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RAREFACTION WAVES IN NONLOCAL CONVECTION-DIFFUSION EQUATIONS

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Abstract. We consider a nonlocal convection-diffusion equation $u_t = J * u - u - uu_x$, where J is a probability density. We supplement this equation with step-like initial conditions and prove the convergence of the corresponding solutions towards a rarefaction wave, i.e. a unique entropy solution of the Riemann problem for the inviscid Burgers equation.

1. Introduction. The goal of this work is to study asymptotic properties of solutions to the Cauchy problem for the nonlocal convection-diffusion equation

$$u_t = \mathcal{L}u - uu_x, \qquad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

where the nonlocal operator $\mathcal{L}$ is defined by the formula

$$\mathcal{L}u = J * u - u \quad \text{with} \quad J \in L^1(\mathbb{R}),\ J \geq 0, \tag{1.2}$$

and " $*$ " denotes convolution with respect to the space variable. We supplement this problem with the step-like initial condition satisfying

$$u(x,0) = u_0(x) \to u_\pm \qquad \text{as } x \to \pm\infty, \tag{1.3}$$

with some constants $u_- < u_+$. The precise meaning of this condition is given in (2.5) and (2.6) below.

Equation (1.1) with kernel $J(x) = \frac{1}{2}e^{-|x|}$ can be obtained from the following system modelling a radiating gas [H]:

$$u_t + uu_x + q_x = 0, \qquad -q_{xx} + q + u_x = 0 \qquad \text{for } \in \mathbb{R},\ t \geq 0. \tag{1.4}$$

Indeed, the second equation in (1.4) can be formally solved to obtain $q = -\tilde{J}u_x$, with kernel $J(x) = \frac{1}{2}e^{-|x|}$, that is, the fundamental solution of the operator $-d^2/dx^2 + I$. Thus, substituting $q_x = -\tilde{J}u_{xx} = u - J * u$ into the

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first equation in (1.4), we obtain an equation which is formally equivalent to (1.1)–(1.2). The derivation of system (1.4) from the Euler system for a perfect compressible fluid coupled with an elliptic equation for the temperature can be found in [KT].

In this work, we consider more general kernels (see our assumptions (2.4) below), because the general integral operator $\mathcal{L}u = J*u-u$ models long range interactions and appears in many problems ranging from micro-magnetism [DGP, DT1, DT2], neural networks [EM], hydrodynamics [R] to ecology [CMS, Cl, DK, KM, M, SSN]. For example, in some population dynamics models, such an operator is used to model the dispersal of individuals in their environment [F1, F2, HMMV]. We also refer the reader to a series of papers [AB, BFRW, C, CV1, CV2, CD, CDM1] on travelling fronts, and to [CDM2] on pulsating fronts for the equation $u_t = J * u - u + f(x,u)$.

The equation in (1.1)–(1.2) with the particular kernel $J(x) = \frac{1}{2}e^{-|x|}$ (thus, in the context of modelling radiating gases) with various classes of initial data has recently been intensively studied. For existence and uniqueness results, we refer the reader to [KN] and [LM]. In [Ch], Chmaj gave an answer to an open problem of Serre [S2] concerning existence of travelling wave solutions to (1.1)–(1.2) with a more general kernel. Here, we refer the reader to the recent work [CHJ] for generalizations of those results and for additional references.

The large time behaviour of solutions to (1.1)–(1.2) was considered e.g. in [KN, S1, L, KT]. In the case of initial data u_0 satisfying $u_0(x) \to u_\pm$ as $x \to \pm\infty$, with $u_- > u_+$, Serre [S1] showed the L^1-stability of shock profiles. Asymptotic stability of smooth travelling waves was proved in [KN]. For initial data $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, Laurençot [L] showed the convergence of integrable and bounded weak solutions of (1.1)–(1.2) towards a source-type solution to the viscous Burgers equation. Here, we also recall recent works [IR, IISD], where a doubly nonlocal version of (1.1) (namely, with the Burgers flux replaced by a nonlinear term in convolution form) was studied with initial conditions from $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

The large time behaviour of solutions to problem (1.1)–(1.3) when $J(x) = \frac{1}{2}e^{-|x|}$ and $u_- < u_+$ was studied by Kawashima and Tanaka [KT], where a specific structure of this model was used to show the convergence of solutions towards rarefaction waves, under suitable smallness conditions on initial data.

