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ON STABLE EQUIVALENCES OF MODULE SUBCATEGORIES OVER A SEMIPERFECT NOETHERIAN RING

BY

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Abstract. Given a semiperfect two-sided noetherian ring Λ, we study two subcategories $\mathcal{A}_k(\Lambda) = \{M \in \operatorname{mod} \Lambda \mid \operatorname{Ext}^j_\Lambda(\operatorname{Tr} M, \Lambda) = 0\ (1 \le j \le k)\}$ and $\mathcal{B}_k(\Lambda) = \{N \in \operatorname{mod} \Lambda \mid \operatorname{Ext}^j_\Lambda(N, \Lambda) = 0\ (1 \le j \le k)\}$ of the category $\operatorname{mod} \Lambda$ of finitely generated right Λ-modules, where $\operatorname{Tr} M$ is Auslander's transpose of M. In particular, we give another convenient description of the categories $\mathcal{A}_k(\Lambda)$ and $\mathcal{B}_k(\Lambda)$, and we study category equivalences and stable equivalences between them. Several results proved in [J. Algebra 301 (2006), 748–780] are extended to the case when Λ is a two-sided noetherian semiperfect ring.

1. Introduction and preliminaries. Throughout this paper we assume that Λ is a semiperfect two-sided noetherian ring. We denote by $\operatorname{mod} \Lambda$ the category of finitely generated right Λ-modules. Following [ABr], given an integer $k \ge 1$, we study two subcategories

$$\mathcal{A}_k(\Lambda) = \{M \in \operatorname{mod} \Lambda \mid \operatorname{Ext}^j_\Lambda(\operatorname{Tr} M, \Lambda) = 0\ (1 \le j \le k)\},$$
$$\mathcal{B}_k(\Lambda) = \{N \in \operatorname{mod} \Lambda \mid \operatorname{Ext}^j_\Lambda(N, \Lambda) = 0\ (1 \le j \le k)\}$$

of the category $\operatorname{mod} \Lambda$, where $\operatorname{Tr} M$ is Auslander's transpose of M (see [ASS], [ABr]). Following [T] we also study the category Gproj-Λ of G-projective Λ-modules. We recall that Λ is *semiperfect* if every module in mod Λ admits a projective cover in $\operatorname{mod} \Lambda$. One of the main tools we use is the minimal approximation technique introduced by Auslander in the 1960s. We recall it in Sections 2–3 and we prove several preparatory results on approximations and the category Gproj-Λ. In particular, we extend several results of Takahashi [T] from the commutative case to the case when Λ is a two-sided noetherian semiperfect ring.

In the second part of the paper (Sections 4–7) we study category equivalences between $\mathcal{A}_k(\Lambda)$ and $\mathcal{B}_k(\Lambda)$. In the particular case when $k = 1$, we show in Theorem 5.5 that $\mathcal{A}_1(\Lambda)$ and $\mathcal{B}_1(\Lambda)$ are stably equivalent. One of the main results of the second part of the paper is a characterization of $\mathcal{A}_k(\Lambda)$

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and $\mathcal{B}_k(\Lambda)$ in Proposition 5.6 and a stable equivalence result in Theorem 5.7. Moreover we prove that the following are equivalent for $M \in \operatorname{mod} \Lambda$:

(a) $M \in \mathcal{A}_k(\Lambda) \cap \mathcal{B}_k(\Lambda)$;
(b) $\operatorname{Ext}^i_\Lambda(M, \Lambda) = 0 = \operatorname{Ext}^i_\Lambda(\operatorname{Tr} M, \Lambda)$ for $1 \le i \le k$;
(c) M admits a k-subcomplete resolution,

where 'k-subcomplete resolution' is defined in Section 6.

The reader is referred to [A]–[AR] and [Y] for details on the minimal approximation technique and its application. Results of a similar nature on classical orders, crossed products, and Cohen–Macaulay modules are discussed in [AM], [B1], [B2], [C], [Dr], [GN], [Si1]–[Si3], [S], and [Y].

2. Approximation and (co)syzygy. Let Λ be a two-sided noetherian ring. Further we assume that it is semiperfect (cf. [AF], [F]). We denote the category of finitely generated right Λ-modules by $\operatorname{mod} \Lambda$ and the one of finitely generated left Λ-modules by $\operatorname{mod} \Lambda^{\mathrm{op}}$.

2.1. Proj Λ-approximation. We recall from [T] the notions of approximation and minimality, and basic facts that are useful for constructing syzygies.

Definition 2.1. Let $M, N \in \operatorname{mod} \Lambda$ and $\rho : M \to N$ a Λ-homomorphism.

(1) We say that ρ is *right minimal* if any $f \in \operatorname{End}_\Lambda(M)$ satisfying $\rho = \rho f$ is an automorphism.
(2) We say that ρ is *left minimal* if any $f \in \operatorname{End}_\Lambda(N)$ satisfying $\rho = g\rho$ is an automorphism.

Definition 2.2. Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} \Lambda$.

(1) Let $X \in \mathcal{X}$ and $M \in \operatorname{mod} \Lambda$, and let $\varphi : X \to M$ be a homomorphism.
   (a) We call φ or X a *right $\mathcal{X}$-approximation* of M if for any homomorphism $\varphi' : X' \to M$ with $X' \in \mathcal{X}$ there exists a homomorphism $f : X' \to X$ such that $\varphi' = \varphi f$.
   (b) We call φ or X a *minimal right $\mathcal{X}$-approximation* of M if φ is a right $\mathcal{X}$-approximation and is right minimal.
(2) Let $X \in \mathcal{X}$ and $M \in \operatorname{mod} \Lambda$, and let $\varphi : M \to X$ be a homomorphism.
   (a) We call φ or X a *left $\mathcal{X}$-approximation* of M if for any homomorphism $\varphi' : M \to X'$ with $X' \in \mathcal{X}$ there exists a homomorphism $f : X \to X'$ such that $\varphi' = f\varphi$.
   (b) We call φ or X a *minimal left $\mathcal{X}$-approximation* of M if φ is a left $\mathcal{X}$-approximation and is left minimal.

By definition, it is easy to see that a minimal right or left $\mathcal{X}$-approximation is uniquely determined up to isomorphism, if it exists. Supposing that $\mathcal{X}$ is closed under direct summands, a Λ-module having a right (resp. left) $\mathcal{X}$-approximation also has a minimal right (resp. left) $\mathcal{X}$-approximation.

2.2. Minimal proj Λ-approximation. When one studies the noncommutative version of [T], the following generalization of [T, Proposition 2.3] is indispensable. It provides a concrete method of constructing a minimal left proj Λ-approximation and cosyzygies, where proj Λ is the full subcategory of mod Λ consisting of all projective Λ-modules.

For $M \in \operatorname{mod} \Lambda$, we denote by θ_M the canonical evaluation map $M \to M^{**}$.

Proposition 2.3. *Let Λ be a semiperfect two-sided noetherian ring and let $M \in \operatorname{mod} \Lambda$.*

(1) *If $\varphi : P \to M$ is a Λ-homomorphism with $P \in \operatorname{proj} \Lambda$, then the following two conditions are equivalent:*
   (a) *φ is a minimal right* proj Λ*-approximation of M;*
   (b) *φ is a projective cover.*
(2) *If $\pi : P \to M^*$ is a projective cover of M^* with $P \in \operatorname{proj} \Lambda^{\mathrm{op}}$, and $\alpha := \pi^* \theta_M : M \to P^*$, then α is a minimal left* proj Λ*-approximation of M.*
(3) *If $\varphi : P \to M^*$ is a minimal right* proj Λ^{op}*-approximation of M^* with $P \in \operatorname{proj} \Lambda^{\mathrm{op}}$, then $\varphi^* \theta_M$ is a minimal left* proj Λ*-approximation of M.*

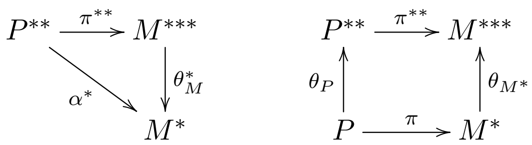
Proof. (1) (a)⇒(b): Suppose $\varphi : P \to M$ is a minimal right proj Λ-approximation of M and $q : Q \to M$ a projective cover of M. By definition, there is a Λ-homomorphism $f : Q \to P$ such that $q = \varphi f$. Thus φ is surjective. By [AF, Lemma 17.17], there exists a decomposition $P = P' \oplus P''$ with $P', P'' \in \operatorname{proj} \Lambda$ such that 1) $P' \simeq Q$, 2) $P'' \subset \operatorname{Ker} \varphi$, 3) $\varphi|_{P'} : P' \to M$ is a projective cover for M. We define a homomorphism $g : P \to P$ by $g(x, y) = (x, 0)$, where $x \in P'$ and $y \in P''$. Let $\varphi = \varphi_1 \oplus \varphi_2$ with $\varphi_1 : P' \to M$ and $\varphi_2 : P'' \to M$. By 2), we have $\varphi(0, y) = 0$. Hence

$$\varphi g(x, y) = \varphi(x, 0) = \varphi_1(x),$$
$$\varphi(x, y) = \varphi_1(x) + \varphi_2(y) = \varphi_1(x).$$

Therefore $\varphi g = \varphi$, and so g is an automorphism. Hence $P'' = 0$, so that $P = P'$. By 3), φ is a projective cover.

