}{\log l} - \frac{\beta}{\alpha} = \frac{\log(1 + t/n_1)}{\alpha \log l}.$$

As before, we reach a contradiction. We leave the details to the reader.

This completes the proof of Theorem 1.2. ∎

Acknowledgements. I would like to thank my supervisor Professor Yong-Gao Chen for posing the problem that I study in this paper, and also for his valuable suggestions and useful discussions. This research was supported by the National Natural Science Foundation of China, Grant No. 11371195, and the Project of Graduate Education Innovation of Jiangsu Province (CXZZ12-0381).

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Received 21 August 2013 (6010)