The goal of this work is to generalize the result from [KT] by considering less regular initial conditions with no smallness assumption and more general kernels J. To deal with such a problem, we develop methods and tools which are inspired by those used in [KMX], where the fractal Burgers equation was studied.

2. Main result. First, we recall that the explicit function

$$w^R(x,t) = \begin{cases} u_-, & x/t \le u_-, \\ x/t, & u_- \le x/t \le u_+, \\ u_+, & x/t \ge u_+, \end{cases} \tag{2.1}$$

is called a *rarefaction wave*, and satisfies the following Riemann problem:

$$\begin{aligned} &w_t^R + w^R w_x^R = 0, \\ &w^R(x,0) = w_0^R(x) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0, \end{cases} \end{aligned}$$

in the weak (distributional) sense. Moreover, this is the *unique entropy* solution. Such rarefaction waves appear as asymptotic profiles as $t \to \infty$ of solutions to the viscous Burgers equation

$$u_t - u_{xx} + uu_x = 0$$

supplemented with an initial datum $u(x,0) = u_0(x)$, satisfying $u_0 - u_- \in L^1((-\infty,0))$ and $u_0 - u_+ \in L^1((0,\infty))$ (cf. [HN, IO] and Lemma 4.3 below). Below, we use also the regularized problem

$$w_t - w_{xx} + ww_x = 0, \tag{2.2}$$

$$w(x,0) = w_0(x) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0, \end{cases} \tag{2.3}$$

with some constants $u_- < u_+$; its solutions are called smooth approximations of the rarefaction wave (2.1).

The purpose of this paper is to show that *weak solutions* of the nonlocal Cauchy problem (1.1)–(1.3) exist for all $t \ge 0$ and converge as $t \to \infty$ to the rarefaction wave.

Here, as usual, a function $u \in L^\infty(\mathbb{R} \times [0,\infty))$ is called a *weak solution* to problem (1.1)–(1.3) if for every test function $\varphi \in C_c^\infty(\mathbb{R} \times [0,\infty))$ we have

$$-\int_{\mathbb{R}}\int_0^\infty u\varphi_t \, dt \, dx - \int_{\mathbb{R}} u_0(x)\varphi(x,0)\, dx = \int_{\mathbb{R}}\int_0^\infty u\mathcal{L}\varphi \, dt\, dx + \frac{1}{2}\int_{\mathbb{R}}\int_0^\infty u^2\varphi_x \, dt\, dx.$$

In the following, we assume that $\mathcal{L}u = J * u - u$ with

$$\begin{aligned} &J, |x|^2 J \in L^1(\mathbb{R}), \quad \int_{\mathbb{R}} J(x)\, dx = 1, \\ &J(x) = J(-x) \quad \text{and} \quad J(x) \ge 0 \quad \text{for all } x \in \mathbb{R}. \end{aligned} \tag{2.4}$$

Moreover, we consider initial conditions satisfying

$$u_0 - u_- \in L^1((-\infty,0)) \quad \text{and} \quad u_0 - u_+ \in L^1((0,\infty)) \tag{2.5}$$

with some constants $u_- < u_+$, as well as

$$u_{0,x} \in L^1(\mathbb{R}) \quad \text{and} \quad u_{0,x}(x) \ge 0 \quad \text{a.e. in } \mathbb{R}. \tag{2.6}$$

Now, we formulate the main result of this work on the rate of convergence of solutions to problem (1.1)–(1.3) towards the rarefaction wave (2.1).