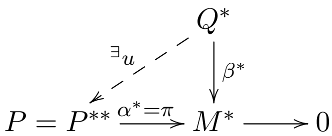
(1) (b)⇒(a): By assumption, φ is a right proj Λ-approximation of M. It is easily shown that φ is right minimal if it is a projective cover.

(2) Since $\alpha^* = \theta_M^* \pi^{**}$, we get the following commutative diagrams:

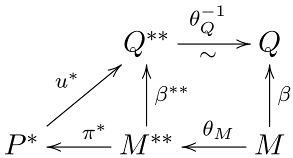


Hence $\pi^{**}\theta_P = \theta_{M^*}\pi$, so that $\pi^{**} = \theta_{M^*}\pi\theta_P^{-1}$. By the definition of α, we have $\alpha^* = \theta_M^*\pi^{**}$. Hence $\alpha^* = \theta_M^*\theta_{M^*}\pi\theta_P^{-1} = \pi$. Take a Λ-homomorphism $h : P^* \to P^*$ with $\alpha = h\alpha$. Then $\alpha^* = \alpha^* h^*$, and since $\alpha^* = \pi$, we have $\pi = \pi h^*$. Since $P \cong P^{**}$, we may think $h^* : P \to P$. Consider the short exact sequence $0 \to \operatorname{Ker}\pi \to P \xrightarrow{\pi} M^* \to 0$. It follows that $\pi(\operatorname{Im} h^* + \operatorname{Ker}\pi) = \pi h^*(P) = \pi(P)$, so that $\operatorname{Im} h^* + \operatorname{Ker}\pi = P$. Since π is a projective cover, we have $\operatorname{Im} h^* = P$. Thus h^* is an automorphism. Hence $h = \theta_{P^*}^{-1} h^{**}\theta_{P^*}$ is also an automorphism. This means that α is left minimal.

We now show that α is a left $\operatorname{proj}\Lambda$-approximation of M. Let $Q \in \operatorname{proj}\Lambda$, and $\beta : M \to Q$ a Λ-homomorphism. Then there is $u : Q^* \to P^{**}$ such that the following is commutative:

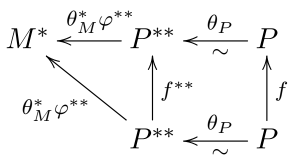


This gives the commutative diagram



Set $v := \theta_Q^{-1}u^* : P^* \to Q$. Then $v\alpha = \theta_Q^{-1}u^*\pi^*\theta_M = \theta_Q^{-1}\beta^{**}\theta_M = \beta$. This implies that α is a left $\operatorname{proj}\Lambda$-approximation of M.

(3) We first show that $\varphi^*\theta_M$ is left minimal. Take $g^* : P^* \to P^*$ with $\varphi^*\theta_M = g\varphi^*\theta_M$. We can write $g = f^*$ for $f : P \to P$. It suffices to show that f is an automorphism. By assumption, we get the commutative diagram



Thus $\theta_M^*\varphi^{**}\theta_P f = \theta_M^*\varphi^{**}\theta_P$. It is well-known that $\varphi^{**}\theta_P = \theta_{M^*}\varphi$ and $\theta_M^*\theta_{M^*} = \mathrm{id}_{M^*}$. Hence $\theta_M^*\varphi^{**}\theta_P = \theta_M^*\theta_{M^*}\varphi = \varphi$. Therefore, $\varphi f = \varphi$. By assumption, φ is right minimal, so that f is an automorphism. Hence, $f^* = g$ is an automorphism. Thus $\varphi^*\theta_M$ is left minimal.

Under the assumption that φ is a minimal right proj Λ^{op}-approximation of M^*, we now show that $\varphi^*\theta_M$ is a left proj Λ-approximation of M. Considering (1) for mod Λ^{op} and since φ is a minimal right proj Λ^{op}-approximation of M^*, it follows that φ is a projective cover of M^*. Hence φ is surjective. For any $\psi : M \to Q'$ with $Q' \in \operatorname{proj}\Lambda$, we will show that there exists $\psi' : P^* \to Q'$ such that $\psi = \psi'\varphi^*\theta_M$. For a technical reason, we set $Q' = Q^*$ with $Q \in \operatorname{proj}\Lambda^{\mathrm{op}}$. Applying $(-)^*$ to $M \xrightarrow{\psi} Q^*$, we get $\psi^*\theta_Q : Q \xrightarrow{\theta_Q} Q^{**} \xrightarrow{\psi^*} M^*$. Since φ is surjective, there exists $h : Q \to P$ such that $\varphi h = \psi^*\theta_Q$. Applying $(-)^*$, we get $\theta_Q^*\psi^{**} = h^*\varphi^*$. Applying $(-)^{**}$ to $\psi : M \to Q^*$, we see that $\theta_{Q^*}\psi = \psi^{**}\theta_M$. Hence

$$\psi = \theta_{Q^*}^{-1}\psi^{**}\theta_M = \theta_Q^*\psi^{**}\theta_M = h^*\varphi^*\theta_M.$$

Set $\psi' = h^*$. Then $\psi = \psi'\varphi^*\theta_M$, so that $\varphi^*\theta_M$ is a minimal left proj Λ-approximation of M. This proves (3). ∎

A Λ-module M is said to be *torsionless* if the canonical evaluation map $M \to M^{**}$ is injective. We now state the equivalence of being torsionless and injectivity of each left proj Λ-approximation. The proof of a noncommutative version will also be given.

2.3. Torsionless modules

Proposition 2.4 ([T, Proposition 2,4]). *Let $M \in \operatorname{mod}\Lambda$. Then the following are equivalent:*

1. *M is torsionless;*
2. *every left* proj *Λ-approximation of M is an injective homomorphism;*
3. *there exists a left* proj *Λ-approximation $\varphi : M \to P^*$ of M which is injective.*

Proof. (1)⇒(2): Let $\psi : M \to P^*$ be a left proj Λ-approximation, and suppose $\psi(m) = 0$ for some $m \in M$. Take any $f \in M^*$. Then there exists $g : P^* \to \Lambda$ with $f = g\psi$. Hence $f(m) = g\psi(m) = 0$. Since f is arbitrary, it follows that $m \in \bigcap\{\operatorname{Ker} f' \mid f' \in M^*\}$. In general, $\operatorname{Ker}\theta_M = \bigcap\{\operatorname{Ker} f' \mid f' \in M^*\}$, and hence $\operatorname{Ker}\theta_M = 0$, so $m = 0$.

(2)⇒(3): This is clear.

(3)⇒(1): Let $\psi : M \to P^*$ be an injective left proj Λ-approximation. For $m \in M$, assume that $\theta_M(m) = 0$. Then $f(m) = \theta_M(m)(f) = 0$ for any $f \in M^*$. There is an injective $i : P^* \to \Lambda^n$ for some positive integer n. Let $p_k : \Lambda^n \to \Lambda$ $(1 \le k \le n)$ be the projection. Then $p_k i\psi \in M^*$. Hence $p_k i\psi(m) = 0$, so that $i\psi(m) = 0$. Since $i\psi$ is injective, we see that $m = 0$. Thus M is torsionless. ∎

2.4. Syzygy and cosyzygy. Following [T], we recall the definition of (co)syzygies.

Definition 2.5. Let $M \in \operatorname{mod} \Lambda$ and $\pi : P \to M$ a minimal right $\operatorname{proj} \Lambda$-approximation of M. The *first syzygy* $\Omega M = \Omega^1 M$ of M is defined as $\operatorname{Ker} \pi$, and the *nth syzygy* $\Omega^n M$ of M is defined inductively: $\Omega^n M = \Omega(\Omega^{n-1} M)$ for $n \geq 2$.

We define cosyzygies by dualizing the above.

Definition 2.6. Let Λ be a semiperfect two-sided noetherian ring and let $M \in \operatorname{mod} \Lambda$.

- Take the minimal left $\operatorname{proj} \Lambda$-approximation $\theta : M \to P$. Then $\Omega^{-1} M = \operatorname{Coker} \theta$ is called the *first cosyzygy* of M.
- For $n \geq 2$, assume that the $(n-1)$th cosyzygy $\Omega^{-(n-1)} M$ is defined. Then $\Omega^{-n} M := \Omega^{-1}(\Omega^{-(n-1)} M)$ is called the *nth cosyzygy* of M.

A module M is called *projective free* if M has no nonzero projective summands. The proof of the following fact is similar to that of [T, Proposition 2.6].

Proposition 2.7. *For any Λ-module M and any positive integer n, the nth cosyzygy $\Omega^{-n} M$ is projective free.*

2.5. A vanishing property. For a subcategory $\mathcal{X}$ of $\operatorname{mod} \Lambda$, the subcategory of $\operatorname{mod} \Lambda$ consisting of all the modules M with $\operatorname{Ext}^1_\Lambda(X, M) = 0$ (respectively, $\operatorname{Ext}^1_\Lambda(M, X) = 0$) for all $X \in \mathcal{X}$ is denoted by $\mathcal{X}^{\llcorner}$ (respectively, ${}^{\llcorner}\mathcal{X}$). Usually, the following is deduced from Wakamatsu's lemma; we give a proof based on another lemma.

Proposition 2.8 ([T, Proposition 3.3(2)]). *Any cosyzygy belongs to ${}^{\llcorner}(\operatorname{proj} \Lambda)$, that is,*

$$\operatorname{Ext}^1_\Lambda(\Omega^{-1} M, \Lambda) = 0 \quad \textit{for any } M \in \operatorname{mod} \Lambda.$$

In the proof of Proposition 2.8, we need the following two lemmata.

Lemma 2.9. *Let $M \in \operatorname{mod} \Lambda$. Then there is an exact sequence*

$$\text{(AF)} \qquad 0 \to \operatorname{Ext}^1_\Lambda(\operatorname{Tr} M, \Lambda) \to M \xrightarrow{\theta_M} M^{**} \to \operatorname{Ext}^2_\Lambda(\operatorname{Tr} M, \Lambda) \to 0$$

(*Auslander formula*).

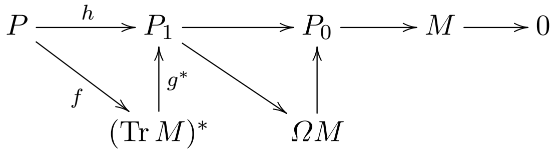
Proof. See [ABr, Chapter 2, §1, p. 48]. ∎

Lemma 2.10. *There are isomorphisms of functors on $\underline{\operatorname{mod}}\ \Lambda$:*

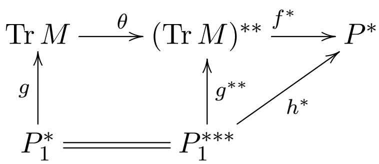
$$\operatorname{Tr} \Omega \cong \Omega^{-1} \operatorname{Tr}, \qquad \Omega \operatorname{Tr} \cong \operatorname{Tr} \Omega^{-1}.$$

Proof. Although the proof might be known, we give it here for the convenience of the reader. Let $M \in \operatorname{mod} \Lambda$ and let $f : P \to (\operatorname{Tr} M)^*$ be a projective cover. For a minimal projective resolution $P_1 \to P_0 \longrightarrow M \to 0$

of M, we have an exact sequence $P_0^* \to P_1^* \xrightarrow{g} \operatorname{Tr} M \to 0$, so $0 \to (\operatorname{Tr} M)^* \xrightarrow{g^*} P_1^{**} \to P_0^{**}$ is exact. Then we get the commutative diagram



with exact top row and $h = g^* f$. Hence $P_1^* \xrightarrow{h^*} P^* \to \operatorname{Tr} \Omega M \to 0$ is exact. Since $h^* = f^* g^{**}$, we get the following commutative diagram:



It follows from Proposition 2.3(2) that $f^*\theta$ is a minimal left proj Λ-approximation of $\operatorname{Tr} M$. Therefore, $\Omega^{-1} \operatorname{Tr} M = \operatorname{Coker}(f^*\theta) = P^*/\operatorname{Im}(f^*\theta)$. By the above diagram, $\operatorname{Im}(f^*\theta) = \operatorname{Im}(f^*\theta g) = \operatorname{Im}(f^* g^{**}) = \operatorname{Im}(h^*)$. Hence

$$\operatorname{Tr} \Omega M = \operatorname{Coker}(h^*) = P^*/\operatorname{Im}(h^*) = P^*/\operatorname{Im}(f^*\theta) = \Omega^{-1} \operatorname{Tr} M.$$

Thus we get $\operatorname{Tr} \Omega \cong \Omega^{-1} \operatorname{Tr}$ on $\underline{\operatorname{mod}}\, \Lambda$. The other isomorphism is obtained by applying the functor Tr on the left and on the right to the first isomorphism. ■

Proof of Proposition 2.8. Let $M \in \operatorname{mod} \Lambda$. By Lemma 2.9 we have the exact sequence

$$0 \to \operatorname{Ext}^1_\Lambda(\operatorname{Tr} \Omega \operatorname{Tr} M, \Lambda) \to \Omega \operatorname{Tr} M \xrightarrow{\theta_{\Omega \operatorname{Tr} M}} (\Omega \operatorname{Tr} M)^{**} \to \operatorname{Ext}^2_\Lambda(\operatorname{Tr} \Omega \operatorname{Tr} M, \Lambda) \to 0.$$

Since $\Omega \operatorname{Tr} M$ is torsionless, $\theta_{\Omega \operatorname{Tr} M}$ is injective, so $\operatorname{Ext}^1_\Lambda(\operatorname{Tr} \Omega \operatorname{Tr} M, \Lambda) = 0$. Since Lemma 2.10 yields $\operatorname{Ext}^1_\Lambda(\operatorname{Tr} \Omega \operatorname{Tr} M, \Lambda) = \operatorname{Ext}^1_\Lambda(\Omega^{-1} \operatorname{Tr} \operatorname{Tr} M, \Lambda) = \operatorname{Ext}^1_\Lambda(\Omega^{-1} M, \Lambda)$, we get $\operatorname{Ext}^1_\Lambda(\Omega^{-1} M, \Lambda) = 0$. ■

3. G-projective modules

3.1. G-projective modules and G-dimension. In this section, we study the basic properties of G-projective modules in the following sense (cf. [ABr], [C]).

DEFINITION 3.1. A Λ-module X is called *G-projective* if the following three conditions hold:

- The canonical homomorphism $\theta_X : X \to X^{**}$ is an isomorphism,
- $\operatorname{Ext}^i_\Lambda(X, \Lambda) = 0$ for any $i > 0$,
- $\operatorname{Ext}^i_\Lambda(X^*, \Lambda) = 0$ for any $i > 0$.

We denote by Gproj-Λ the full subcategory of $\operatorname{mod}\Lambda$ consisting of all G-projective modules. In relation with [ABr], we introduce the following definition.

DEFINITION 3.2. Let $M \in \operatorname{mod}\Lambda$. If for some positive integer n there exists an exact sequence

$$0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0$$

of Λ-modules with $X_i \in$ Gproj-Λ $(0 \leq i \leq n)$, then we say that M has *G-dimension* at most n and write $\text{G-dim}_\Lambda M \leq n$. If such an integer n does not exist, then we say that M has *infinite G-dimension*, $\text{G-dim}_\Lambda M = \infty$.

If $M \in \operatorname{mod}\Lambda$ has G-dimension at most n but does not have G-dimension at most n–1, then we say that M has *G-dimension* n, and write $\text{G-dim}_\Lambda M = n$. In [ABr], G-projective modules are called modules of G-dimension zero and are extensively studied.

For $M \in \operatorname{mod}\Lambda$, a complex of Λ-modules

$$P_\bullet = (\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_{-1} \xrightarrow{d_{-1}} P_{-2} \xrightarrow{d_{-2}} \cdots)$$

is called a *complete resolution* of M if the following conditions are satisfied:

- $P_i \in \operatorname{proj}\Lambda$ for any $i \in \mathbb{Z}$,
- $\mathrm{H}_i(P_\bullet) = 0 = \mathrm{H}^i((P_\bullet)^*)$ for any $i \in \mathbb{Z}$,
- $\operatorname{Im} d_0 = M$.

3.2. A characterization of G-projective modules. We give the following characterization

PROPOSITION 3.3. *Let $M \in \operatorname{mod}\Lambda$. Then the following are equivalent:*

(a) *M is G-projective;*
(b) $\operatorname{Ext}^i_\Lambda(M, \Lambda) = 0 = \operatorname{Ext}^i_\Lambda(\operatorname{Tr} M, \Lambda)$ *for any $i > 0$;*
(c) *M has a complete resolution.*

Proof. The equivalence (a)⇔(b) is shown in [ABr, Proposition 3.8]. To prove (b)⇔(c), we need the following from [ABr, Theorem 2.17].

PROPOSITION 3.4. *The following are equivalent for any $M \in \operatorname{mod}\Lambda$ and $n > 0$:*

- *M is n-torsion free, that is,* $\operatorname{Ext}^i_\Lambda(\operatorname{Tr} M, \Lambda) = 0$ *for any $1 \leq i \leq n$;*
- *There exists an exact sequence* $0 \to M \to P_1 \to \cdots \to P_n$ $(P_1, \ldots, P_n \in \operatorname{proj}\Lambda)$ *such that* $P_n^* \to \cdots \to P_1^* \to M^* \to 0$ *is also exact.*

Assume (b) holds. It follows from Proposition 3.4 that there exists an exact sequence $0 \to M \to P_{-1} \to \cdots \to P_{-n} \to \cdots$ such that $\cdots \to P_{-n}^* \to \cdots \to P_{-1}^* \to M^* \to 0$ is exact. Having a minimal projective resolution, $\cdots \to P_n \to \cdots \to P_1 \to M \to 0$, we apply $(-)^*$, and get an exact sequence

$0 \to M^* \to P_1^* \to \cdots \to P_n^* \to \cdots$. Then we get a complete resolution: $\cdots \to P_1 \to P_0 \to P_{-1} \to \cdots$ of M, which implies (c).

Using Proposition 3.4, we can show (c)⇒(b). ∎

G-projective modules are invariant under some functors.

PROPOSITION 3.5 (cf. [T, Proposition 3.3(2)]). *If a module $M \in \operatorname{mod} \Lambda$ is G-projective, then so are M^*, $\operatorname{Tr} M$, ΩM, and $\Omega^{-1} M$.*

Proof. Assume that M is G-projective. Then we can easily see that M^* and $\operatorname{Tr} M$ are G-projective. We have an exact sequence $0 \to \Omega M \to P \to M \to 0$ ($P \in \operatorname{proj} \Lambda$). By [ABr, Lemma 3.10], ΩM is G-projective.