THEOREM 2.1. *Assume that the kernel J satisfies* (2.4) *and the initial datum u_0 has the properties stated in* (2.5) *and* (2.6). *Then there exists a unique weak solution $u = u(x,t)$ of problem* (1.1)–(1.3) *with the following property: for every $p \in [1,\infty]$ there is a constant $C > 0$ such that*

$$\|u(t) - w^R(t)\|_p \le Ct^{-(1-1/p)/2}[\log(2+t)]^{(1+1/p)/2} \tag{2.7}$$

for all $t > 0$.

REMARK 2.1. Although the nonlocal operator $\mathcal{L}u = J * u - u$ has no regularizing properties like *e.g.* the Laplace operator, we still have global-in-time continuous solutions, because, for a nondecreasing initial condition, the nonlinear term in equation (1.1) does not develop shocks in finite time.

The paper is organized as follows. In the next section, we gather results concerning the equation regularized by the usual viscosity term, and auxiliary lemmas on the properties of the nonlocal operator $\mathcal{L}$. The main result on the large time behaviour of solutions to the regularized problem is proved in Section 4. The convergence of regularized solutions to a weak solution of the nonlocal problem (1.1)–(1.3) and Theorem 2.1 are proved in Section 5.

NOTATION. We denote by $\|\cdot\|_p$ the L^p-norm of a function defined on $\mathbb{R}$. Integrals without integration limits are over $\mathbb{R}$. Several numerical constants are denoted by C.

3. Regularized problem. In this section, we consider the regularized problem

$$u_t = \varepsilon u_{xx} + \mathcal{L}u - uu_x, \qquad x \in \mathbb{R},\ t > 0, \tag{3.1}$$

$$u(x,0) = u_0(x), \tag{3.2}$$

with fixed $\varepsilon > 0$. Our first goal is to show that this initial value problem has a unique smooth global-in-time solution.

THEOREM 3.1 (Existence of solutions). *If $u_0 \in L^\infty(\mathbb{R})$ and $\mathcal{L}u = J * u - u$, where the kernel J satisfies* (2.4), *then the regularized problem* (3.1)–(3.2) *has a solution $u^\varepsilon \in L^\infty(\mathbb{R}\times[0,\infty])$. Moreover, this solution satisfies:*

(i) *$u \in C^\infty(\mathbb{R}\times(0,\infty))$ and all its derivatives are bounded on $\mathbb{R}\times(t_0,\infty)$ for all $t_0 > 0$,*

(ii) *for all $(x,t) \in \mathbb{R}\times[0,\infty)$,*

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}} u_0 \le u^\varepsilon(x,t) \le \operatorname*{ess\,sup}_{x\in\mathbb{R}} u_0, \tag{3.3}$$

(iii) *u satisfies equation* (3.1) *in the classical sense,*
(iv) $u(t) \to u_0$ *as* $t \to 0$, *in* $L^\infty(\mathbb{R})$ *weak-** *and in* $L^p_{\mathrm{loc}}(\mathbb{R})$ *for all* $p \in [1, \infty)$.

This is a unique solution of problem (3.1)–(3.2) *in the sense of the integral formulation* (3.6) *below.*

In the following theorem we collect other properties of solutions to the regularized problem.

THEOREM 3.2. *Assume that the kernel J satisfies* (2.4). *Let u^ε be the solution of the regularized problem corresponding to an initial condition u_0 satisfying* (2.5). *If $u_{0,x} \in L^1(\mathbb{R})$ then*

$$\int u^\varepsilon_x(x,t)\,dx = \int u_{0,x}(x)\,dx. \tag{3.4}$$

If $u_{0,x} \geq 0$ then

$$u^\varepsilon_x(x,t) \geq 0 \quad \text{for all } x \in \mathbb{R} \text{ and } t \geq 0.$$

Moreover, for two initial conditions u_0, $\bar{u}_0$ satisfying (2.5)–(2.6), *the corresponding solutions u^ε, $\bar{u}^\varepsilon$ satisfy*

$$\|u^\varepsilon(t) - \bar{u}^\varepsilon(t)\|_1 \leq \|u_0 - \bar{u}_0\|_1. \tag{3.5}$$