Finally, we show that $\Omega^{-1} M$ is G-projective. Let $\varphi : P \to M^*$ be a projective cover. By Proposition 2.3(1)&(3), $\varphi^* \theta_M$ is a minimal left $\operatorname{proj} \Lambda$-approximation of M. By definition, $\Omega M^* = \operatorname{Ker} \varphi$ and $\Omega^{-1} M = \operatorname{Coker}(\varphi^* \theta_M)$. Applying $(-)^*$ to the exact sequence $0 \to \Omega M^* \to P \xrightarrow{\varphi} M^* \to 0$, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{\varphi^*\theta_M} & P^* & \longrightarrow & \Omega^{-1}M & \longrightarrow & 0 \\
 & & {\scriptstyle \theta_M}\downarrow\wr & & \| & & & & \\
0 & \longrightarrow & M^{**} & \xrightarrow{\varphi^*} & P^* & \longrightarrow & (\Omega M^*)^* & \longrightarrow & 0
\end{array}$$

Therefore, $(\Omega M^*)^* \cong \Omega^{-1} M$, so that $\Omega^{-1} M$ is also G-projective. ∎

3.3. The category of G-projective modules. Before studying the properties of the category Gproj-Λ, we fix some notation.

DEFINITION 3.6. A subcategory $\mathcal{X}$ of $\operatorname{mod} \Lambda$ is called *resolving* if:

- $\mathcal{X}$ contains $\operatorname{proj} \Lambda$,
- $\mathcal{X}$ is closed under direct summands,
- $\mathcal{X}$ is closed under extensions,
- $\mathcal{X}$ is closed under kernels of epimorphisms.

Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} \Lambda$. We will use several subcategories of $\operatorname{mod} \Lambda$ connected with $\mathcal{X}$ (see [ASS] for more details). We set

$$\mathcal{X}^{\perp} := \{M \in \operatorname{mod} \Lambda \mid \operatorname{Ext}^i_\Lambda(X, M) = 0 \text{ for any } X \in \mathcal{X} \text{ and } i > 0\},$$

$$^{\perp}\mathcal{X} := \{M \in \operatorname{mod} \Lambda \mid \operatorname{Ext}^i_\Lambda(M, X) = 0 \text{ for any } X \in \mathcal{X} \text{ and } i > 0\},$$

$$\widehat{\mathcal{X}} := \{M \in \operatorname{mod} \Lambda \mid \text{there exists } n \geq 0 \text{ and an exact sequence } 0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0 \text{ with } X_i \in \mathcal{X} \text{ for } 0 \leq i \leq n\}.$$

A subcategory $\mathcal{Y}$ of $\mathcal{X}$ is called *Ext-injective* in $\mathcal{X}$ if $\mathcal{Y}$ is contained in $\mathcal{X}^{\perp}$. A subcategory $\mathcal{Y}$ of $\mathcal{X}$ is called a *cogenerator* of $\mathcal{X}$ if there exists an exact sequence $0 \to X \to Y \to X' \to 0$ with $Y \in \mathcal{Y}$ and $X' \in \mathcal{X}$ for any $X \in \mathcal{X}$.

We recall the following result due to Auslander and Buchweitz [ABu, Theorem 1.1, Proposition 3.6].

LEMMA 3.7. *Let $\mathcal{X}$ be a resolving subcategory of* $\operatorname{mod}\Lambda$ *with Ext-injective cogenerator $\mathcal{W}$. Then:*

(1) *$\mathcal{X}$ is contravariantly finite in $\widehat{\mathcal{X}}$,*
(2) *$\widehat{\mathcal{W}} = \mathcal{X}^{\perp} \cap \widehat{\mathcal{X}}$.*

Recall that, for subcategories $\mathcal{X} \subset \mathcal{X}' \subset \operatorname{mod}\Lambda$, $\mathcal{X}$ is *contravariantly finite* in $\mathcal{X}'$ if any $M \in \mathcal{X}'$ has a right $\mathcal{X}$-approximation.

Concerning the category Gproj-Λ, we have

PROPOSITION 3.8 ([T, Proposition 3.7]). *The category* Gproj-Λ *is a resolving subcategory of* mod Λ *with Ext-injective cogenerator* proj Λ.

Proof. An easy calculation shows that Gproj-Λ is a resolving subcategory of $\operatorname{mod}\Lambda$. Since $\operatorname{proj}\Lambda$ is contained in $(\text{Gproj-}\Lambda)^{\perp}$, it is Ext-injective in Gproj-Λ. Take any $X \in$ Gproj-Λ. Then X is torsionless, hence we have an exact sequence $0 \to X \to P \to \Omega^{-1}X \to 0$ with $P \in \operatorname{proj}\Lambda$ by Proposition 2.4. Since $\Omega^{-1}X \in$ Gproj-Λ, $\operatorname{proj}\Lambda$ is a cogenerator for Gproj-Λ. ∎

4. Stable categories $\underline{\mathcal{A}_k}$, $\underline{\mathcal{B}_k}$, $\underline{\text{Gproj-}\Lambda}$. In this section, we study categories containing Gproj-Λ. We follow the results in [T, §7] on stable categories. For a subcategory $\mathcal{C}$ of mod Λ, we denote by $\underline{\mathcal{C}}$ the *stable category* of $\mathcal{C}$, that is, the objects of $\underline{\mathcal{C}}$ are the same as those of $\mathcal{C}$, and for objects $M, N \in \mathcal{C}$, the set of morphisms from M to N is defined by

$$\underline{\operatorname{Hom}}_{\Lambda}(M,N) = \operatorname{Hom}_{\Lambda}(M,N)/\mathfrak{P}_{\Lambda}(M,N),$$

where $\mathfrak{P}_{\Lambda}(M,N)$ is the submodule of $\operatorname{Hom}_{\Lambda}(M,N)$ consisting of all homomorphisms from M to N factoring through some projective Λ-module.

4.1. Preliminaries. We record some elementary results for projective covers and syzygies. Let $M \in \operatorname{mod}\Lambda$. Suppose that an exact sequence $0 \to K' \xrightarrow{\iota'} P' \xrightarrow{q} M \to 0$ with $P' \in \operatorname{proj}\Lambda$ is given. Let $0 \to K \xrightarrow{\iota} P \xrightarrow{p} M \to 0$ be a projective cover of M. Then it follows from [AF, 17.17] that there exist $\alpha : P \to P'$ and $\pi : P' \to P$ such that $q = p\pi$, $\pi\alpha = \mathrm{id}_P$, and $P' = \operatorname{Ker}\pi \oplus \operatorname{Im}\alpha$ with $\operatorname{Im}\alpha \cong P$. We set $P'' = \operatorname{Ker}\pi \in \operatorname{proj}\Lambda$ and identify $K = \iota(K)$, respectively, $K' = \iota'(K')$. Then the following holds.

LEMMA 4.1. *We have $K' = \alpha(K) \oplus P$, consequently $K' \cong \Omega M \oplus P''$.*

4.2. On syzygy functors. The functors Ω and Ω^{-1} are well-behaved on Gproj-Λ.

Proposition 4.2 ([T, Proposition 7.1]). *For $M, N \in \operatorname{Gproj-}\Lambda$, the homomorphisms*

$$\begin{cases} \underline{\operatorname{Hom}}_\Lambda(M,N) \to \underline{\operatorname{Hom}}_\Lambda(\Omega M, \Omega N), \\ \underline{\operatorname{Hom}}_\Lambda(M,N) \to \underline{\operatorname{Hom}}_\Lambda(\Omega^{-1} M, \Omega^{-1} N) \end{cases}$$

defined by Ω and Ω^{-1} are isomorphisms.

Proof. The first isomorphism follows from [ABr, Proposition 2.43]. To prove the second, we show that for $M \in \operatorname{mod}\Lambda$ there is an exact sequence

$$(1) \qquad 0 \to \operatorname{Ext}^1_\Lambda(\operatorname{Tr} M, \Lambda) \to M \to \Omega\Omega^{-1} M \oplus P \to 0$$

with some $P \in \operatorname{proj}\Lambda$.

Let $\pi : P \to M^*$ be a projective cover of M^*. Set $f = \pi^* \theta_M : M \to P^*$. It follows from Proposition 2.3 that f is a minimal left $\operatorname{proj}\Lambda$-approximation of M, so that $\Omega^{-1} M = \operatorname{Coker} f$. Hence, $M \xrightarrow{f} P^* \xrightarrow{g} \Omega^{-1} M \to 0$ is exact. Since $\operatorname{Im}\theta_M \cong \operatorname{Im} f = \operatorname{Ker} g$, we get an exact sequence $0 \to \operatorname{Im}\theta_M \to P^* \xrightarrow{g} \Omega^{-1} M \to 0$. Since P^* is projective, $\operatorname{Im}\theta_M \cong \Omega\Omega^{-1} M \oplus P$ with $P \in \operatorname{proj}\Lambda$ by Lemma 4.1. From (AF), we have an exact sequence $0 \to \operatorname{Ext}^1_\Lambda(\operatorname{Tr} M, \Lambda) \to M \xrightarrow{\theta_M} M^{**}$, which provides an exact sequence (1).

Let $M, N \in \operatorname{Gproj-}\Lambda$. Then $M \cong \Omega\Omega^{-1} M \oplus P$ and $N \cong \Omega\Omega^{-1} N \oplus Q$, $P, Q \in \operatorname{proj}\Lambda$. Since $\Omega^{-1} M, \Omega^{-1} N \in \operatorname{Gproj-}\Lambda$ by Proposition 3.5, we can apply the first isomorphism:

$$(2) \quad \underline{\operatorname{Hom}}_\Lambda(\Omega^{-1} M, \Omega^{-1} N) \cong \underline{\operatorname{Hom}}_\Lambda(\Omega\Omega^{-1} M, \Omega\Omega^{-1} N) \cong \underline{\operatorname{Hom}}_\Lambda(M, N). \ \blacksquare$$

Remark 4.3. The assumption in Proposition 4.2 that $M, N \in \operatorname{Gproj-}\Lambda$ is too strong. The conditions which we will find have wider applications.

4.3. Categories $\mathcal{A}_k$ and $\mathcal{B}_k$. In what follows, we write $\mathcal{A}_k = \mathcal{A}_k(\Lambda)$ and $\mathcal{B}_k = \mathcal{B}_k(\Lambda)$, for short. Note that, by Proposition 3.3, $\mathcal{A}_\infty \cap \mathcal{B}_\infty = \operatorname{Gproj-}\Lambda$. Following [ABr], we call modules in $\mathcal{A}_k$ *k-torsion free modules.* The first isomorphism of Proposition 4.2 is valid if $\operatorname{Ext}^1_\Lambda(M, \Lambda) = 0$, i.e., $M \in \mathcal{B}_1$, due to [ABr, Proposition 2.43]. If $M \in \mathcal{A}_1$, then $M \cong \Omega\Omega^{-1} M \oplus P$ for $P \in \operatorname{proj}\Lambda$ by (1) in the proof of Proposition 4.2. Due to Proposition 2.8, we have $\Omega^{-1} M \in \mathcal{B}_1$. Assume further $N \in \mathcal{A}_1$; then applying the first isomorphism to $\Omega^{-1} M$ and $\Omega^{-1} N$, we get (2) above.

Thus, we have shown that if $M \in \mathcal{A}_1 \cap \mathcal{B}_1$ and $N \in \mathcal{A}_1$ then the two homomorphisms of Proposition 4.2 are isomorphisms.

Similarly, we can ease the assumption of [T, Lemma 7.2].

Lemma 4.4. *Let $M \in \operatorname{mod}\Lambda$ and $X \in \mathcal{A}_1 \cap \mathcal{B}_1$. Then*

$$\underline{\operatorname{Hom}}_\Lambda(X, M) \cong \operatorname{Ext}^1_\Lambda(X, \Omega M) \cong \operatorname{Ext}^1_\Lambda(\Omega^{-1} X, M).$$

Proof. $X \in \mathcal{B}_1$ implies $\underline{\operatorname{Hom}}_\Lambda(X, M) \cong \operatorname{Ext}^1_\Lambda(X, \Omega M)$, and $X \in \mathcal{A}_1$ implies $\underline{\operatorname{Hom}}_\Lambda(X, M) \cong \operatorname{Ext}^1_\Lambda(\Omega^{-1} X, M)$. $\blacksquare$

5. Category equivalence between $\underline{\mathcal{A}_k}$ and $\underline{\mathcal{B}_k}$. In the previous section, we have introduced the categories $\mathcal{A}_k$ and $\mathcal{B}_k$ for $k \geq 1$ and considered some facts on Gproj-Λ using these categories. In this section, we prove the category equivalence $\mathcal{A}_k$ and $\mathcal{B}_k$. We expect that these equivalences can be used to generalize the category Gproj-Λ. For $M, N \in \operatorname{mod} \Lambda$, we write $M \sim N$ whenever M and N are stably isomorphic. Thus $M \sim N$ if and only if $M \oplus P \cong N \oplus Q$ for $P, Q \in \operatorname{proj} \Lambda$.

5.1. Category $\mathcal{A}_1$

THEOREM 5.1. *The following are equivalent for $M \in \operatorname{mod} \Lambda$:*

(1) $M \in \mathcal{A}_1$;
(2) *M is torsionless;*
(3) $M \sim \Omega\Omega^{-1}M$.

Proof. We show the following lemma.

LEMMA 5.2.
(1) *For any $M \in \operatorname{mod} \Lambda$, we have $\Omega^{-1}M \in \mathcal{B}_1$.*
(2) *For any $M \in \operatorname{mod} \Lambda$, we have $\Omega M \in \mathcal{A}_1$.*

Proof. (1) This is nothing but Proposition 2.8.
(2) $\operatorname{Ext}^1_\Lambda(\operatorname{Tr} \Omega M, \Lambda) \cong \operatorname{Ext}^1_\Lambda(\Omega^{-1} \operatorname{Tr} M, \Lambda) = 0$ by Lemma 2.10 and (1). ∎

Proof of Theorem 5.1. (1)⇔(2): This is an easy consequence of the definitions.

(1)⇒(3): In the proof of Proposition 4.2, we have provided the exact sequence

$$0 \to \operatorname{Ext}^1_\Lambda(\operatorname{Tr} M, \Lambda) \to M \to \Omega\Omega^{-1}M \oplus P \to 0,$$

with some $P \in \operatorname{proj} \Lambda$. Hence (1)⇒(3) holds.

(3)⇒(1): Suppose that $M \sim \Omega\Omega^{-1}M$. Then $M \oplus P \cong \Omega\Omega^{-1}M \oplus Q$ for all $P, Q \in \operatorname{proj} \Lambda$. By Lemma 5.2, $\Omega\Omega^{-1}M \in \mathcal{A}_1$, so $\operatorname{Ext}^1_\Lambda(\operatorname{Tr}(\Omega\Omega^{-1}M), \Lambda) = 0$. Therefore, $\operatorname{Ext}^1_\Lambda(\operatorname{Tr} M, \Lambda) = 0$. Thus $M \in \mathcal{A}_1$. ∎

5.2. Category $\mathcal{B}_1$. We will show that the category $\mathcal{B}_1$ is the counterpart of $\mathcal{A}_1$.

THEOREM 5.3. *The following are equivalent for $M \in \operatorname{mod} \Lambda$:*

(1) $M \in \mathcal{B}_1$;
(2) $M \sim \Omega^{-1}\Omega M$.

Proof. (1)⇒(2): Let $0 \to \Omega M \xrightarrow{\varphi} P(M) \to M \to 0$ be a projective cover of M. Applying $(-)^*$, we get an exact sequence $0 \to M^* \to P(M)^* \xrightarrow{\varphi^*} (\Omega M)^* \to 0$. Let $0 \to K \to P \xrightarrow{p} (\Omega M)^* \to 0$ be a projective cover of $(\Omega M)^*$. By a standard argument [ASS, 17.17], we get the diagram

$$
(*)\qquad
\begin{array}{ccc}
P(M)^* & \xrightarrow{\varphi^*} & (\Omega M)^* \\
{\scriptstyle \pi}\Big\Downarrow\Big\Downarrow{\scriptstyle \pi'} & & \Big\| \\
P \oplus P' & \xrightarrow{(p,0)} & (\Omega M)^*
\end{array}
$$

where $P(M)^* = P \oplus P'$ with $\pi : P(M)^* \to P$ such that $p\pi = \varphi^*$ and $\pi' : P(M)^* \to P'$ by $\pi'(x) = v_x$, since any $x \in P(M)^*$ is uniquely represented as $x = (u_x, v_x)$ with $u_x \in P$, $v_x \in P'$. Let $\lambda = (\pi, \pi') : P(M)^* \to P \oplus P'$. Then λ is an isomorphism. Let $(p, 0) : P \oplus P' \to (\Omega M)^*$ be $(p, 0)(u, v) = p(u)$ for $u \in P$, $v \in P'$. Take $x \in P(M)^*$. Then

$$
\begin{aligned}
(p,0) \circ (\pi, \pi')(x) &= (p,0) \circ (\pi, \pi')(u_x, v_x) = (p,0) \circ (\pi(u_x), \pi'(v_x)) \\
&= p\pi(u_x) = \varphi^*(u_x) \\
&= \varphi^*(u_x, v_x) \qquad (v_x \in P' = \operatorname{Ker}\pi \subset \operatorname{Ker}\varphi^*) \\
&= \varphi^*(x).
\end{aligned}
$$

Hence $\varphi^* = (p, 0)(\pi, \pi') = \mu\lambda$, where $\mu = (p, 0)$. Applying $(-)^*$ to the bottom row of the diagram $(*)$, we get the following commutative diagram with exact rows:

$$
(**)\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & (\Omega M)^{**} & \xrightarrow{\mu^*} & (P \oplus P')^* & & & & \\
 & & {\scriptstyle \theta_{\Omega M}}\Big\uparrow & & \Big\| & & & & \\
0 & \longrightarrow & \Omega M & \xrightarrow{\mu^*\theta_{\Omega M}} & (P \oplus P')^* & \longrightarrow & \operatorname{Coker}(\mu^*\theta_{\Omega M}) & \longrightarrow & 0
\end{array}
$$

Note that

$$
P^* \oplus P'^* = (P \oplus P')^* \underset{\sim}{\xrightarrow{\lambda^*}} P(M)^{**} \underset{\sim}{\xrightarrow{\theta^{-1}_{P(M)}}} P(M).
$$

By calculation, we find that

$$
(***)\qquad \theta^{-1}_{P(M)}\lambda^*\mu^*\theta_{\Omega M} = \theta^{-1}_{P(M)}\varphi^{**}\theta_{\Omega M} = \theta^{-1}_{P(M)}\theta_{P(M)}\varphi = \varphi.
$$

Since $p^*\theta_{\Omega M}$ is a minimal left proj Λ-approximation of ΩM, we see that $\operatorname{Coker}(\mu^*\theta_{\Omega M}) = \operatorname{Coker}(p^*\theta_{\Omega M}) \oplus P'^* = \Omega^{-1}\Omega M \oplus P'^*$. From the exact sequence $0 \to \Omega M \xrightarrow{\varphi} P(M) \to M \to 0$ and the bottom row of $(**)$, we get the following diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega M & \xrightarrow{\mu^*\theta_{\Omega M}} & P^* \oplus P'^* & \longrightarrow & \Omega^{-1}\Omega M \oplus P'^* & \longrightarrow & 0 \\
 & & \Big\| & & \Big\downarrow{\scriptstyle \theta^{-1}_{P(M)}\lambda^*} & & & & \\
0 & \longrightarrow & \Omega M & \xrightarrow{\varphi} & P(M) & \longrightarrow & M & \longrightarrow & 0
\end{array}
$$

By $(***)$, the left square of this diagram commutes. Since $\theta^{-1}_{P(M)}\lambda^*$ is an isomorphism, we have $\Omega^{-1}\Omega M \oplus P'^* \cong M$. Thus $\Omega^{-1}\Omega M \sim M$.