Proof of Theorem 3.1. Following the usual procedure based on the Duhamel principle, we rewrite problem (3.1)–(3.2) in the integral form

$$u^\varepsilon(x,t) = (G^\varepsilon(\cdot,t) * u^\varepsilon_0)(x) + \int_0^t (G^\varepsilon(\cdot,t-s) * \mathcal{L}u^\varepsilon(\cdot,s))(x)\,ds - \int_0^t (G^\varepsilon(\cdot,t-s) * u^\varepsilon(\cdot,s)u^\varepsilon_x(\cdot,s))(x)\,ds, \tag{3.6}$$

where $G^\varepsilon(x,t) = (4\pi\varepsilon t)^{-1/2} e^{-|x|^2/4\varepsilon t}$ is the fundamental solution of the heat equation $u_t = \varepsilon u_{xx}$. It is a completely standard reasoning (details can be found for example in [DGV, Section 5]), based on the Banach contraction principle, that the integral equation (3.6) has a unique local-in-time regular solution on $[0,T]$ with properties stated in (i), (iii) and (iv). Here, one should notice that the second term on the right hand side of (3.6) does not pose any problem in adapting the arguments from [DGV, Section 5]. This is due to the fact that the convolution operator $\mathcal{L}$ is bounded on $L^\infty(\mathbb{R})$. Hence, we skip these details. This solution is global-in-time because of estimates (3.3) which we are going to prove below. ∎

In the proof of the comparison principle expressed by inequalities (3.3) we adapt ideas described in [K]. This argument is based on the following auxiliary results.

Lemma 3.1. *Let $\varphi \in C_b^3(\mathbb{R})$. If a sequence $\{x_n\} \subset \mathbb{R}$ satisfies $\varphi(x_n) \to \sup_{x\in\mathbb{R}} \varphi(x)$, then*

(i) $\lim_{n\to\infty} \varphi'(x_n) = 0$,
(ii) $\limsup_{n\to\infty} \varphi''(x_n) \le 0$,
(iii) $\limsup_{n\to\infty} \mathcal{L}\varphi(x_n) \le 0$.

Proof. Since φ'' is bounded, there exists $C > 0$ such that

$$\sup_{x\in\mathbb{R}} \varphi(x) \ge \varphi(x_n - z) \ge \varphi(x_n) - \varphi'(x_n)z - Cz^2 \tag{3.7}$$

for every $z \in \mathbb{R}$. Since the sequence $\{\varphi'(x_n)\}$ is bounded, passing to a subsequence we can assume that $\varphi'(x_n) \to p$. Consequently, passing to the limit in (3.7) we obtain

$$0 \ge -pz - Cz^2 \quad \text{for every } z \in \mathbb{R},$$

which immediately implies $p = 0$.

To prove (ii), we use an analogous argument involving the inequality

$$\sup_{x\in\mathbb{R}} \varphi(x) \ge \varphi(x_n - z) \ge \varphi(x_n) - \varphi'(x_n)z + \frac{1}{2}\varphi''(x_n)z^2 - Cz^3 \tag{3.8}$$

for all $z > 0$, where $C = \frac{1}{6}\|\varphi'''\|_\infty$. Passing to the upper limit in (3.8), denoting $q = \limsup_{n\to\infty} \varphi''(x_n)$ and using (i) we obtain

$$0 \ge \frac{1}{2}qz - Cz^3 \quad \text{for all } z > 0.$$

Choosing $z > 0$ arbitrarily small we deduce that $q \le 0$, which completes the proof of (ii).

Now, we prove that $\limsup_{n\to\infty} \mathcal{L}\varphi(x_n) \le 0$. Note first that by the definition of $\{x_n\}$ we have

$$\varphi(x_n - z) - \varphi(x_n) \le \sup_{x\in\mathbb{R}} \varphi(x) - \varphi(x_n) \to 0 \quad \text{as } n \to \infty.$$

Hence, $\limsup_{n\to\infty}(\varphi(x_n - z) - \varphi(x_n)) \le 0$. Applying the Fatou lemma to

$$\mathcal{L}\varphi(x_n) = \int (\varphi(x_n - z) - u(x_n))J(z)\,dz$$

ends the proof of (iii). ∎

We are now in a position to prove the comparison principle for equations with the nonlocal operator $\mathcal{L}$.