(2)⇒(1): This follows from Lemma 5.2(1). ∎

Corollary 5.4. *Let $M \in \mathcal{A}_1$. Then there is an isomorphism of functors* $\underline{\mathrm{Hom}}_\Lambda(\Omega^{-1}M, -) \cong \underline{\mathrm{Hom}}_\Lambda(M, \Omega(-))$ *on* $\underline{\mathrm{mod}}\ \Lambda$.

Proof. By assumption, we have $\underline{\mathrm{Hom}}_\Lambda(M, \Omega N) \cong \underline{\mathrm{Hom}}_\Lambda(\Omega\Omega^{-1}M, \Omega N)$. By Proposition 2.8, we can apply [ABr, Proposition 2.43] to obtain $\underline{\mathrm{Hom}}_\Lambda(\Omega^{-1}M, N) \cong \underline{\mathrm{Hom}}_\Lambda(\Omega\Omega^{-1}M, \Omega N)$. Combining these isomorphisms, we get $\underline{\mathrm{Hom}}_\Lambda(\Omega^{-1}M, N) \cong \underline{\mathrm{Hom}}_\Lambda(M, \Omega N)$. ∎

5.3. A stable equivalence for $k = 1$. We summarize the above in

Theorem 5.5. *The functors $\Omega^{-1} : \underline{\mathcal{A}_1} \to \underline{\mathcal{B}_1}$ and $\Omega : \underline{\mathcal{B}_1} \to \underline{\mathcal{A}_1}$ give a category equivalence between $\underline{\mathcal{A}_1}$ and $\underline{\mathcal{B}_1}$.*

Proof. Take $M \in \underline{\mathcal{B}_1}$. Since $\Omega M \in \underline{\mathcal{A}_1}$, by Lemma 5.2 we see that

$$\underline{\mathrm{Hom}}_\Lambda(\Omega M, \Omega N) \cong \underline{\mathrm{Hom}}_\Lambda(\Omega^{-1}\Omega M, N) \cong \underline{\mathrm{Hom}}_\Lambda(M, N)$$

for any $N \in \mathcal{B}_1$ by Corollary 5.4. Thus Ω is fully faithful. It is dense by Theorem 5.1. ∎

We denote by $\mathrm{mod}_P\,\Lambda$ the full subcategory consisting of all $M \in \mathrm{mod}\,\Lambda$ without projective direct summands. Note that, for $M, N \in \mathrm{mod}_P\,\Lambda$, we have $M \sim N$ if and only if $M \cong N$.

5.4. A characterization of $\mathcal{A}_k$ and $\mathcal{B}_k$. We give the following characterization:

Proposition 5.6. *Let $k \geq 1$.*

(1) *The following are equivalent for $M \in \mathrm{mod}_P\,\Lambda$:*
  (1.1) *$M \in \mathcal{A}_k$;*
  (1.2) *$\Omega^{-i}M \in \mathcal{A}_1$ for $0 \leq i \leq k-1$;*
  (1.3) *there is an exact sequence*
$$0 \to M \to P_{-1} \to \cdots \to P_{-k} \to \Omega^{-k}M \to 0$$
  with $P_{-j} \in \mathrm{proj}\,\Lambda$ $(1 \leq j \leq k)$ such that
$$P_{-k}^* \to \cdots \to P_{-1}^* \to M^* \to 0$$
  is exact.

(2) *The following are equivalent for $N \in \mathrm{mod}_P\,\Lambda$:*
  (2.1) *$N \in \mathcal{B}_k$;*
  (2.2) *$\Omega^i N \in \mathcal{B}_1$ for $0 \leq i \leq k-1$;*
  (2.3) *for the projective resolution $\cdots \to P_1 \to P_0 \to N \to 0$ of N, the dual*
$$0 \to N^* \to P_0^* \to \cdots \to P_{k-1}^* \to (\Omega^k N)^* \to 0$$
  is exact.

Proof. (1) We have the following equivalences:

$$\begin{aligned} M \in \mathcal{A}_k &\Leftrightarrow \operatorname{Ext}^1_\Lambda(\Omega^i \operatorname{Tr} M, \Lambda) = 0,\ 0 \le i \le k-1 \\ &\Leftrightarrow \operatorname{Ext}^1_\Lambda(\operatorname{Tr} \Omega^{-i} M, \Lambda) = 0,\ 0 \le i \le k-1 \\ &\Leftrightarrow \Omega^{-i} M \in \mathcal{A}_1,\ 0 \le i \le k-1. \end{aligned}$$

Hence (1.1)⇔(1.2) holds. The equivalence (1.1)⇔(1.3) is proved in [ABr, Chapter II, §3, Theorem (2.17)].

(2) We have the following equivalences:

$$\begin{aligned} N \in \mathcal{B}_k &\Leftrightarrow \operatorname{Ext}^1_\Lambda(\Omega^i M, \Lambda) = 0,\ 0 \le i \le k-1 \\ &\Leftrightarrow \Omega^i M \in \mathcal{B}_1,\ 0 \le i \le k-1. \end{aligned}$$

Hence (2.1)⇔(2.2) holds. Dualizing a projective resolution of N

$$\cdots \to P_k \xrightarrow{f_k} P_{k-1} \xrightarrow{f_{k-1}} P_{k-2} \to \cdots \xrightarrow{f_1} P_0 \to N \to 0,$$

we get a complex

$$0 \to N^* \to P_0^* \to \cdots \to P_{k-2}^* \xrightarrow{f_{k-1}^*} P_{k-1} \xrightarrow{f_k^*} P_k^* \to \cdots .$$

Dualizing the two exact sequences

$$P_k \xrightarrow{h} \Omega^k N \to 0 \quad \text{and} \quad 0 \to \Omega^k N \xrightarrow{g} P_{k-1} \to \Omega^{k-1} N \to 0,$$

we get the exact sequences

$$0 \to (\Omega^k N)^* \xrightarrow{h^*} P_k^*$$

and

$$(*) \qquad 0 \to (\Omega^{k-1} N)^* \to P_{k-1}^* \xrightarrow{g^*} (\Omega^k N)^* \to \operatorname{Ext}^1_\Lambda(\Omega^{k-1} N, \Lambda) \to 0.$$

Since $f_k^* = h^* g^*$, there exists a commutative diagram

$$(**) \qquad \begin{array}{ccccc} \xrightarrow{f_{k-1}^*} & P_{k-1}^* & \xrightarrow{\quad f_k^* \quad} & P_k^* & \\ & {\scriptstyle g^*}\downarrow & \nearrow{\scriptstyle h^*} & & \\ & (\Omega^k N)^* & \longrightarrow & \operatorname{Ext}^k_\Lambda(N, \Lambda) & \longrightarrow 0 \end{array}$$

(2.1)⇒(2.3): By assumption, $\operatorname{Ext}^i_\Lambda(N, \Lambda) = 0$ $(1 \le i \le k)$, and hence $\operatorname{Ext}^k_\Lambda(N, \Lambda) = 0$ and g^* is surjective. Since

$$\operatorname{Im} f_{k-1}^* = \operatorname{Ker} f_k^* = \operatorname{Ker} h^* g^* = \operatorname{Ker} g^*,$$

the sequence $0 \to N^* \to P_0^* \to \cdots \xrightarrow{f_{k-1}^*} P_{k-1} \xrightarrow{g^*} (\Omega^k N)^* \to 0$ is exact.

(2.3)⇒(2.1): By assumption, $\operatorname{Ext}^i_\Lambda(N, \Lambda) = 0$ $(1 \le i \le k-2)$. We also get $\operatorname{Ext}^k_\Lambda(N, \Lambda) = 0$, by assumption and $(*)$. By $(**)$, we have $\operatorname{Im} f_{k-1}^* = \operatorname{Ker} g^* = \operatorname{Ker} f_k^*$, so that $\operatorname{Ext}^{k-1}_\Lambda(N, \Lambda) = 0$. Therefore, (2.1) holds. ∎

5.5. A stable equivalence for $k \geq 1$. Now, we show that the functors Ω^k and Ω^{-k} define a category equivalence of $\underline{\mathcal{A}_k}$ to $\underline{\mathcal{B}_k}$ for $k \geq 1$.