Proposition 3.1. *Assume that $u \in C_b(\mathbb{R}\times[0,T]) \cap C_b^3(\mathbb{R}\times[\varepsilon,T])$ is a solution of the equation*

$$u_t = u_{xx} + \mathcal{L}u - b(x,t)u_x, \tag{3.9}$$

where $\mathcal{L}$ is the nonlocal convolution operator given by (1.2) and $b = b(x,t)$ is a given and sufficiently regular real-valued function. Then

$$u(x,0) \le 0 \quad \textit{implies} \quad u(x,t) \le 0 \quad \textit{for all } x \in \mathbb{R},\ t \in [0,T].$$

Proof. The function $\Phi(t) = \sup_{x\in\mathbb{R}} u(x,t)$ is well-defined and continuous. Our goal is to show that Φ is locally Lipschitz and $\Phi'(t) \le 0$ almost everywhere. To show the Lipschitz continuity of Φ, for every $\varepsilon > 0$ we choose x_ε such that

$$\sup_{x\in\mathbb{R}} u(x,t) = u(x_\varepsilon, t) + \varepsilon.$$

Now, we fix $t, s \in I$, where $I \subset (0,T)$ is a closed bounded interval and we suppose (without loss of generality) that $\Phi(t) \ge \Phi(s)$. Using the definition of Φ and regularity of u we obtain

$$\begin{aligned} 0 \le \Phi(t) - \Phi(s) &= \sup_{x\in\mathbb{R}} u(x,t) - \sup_{x\in\mathbb{R}} u(x,s) \\ &\le \varepsilon + u(x_\varepsilon,t) - u(x_\varepsilon,s) \\ &\le \varepsilon + \sup_{x\in\mathbb{R}} |u(x,t) - u(x,s)| \\ &\le \varepsilon + |t-s| \sup_{x\in\mathbb{R},\, t\in I} |u_t(x,t)|. \end{aligned}$$

Since $\varepsilon > 0$ and $t, s \in I$ are arbitrary, we immediately see that Φ is locally Lipschitz, hence, by the Rademacher theorem, differentiable almost everywhere.

Now, let us differentiate $\Phi(t) = \sup_{x\in\mathbb{R}} u(x,t)$ with respect to $t > 0$. By the Taylor expansion, for $0 < s < t$, we have

$$u(x,t) = u(x,t-s) + s u_t(x,t) + Cs^2.$$

Hence, using (3.9), we obtain

$$u(x,t) \le \sup_{x\in\mathbb{R}} u(x,t-s) + s\big(u_{xx}(x,t) + \mathcal{L}u(x,t) - b(x,t)u_x(x,t)\big) + Cs^2. \tag{3.10}$$

Substituting in (3.10) $x = x_n$, where $u(x_n,t) \to \sup_{x\in\mathbb{R}} u(x,t)$ as $n \to \infty$, and passing to the limit using Lemma 3.1, we obtain

$$\sup_{x\in\mathbb{R}} u(x,t) \le \sup_{x\in\mathbb{R}} u(x,t-s) + Cs^2,$$

which can be transformed into

$$\frac{\Phi(t) - \Phi(t-s)}{s} \le Cs.$$

For $s \searrow 0$, we obtain $\Phi'(t) \le 0$ for those t at which Φ is differentiable. ■

Proof of (3.3). Let $m = \operatorname{ess\,sup}_{x\in\mathbb{R}} u_0$. Then, since $\mathcal{L}m = 0$, the function $v^\varepsilon(x,t) = u^\varepsilon(x,t) - m$ satisfies

$$v_t^\varepsilon = v_{xx}^\varepsilon + \mathcal{L}v^\varepsilon - (v^\varepsilon + m)v_x^\varepsilon.$$