THEOREM 5.7. *Let $k \geq 1$.*

(a) *If $M \in \underline{\mathcal{A}_k}$, then $\Omega^{-k}M \in \underline{\mathcal{B}_k}$ and $\Omega^k\Omega^{-k}M \sim M$.*
(b) *If $N \in \underline{\mathcal{B}_k}$, then $\Omega^k N \in \underline{\mathcal{A}_k}$ and $\Omega^{-k}\Omega^k N \sim N$.*
(c) *Ω^k and Ω^{-k} define equivalences between the categories $\underline{\mathcal{A}_k}$ and $\underline{\mathcal{B}_k}$, inverse to each other.*

Proof. (a) Let $M \in \underline{\mathcal{A}_k}$. Then $\Omega^{-i+1}M \in \underline{\mathcal{A}_1}$ $(1 \leq i \leq k)$, so that

$$\Omega^{i-1}\Omega^{-k}M \sim \Omega^{i-2}(\Omega\Omega^{-1})\Omega^{-(k-1)}M \sim \Omega^{i-2}\Omega^{-(k-1)}M \sim \cdots \sim \Omega^{-(k-i+1)}M.$$

For $1 \leq i \leq k$, we have

$$\operatorname{Ext}^i_\Lambda(\Omega^{-k}M, \Lambda) = \operatorname{Ext}^1_\Lambda(\Omega^{i-1}\Omega^{-k}M, \Lambda) = \operatorname{Ext}^1_\Lambda(\Omega^{-(k-i+1)}M, \Lambda) = 0,$$

because $-(k-i+1) < 0$. Thus $\Omega^{-k}M \in \underline{\mathcal{B}_k}$. Since $\Omega^{-i+1}M \in \underline{\mathcal{A}_1}$ $(1 \leq i \leq k)$, we have

$$\Omega^i\Omega^{-i}M = \Omega^{i-1}(\Omega\Omega^{-1})\Omega^{-i+1}M \sim \Omega^{i-1}\Omega^{1-i}M,$$

by Theorem 5.1. Continuing this process, we get

$$\Omega^i\Omega^{-i}M \sim \Omega\Omega^{-1}M \sim M.$$

(b) Let $X \in \operatorname{mod}\Lambda$. Then $\operatorname{Ext}^1_\Lambda(\operatorname{Tr}\Omega X, \Lambda) = 0$ by Lemma 5.2. Hence $\operatorname{Ext}^1_\Lambda(\operatorname{Tr}\Omega^j X, \Lambda) = 0$ for $j \geq 1$. Let $N \in \underline{\mathcal{B}_k}$. Then $\Omega^i N \in \underline{\mathcal{B}_1}$ for $0 \leq i \leq k-1$, by Proposition 5.6(2).

We now show $\Omega^{-i+1}\Omega^k N \sim \Omega^{k-i+1}N$. Set $i = k-1$; then $\Omega^{k-1}N \in \underline{\mathcal{B}_1}$, so $\Omega\Omega^{k-1}N \in \underline{\mathcal{A}_1}$. Thus $\Omega^{-1}\Omega\Omega^{k-1}N \sim \Omega^{k-1}N$, by Theorem 5.3. Therefore,

$$\begin{aligned}\Omega^{-i+1}\Omega^k N &= \Omega^{-i+2}(\Omega^{-1}\Omega)\Omega^{k-1}N \\ &\sim \Omega^{-i+2}\Omega^{k-1}N \sim \cdots \sim \Omega^{-i+k+1}N.\end{aligned}$$

For $1 \leq i \leq k$, we have

$$\begin{aligned}\operatorname{Ext}^i_\Lambda(\operatorname{Tr}\Omega^k N, \Lambda) &\cong \operatorname{Ext}^1_\Lambda(\Omega^{i-1}\operatorname{Tr}\Omega^k N, \Lambda) \cong \operatorname{Ext}^1_\Lambda(\operatorname{Tr}\Omega^{-i+1}\Omega^k N, \Lambda) \\ &\cong \operatorname{Ext}^1_\Lambda(\operatorname{Tr}\Omega^{k-i+1}N, \Lambda) = 0,\end{aligned}$$

since $k-i+1 \geq 1$. Hence $\Omega^k N \in \underline{\mathcal{A}_k}$. Since $\Omega^j N \in \underline{\mathcal{B}_1}$ for $0 \leq j \leq k-1$, we can prove $\Omega^{-1}\Omega^i N \sim \Omega^{i-1}N$ for $0 \leq i \leq k$. Indeed, we have $\Omega^j N \sim \Omega^{-1}\Omega\Omega^j N \sim \Omega^{-1}\Omega^{j+1}N$ by Theorem 5.3. Thus $\Omega^{i-1}N \sim \Omega^{-1}\Omega^i N$ for $1 \leq i \leq k$. This holds for $i = 0$ too.

Thus $\Omega^{-i}\Omega^i N \sim \Omega^{-i+1}(\Omega^{-1}\Omega^i N) \sim \Omega^{-i+1}\Omega^{i-1}N \sim \cdots \sim \Omega^{-1}\Omega^1 N \sim N$ for $0 \leq i \leq k$. Therefore $\Omega^{-k}\Omega^k N \sim N$. Since (c) is a consequence of (a) and (b), the proof is complete. ∎

6. The properties of a module in $\mathcal{A}_k \cap \mathcal{B}_k$. We finish the paper by some observations on the categories $\mathcal{A}_k \cap \mathcal{B}_k$. Following [T], we study k-subcomplete resolutions of Λ-modules.

DEFINITION 6.1. Let $M \in \operatorname{mod} \Lambda$ and $k \geq 1$. A complex

$$P_\bullet = (P_k \to P_{k-1} \to \cdots \to P_1 \to P_0 \xrightarrow{d_0} P_{-1} \to \cdots \to P_{-k})$$

is said to be a *k-subcomplete resolution* of M if:

(a) $P_i \in \operatorname{proj} \Lambda$ for $-k \leq i \leq k$,
(b) $\mathrm{H}_i(P_\bullet) = 0 = \mathrm{H}^i((P_\bullet)^*)$ for $-k < i < k$,
(c) $\operatorname{Im} d_0 = M$.

The following 'k-subcomplete version' of Proposition 3.3 holds.

PROPOSITION 6.2. *The following are equivalent for $M \in \operatorname{mod} \Lambda$:*

(a) $M \in \mathcal{A}_k \cap \mathcal{B}_k$;
(b) $\operatorname{Ext}^i_\Lambda(M, \Lambda) = 0 = \operatorname{Ext}^i_\Lambda(\operatorname{Tr} M, \Lambda)$ *for* $1 \leq i \leq k$;
(c) M *admits a k-subcomplete resolution.*

Proof. Apply the arguments used in the proof of Proposition 3.3. ∎

We now observe the behavior of $\mathcal{A}_k \cap \mathcal{B}_k$ under the action of some functors.

LEMMA 6.3. *Let $M \in \mathcal{A}_k \cap \mathcal{B}_k$. Then* $\operatorname{Tr} M \in \mathcal{A}_k \cap \mathcal{B}_k$.

Proof. Since $\operatorname{Ext}^i_\Lambda(\operatorname{Tr}(\operatorname{Tr} M), \Lambda) \cong \operatorname{Ext}^i_\Lambda(M, \Lambda) = 0$, we have $\operatorname{Tr} M \in \mathcal{A}_k$; and $\operatorname{Tr} M \in \mathcal{B}_k$ is obvious. ∎

LEMMA 6.4.

(a) $\Omega(\mathcal{A}_k \cap \mathcal{B}_k) = \mathcal{A}_{k+1} \cap \mathcal{B}_{k-1}$.
(b) $\Omega^{-1}(\mathcal{A}_{k+1} \cap \mathcal{B}_{k-1}) = \mathcal{A}_k \cap \mathcal{B}_k$.

Proof. Take any $M \in \mathcal{A}_k \cap \mathcal{B}_k$. We will show $\Omega M \in \mathcal{A}_{k+1} \cap \mathcal{B}_{k-1}$. Since $M \in \mathcal{A}_1$, we have an exact sequence $0 \to M \to P \to \Omega^{-1} M \to 0$ with $P \in \operatorname{proj} \Lambda$. The long exact sequence obtained from this short exact sequence by applying $(-)^*$ provides the isomorphism

$$\operatorname{Ext}^i_\Lambda(\Omega^{-1} M, \Lambda) \cong \operatorname{Ext}^{i-1}_\Lambda(M, \Lambda) \quad \text{for } i \geq 2.$$

Then

$$\operatorname{Ext}^i_\Lambda(\operatorname{Tr} \Omega M, \Lambda) = \operatorname{Ext}^i_\Lambda(\Omega^{-1} \operatorname{Tr} M, \Lambda) \cong \operatorname{Ext}^{i-1}_\Lambda(\operatorname{Tr} M, \Lambda) = 0$$

for $2 \leq i \leq k+1$. For $i = 1$, we obtain $\operatorname{Ext}^1_\Lambda(\operatorname{Tr} \Omega M, \Lambda) = 0$, by Lemma 2.10 and Proposition 2.8. Thus $\Omega M \in \mathcal{A}_{k+1}$; and showing $\Omega M \in \mathcal{B}_{k-1}$ is easy.

To prove the converse, take $N \in \mathcal{A}_{k+1} \cap \mathcal{B}_{k-1}$. For $1 \leq i \leq k$, we obtain

$$\operatorname{Ext}^i_\Lambda(\operatorname{Tr} \Omega^{-1} N, \Lambda) \cong \operatorname{Ext}^i_\Lambda(\Omega \operatorname{Tr} N, \Lambda) \cong \operatorname{Ext}^{i+1}_\Lambda(\operatorname{Tr} N, \Lambda).$$

Since $N \in \mathcal{A}_{k+1}$, we get $\mathrm{Ext}^j_\Lambda(\mathrm{Tr}\, N, \Lambda) = 0$ for $1 \le j \le k+1$. Thus $\mathrm{Ext}^{i+1}_\Lambda(\mathrm{Tr}\, N, \Lambda) = 0$ for $1 \le i+1 \le k+1$, and hence in particular for $1 \le i \le k$. Thus $\mathrm{Ext}^i_\Lambda(\mathrm{Tr}\, \Omega^{-1} N, \Lambda) = 0$ for $1 \le i \le k$. Hence $\Omega^{-1}N \in \mathcal{A}_k$. To show that $\Omega^{-1}N \in \mathcal{B}_k$, we note that

$$(*) \qquad \mathrm{Ext}^i_\Lambda(\Omega^{-1}N, \Lambda) = \mathrm{Ext}^{i-1}_\Lambda(\Omega\Omega^{-1}N, \Lambda) = \mathrm{Ext}^{i-1}_\Lambda(N, \Lambda).$$

Also, $N \in \mathcal{A}_{k+1} \subset \mathcal{A}_1$. By assumption, $N \in \mathcal{B}_{k-1}$, so we obtain $\mathrm{Ext}^j_\Lambda(N, \Lambda) = 0$ for $1 \le j \le k-1$. Hence, in $(*)$, $\mathrm{Ext}^{i-1}_\Lambda(N, \Lambda) = 0$ for $1 \le i-1 \le k-1$, i.e., for $2 \le i \le k$. Hence $\mathrm{Ext}^i_\Lambda(\Omega^{-1}N, \Lambda) = 0$ for $2 \le i \le k$. Proposition 2.8 yields $\mathrm{Ext}^1_\Lambda(\Omega^{-1}N, \Lambda) = 0$, and therefore $\mathrm{Ext}^i_\Lambda(\Omega^{-1}N, \Lambda) = 0$ for $1 \le i \le k$. Thus $\Omega^{-1}N \in \mathcal{B}_k$. This finishes the proof of (a). Since (b) is a consequence of (a), the proof is complete. ∎

Now, we prove some other properties of the category $\mathcal{A}_k \cap \mathcal{B}_k$.

Proposition 6.5. *Let $M \in \mathcal{A}_k \cap \mathcal{B}_k$, and*

$$P_\bullet = (P_k \to P_{k-1} \to \cdots \to P_1 \to P_0 \xrightarrow{d_0} P_{-1} \to \cdots \to P_{-k})$$

be a k-subcomplete resolution of M. Let $\alpha : P_0 \to M$ be the surjective homomorphism induced by d_0, and $\beta : M \to P_{-1}$ be the inclusion map. Then α (respectively, β) is a right (respectively, left) proj Λ*-approximation of M.*

Proof. It is clear that α is a right proj Λ-approximation. To show that β is a left proj Λ-approximation, take a projective Λ-module P. Then we have the commutative diagram

$$\begin{array}{ccccccc} \cdots \longrightarrow & \mathrm{Hom}_\Lambda(P_{-1}, P) & \xrightarrow{\mathrm{Hom}(d_0,P)} & \mathrm{Hom}_\Lambda(P_0, P) & \xrightarrow{\mathrm{Hom}(d_1,P)} & \mathrm{Hom}_\Lambda(P_1, P) & \longrightarrow \cdots \\ & {\scriptstyle \mathrm{Hom}(\beta,P)}\downarrow & \nearrow {\scriptstyle \mathrm{Hom}(\alpha,P)} & & & & \\ & \mathrm{Hom}_\Lambda(M, P) & & & & & \end{array}$$

with exact top row. Take $f \in \mathrm{Hom}_\Lambda(M, P)$. Since $\alpha d_1 = 0$, we have

$$0 = \mathrm{Hom}(\alpha d_1, P)(f) = \mathrm{Hom}(d_1, P) \circ \mathrm{Hom}(\alpha, P)(f).$$

Hence

$$\mathrm{Hom}(\alpha, P)(f) \in \mathrm{Ker}\, \mathrm{Hom}(d_1, P) = \mathrm{Im}\, \mathrm{Hom}(d_0, P).$$

Therefore there exists $g \in \mathrm{Hom}_\Lambda(P_{-1}, P)$ such that $\mathrm{Hom}(d_0, P)(g) = \mathrm{Hom}(\alpha, P)(f)$. We have $f\alpha = gd_0 = g\beta\alpha$ because $d_0 = \beta\alpha$. Since α is surjective, one has $f = g\beta$. Thus $\mathrm{Hom}(\beta, P)$ is surjective, which means that β is a left proj Λ-approximation of M. ∎

Proposition 6.6. *Let $k \ge 1$. Then:*

(a) $\mathcal{A}_k \cap \mathcal{B}_k$ *contains* proj Λ *and* Gproj-Λ*,*
(b) $\mathcal{A}_k \cap \mathcal{B}_k$ *is closed under finite direct sums,*
(c) $\mathcal{A}_k \cap \mathcal{B}_k$ *is closed under direct summands.*

Proof. (a) This is an easy consequence of the definitions.

(b) Let $M, N \in \operatorname{mod} \Lambda$ be in $\mathcal{A}_k \cap \mathcal{B}_k$. Since $\operatorname{Ext}^i_\Lambda(M \oplus N, \Lambda) \cong \operatorname{Ext}^i_\Lambda(M, \Lambda) \oplus \operatorname{Ext}^i_\Lambda(N, \Lambda)$ for all i, and by definition of $\mathcal{B}_k$, we have $\operatorname{Ext}^i_\Lambda(M \oplus N, \Lambda) = 0$ for $1 \le i \le k$. Similarly, $\operatorname{Ext}^i_\Lambda(\operatorname{Tr}(M \oplus N), \Lambda) \cong \operatorname{Ext}^i_\Lambda(\operatorname{Tr} M, \Lambda) \oplus \operatorname{Ext}^i_\Lambda(\operatorname{Tr} N, \Lambda)$ for all i, and we get $M \oplus N \in \mathcal{A}_k \cap \mathcal{B}_k$.

(c) Let $M, N \in \operatorname{mod} \Lambda$ with $M \oplus N \in \mathcal{A}_k \cap \mathcal{B}_k$. Since $0 = \operatorname{Ext}^i_\Lambda(M \oplus N, \Lambda) \cong \operatorname{Ext}^i_\Lambda(M, \Lambda) \oplus \operatorname{Ext}^i_\Lambda(N, \Lambda)$ for $1 \le i \le k$, we have $M \in \mathcal{B}_k$. Similarly, $0 = \operatorname{Ext}^i_\Lambda(\operatorname{Tr}(M \oplus N), \Lambda) \cong \operatorname{Ext}^i_\Lambda(\operatorname{Tr} M, \Lambda) \oplus \operatorname{Ext}^i_\Lambda(\operatorname{Tr} N, \Lambda)$ for $1 \le i \le k$. Thus, we have $M \in \mathcal{A}_k$ and hence $M \in \mathcal{A}_k \cap \mathcal{B}_k$. ■

7. Gorenstein dimension of a module in $\mathcal{A}_k$ or $\mathcal{B}_k$. We show that a module M in $\mathcal{A}_k$ is G-projective whenever $\operatorname{G-dim} M \le k$, and $N \in \mathcal{B}_k$ is G-projective whenever $\operatorname{G-dim} N \le k$.

PROPOSITION 7.1. *Let $0 < k < \infty$. The following conditions are equivalent for $M \in \mathcal{A}_k$:*

(a) $\operatorname{G-dim} \operatorname{Tr} M \le k$;
(b) $\operatorname{G-dim} \operatorname{Tr} M = 0$.

Proof. It is sufficient to prove that (a) implies (b), because the converse is obvious. By (a), we have $\operatorname{G-dim} \Omega^k(\operatorname{Tr} M) = 0$. Since Gproj-$\Lambda$ is closed under Ω^{-1}, we get $\operatorname{G-dim} \Omega^{-k}\Omega^k(\operatorname{Tr} M) = 0$. By Lemma 2.10, we have $\Omega^{-k}\Omega^k(\operatorname{Tr} M) \cong \operatorname{Tr} \Omega^k\Omega^{-k} M$. Since $M \in \mathcal{A}_k$, it follows that $\Omega^k\Omega^{-k}M \sim M$, by Theorem 5.7. Hence $\operatorname{G-dim} \operatorname{Tr} M = \operatorname{G-dim} \operatorname{Tr} \Omega^k\Omega^{-k}M = 0$. ■

PROPOSITION 7.2. *Let $0<k<\infty$. The following are equivalent for $N \in \mathcal{B}_k$:*

(a) $\operatorname{G-dim} N \le k$;
(b) $\operatorname{G-dim} N = 0$.

Proof. (a)⇒(b): Since $\operatorname{G-dim} \Omega^k N = 0$, we see $\operatorname{G-dim} \Omega^{-k}\Omega^k N = 0$. By assumption, we have $\Omega^{-k}\Omega^k N \sim N$, so that $\operatorname{G-dim} N = 0$. ■

Note added in proof. Our Theorems 5.1, 5.3, 5.5, 5.7 are consequences of Proposition 1.1.1 of O. Iyama, *Higher-dimensional Auslander–Reiten theory on maximal orthogonal subcategories*, Adv. Math. 210 (2007), 22–50. For the convenience of the readers, we gave independent direct proofs.

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