Now, we use Proposition 3.1 with $b(x,t) = v^\varepsilon(x,t) + m$ to conclude that $v^\varepsilon(x,t) \le 0$, so $u^\varepsilon(x,t) \le m$ for all $x \in \mathbb{R}$ and $t \in [0,T]$, for arbitrary $T > 0$. The proof of $\operatorname{ess\,inf}_{x\in\mathbb{R}} u_0 \le u^\varepsilon(x,t)$ is completely analogous, hence we skip it. ■

Proof of Theorem 3.2. In order to show (3.4), we differentiate the Duhamel formula (3.6) to obtain

$$u^\varepsilon_x(x,t) = (G^\varepsilon(\cdot,t) * u^\varepsilon_{0,x})(x) + \int_0^t (G^\varepsilon(\cdot,t-s) * \mathcal{L}u^\varepsilon_x(\cdot,s))(x)\,ds - \int_0^t (G^\varepsilon_x(\cdot,t-s) * u^\varepsilon(\cdot,s)u^\varepsilon_x(\cdot,s))(x)\,ds. \tag{3.11}$$

Then, integrating (3.11) over $\mathbb{R}$, we have

$$\begin{aligned}\int u^\varepsilon_x(x,t)\,dx &= \int (G^\varepsilon(\cdot,t) * u^\varepsilon_{0,x})(x)\,dx + \int_0^t\int (G^\varepsilon(\cdot,t-s) * \mathcal{L}u^\varepsilon_x(\cdot,s))(x)\,dx\,ds \\ &\quad - \int_0^t\int (G^\varepsilon_x(\cdot,t-s) * u^\varepsilon(\cdot,s)u^\varepsilon_x(\cdot,s))(x))\,dx\,ds.\end{aligned} \tag{3.12}$$

Since $\int G^\varepsilon(x,t)\,dx = 1$, the second term on the right hand side of (3.12) is zero by (4.3). Now, the equality $\int G^\varepsilon_x(x,t)\,dx = 0$ implies that the last term on the right hand side of (3.12) is zero, and that ends the proof of (3.4).

To prove nonnegativity of u^ε_x, we first differentiate (3.1) with respect to x to obtain

$$(u^\varepsilon_x)_t = \varepsilon(u^\varepsilon_{xx})_x + \mathcal{L}u^\varepsilon_x - (u^\varepsilon u^\varepsilon_x)_x, \qquad x \in \mathbb{R},\ t > 0. \tag{3.13}$$

Next, we multiply (3.13) by $(u^\varepsilon_x)^- = \max\{-u^\varepsilon_x, 0\}$, and we integrate the resulting equation over $\mathbb{R}$, to obtain

$$\int (u^\varepsilon_x)_t (u^\varepsilon_x)^-\,dx = \varepsilon\int (u^\varepsilon_{xx})_x (u^\varepsilon_x)^-\,dx + \int (u^\varepsilon_x)^-\mathcal{L}u^\varepsilon_x\,dx - \int (u^\varepsilon u^\varepsilon_x)_x (u^\varepsilon_x)^-\,dx. \tag{3.14}$$

Now, the left hand side of (3.14) is equal to $\frac{1}{2}\frac{d}{dt}\int_{u^\varepsilon_x \le 0}[(u^\varepsilon_x)^-]^2\,dx$. Straightforward calculations, based on integration by parts in the first and third terms of the right hand side of (3.14), lead to

$$\frac{1}{2}\frac{d}{dt}\int\limits_{u^\varepsilon_x\le 0}[(u^\varepsilon_x)^-]^2\,dx = -\int\limits_{u^\varepsilon_x\le 0}[(u^\varepsilon_x)^-]^2\,dx + \int\limits_{u^\varepsilon_x\le 0}(u^\varepsilon_x)^-\mathcal{L}u^\varepsilon_x\,dx - \frac{1}{2}\int\limits_{u^\varepsilon_x\le 0}[(u^\varepsilon_x)^-]^3\,dx.$$

By Lemmas 4.2 and 4.1, $\int_{u^\varepsilon_x\le 0}(u^\varepsilon_x)^-\mathcal{L}u^\varepsilon_x\,dx \le 0$. As a consequence,

$$\frac{1}{2}\frac{d}{dt}\int\limits_{u^\varepsilon_x\le 0}[(u^\varepsilon_x)^-]^2\,dx \le 0,$$

which immediately implies

$$\int_{u_x^\varepsilon \le 0} [(u_x^\varepsilon)^-]^2\,dx \le \int_{u_x^\varepsilon \le 0} [(u_{0,x})^-]^2\,dx.$$

By nonnegativity of $u_{0,x}$ imposed in (2.6) we have $(u_x^\varepsilon)^- = 0$ on $\{u_x^\varepsilon \le 0\}$, and thus $u_x^\varepsilon(x,t) \ge 0$ for all $x \in \mathbb{R}$ and $t > 0$.

To prove the L^1-contraction property (3.5) it is sufficient to repeat the reasoning from Lemma 4.4 below, hence we do not reproduce it here. ■

4. Convergence of regularized solutions to a rarefaction wave. Now, we show that a solution to the regularized problem satisfies certain decay estimates and converges to a rarefaction wave with all estimates independent of $\varepsilon > 0$. The main result of this section reads as follows.

THEOREM 4.1. *Let $u = u^\varepsilon(x,t)$ be the solution of the regularized problem* (3.1)–(3.2), *with the kernel J satisfying* (2.4) *and the initial data u_0 satisfying* (2.5)–(2.6), *from Theorem* 3.1. *For every $p \in [1,\infty]$ there exists $C = C(p) > 0$ independent of t and of $\varepsilon > 0$ such that*

$$\|u_x^\varepsilon(t)\|_p \le t^{-1+1/p}\|u_{0,x}\|_1^{1/p} \tag{4.1}$$

and

$$\|u^\varepsilon(t) - w^R(t)\|_p \le Ct^{-(1-1/p)/2}[\log(2+t)]^{(1+1/p)/2} \tag{4.2}$$

for all $t > 0$, where $w^R = w^R(x,t)$ is the rarefaction wave (2.1).

Before proving this theorem, we establish preliminary inequalities involving the nonlocal operator $\mathcal{L}$.

LEMMA 4.1. *For every $\varphi \in L^1(\mathbb{R})$ we have $\mathcal{L}\varphi \in L^1(\mathbb{R})$. Moreover,*

$$\int \mathcal{L}\varphi\,dx = 0, \tag{4.3}$$

$$\int \mathcal{L}\varphi \operatorname{sgn}\varphi\,dx \le 0. \tag{4.4}$$

Proof. The function $\mathcal{L}\varphi$ is integrable by the Young inequality and the calculation

$$\|\mathcal{L}\varphi\|_1 \le \|\varphi\|_1 + \|J * \varphi\|_1 \le \|\varphi\|_1(1 + \|J\|_1).$$

Since $\int J(x)\,dx = 1$, we obtain (4.3) immediately by applying the Fubini theorem. Since $\mathcal{L}\varphi = J * \varphi - \varphi$, to prove (4.4) it is sufficient to use the estimates

$$\left|\int J * \varphi \cdot \operatorname{sgn}\varphi\,dx\right| \le \iint J(y)|\varphi(x-y)|\,dx\,dy = \int |\varphi(x)|\,dx$$

by the Fubini theorem and assumptions (2.4). ■

Lemma 4.2. *Let $\varphi \in L^1(\mathbb{R})$ and let $g \in C^2(\mathbb{R})$ be a convex function. Then*

$$\mathcal{L}g(\varphi) \geq g'(\varphi)\mathcal{L}\varphi \quad a.e. \tag{4.5}$$

Proof. The convexity of g leads to the inequality

$$g(\varphi(x-y)) - g(\varphi(x)) \geq g'(\varphi(x))[\varphi(x-y) - \varphi(x)].$$

Multiplying it by $J(y)$ and integrating with respect to y over $\mathbb{R}$ we obtain (4.5). ∎

For simplicity of exposition, we first formulate some auxiliary lemmas. We start with known results concerning the initial value problem for the viscous Burgers equation (2.2)–(2.3). The following estimates can be deduced from the explicit formula for solutions to problem (2.2)–(2.3). We refer the reader to [HN] for detailed calculations, and for additional improvements to [KT].

Lemma 4.3. *Problem* (2.2)–(2.3) *with $u_- < u_+$ has a unique solution $w(x,t)$ satisfying $u_- < w(t,x) < u_+$ and $w_x(t,x) > 0$ for all $(x,t) \in \mathbb{R} \times (0,\infty)$. Moreover, for every $p \in [1, \infty]$, there is a constant $C = C(p,u_-,u_+) > 0$ such that*

$$\|w_x(t)\|_p \leq Ct^{-1+1/p}, \quad \|w_{xx}(t)\|_p \leq Ct^{-3/2+1/(2p)}$$

and

$$\|w(t) - w^R(t)\|_p \leq Ct^{-(1-1/p)/2},$$

for all $t > 0$, where $w^R(x,t)$ is the rarefaction wave (2.1).

Our goal is to estimate $\|u^\varepsilon(t) - w(t)\|_p$ where $u^\varepsilon = u^\varepsilon(x,t)$ is a solution of the regularized problem (3.1)–(3.2) and $w = w(x,t)$ is a smooth approximation of the rarefaction wave w^R. First, we deal with the L^1-norm.

Lemma 4.4. *Assume that $u^\varepsilon = u^\varepsilon(x,t)$ is the solution of problem* (3.1)–(3.2) *from Theorem* 3.1. *Let $w = w(x,t)$ be a smooth approximation of a rarefaction wave. Then there exists a constant $C > 0$ independent of t and of $\varepsilon > 0$ such that*

$$\|u^\varepsilon(t) - w(t)\|_1 \leq C\log(2+t) \quad \text{for all } t > 0.$$

Proof. The function $v^\varepsilon(x,t) = u^\varepsilon(x,t) - w(x,t)$ satisfies the equation

$$v_t^\varepsilon - \mathcal{L}v^\varepsilon + \left(\frac{(v^\varepsilon)^2}{2} + v^\varepsilon w\right)_x = \mathcal{L}w - w_{xx}.$$

We multiply it by $\operatorname{sgn} v^\varepsilon$ and integrate over $\mathbb{R}$ to obtain

$$\frac{d}{dt}\int |v^\varepsilon|\,dx - \int \mathcal{L}v^\varepsilon \ \operatorname{sgn} v^\varepsilon\,dx + \frac{1}{2}\int ((v^\varepsilon)^2 + 2v^\varepsilon w)_x \operatorname{sgn} v^\varepsilon\,dx = \int (\mathcal{L}w - w_{xx}) \operatorname{sgn} v^\varepsilon\,dx. \tag{4.6}$$

By Lemma 4.1, the second term on the left hand side of (4.6) is nonnegative. For the third term, we approximate the sgn function by a smooth and nondecreasing function $\varphi = \varphi(x)$. Thus, we obtain

$$\begin{aligned}\int [(v^\varepsilon)^2 + 2v^\varepsilon w]_x \varphi(v^\varepsilon)\,dx &= -\int ((v^\varepsilon)^2 + 2v^\varepsilon w)\varphi'(v^\varepsilon) v^\varepsilon_x\,dx \\ &= -\int \Psi(v^\varepsilon)_x\,dx + \int w_x \Phi(v^\varepsilon)\,dx,\end{aligned}$$

where $\Psi(s) = \int_0^s z^2\varphi'(z)\,dz$ and $\Phi(s) = \int_0^s 2z\varphi'(z)\,dz$. Here, the first term on the right hand side equals zero and the second one is nonnegative because $w_x \geq 0$ and $\Phi(s) \geq 0$ for all $s \in \mathbb{R}$. Hence, an approximation argument gives $\int [(v^\varepsilon)^2 + \ 2v^\varepsilon w]_x \operatorname{sgn} v^\varepsilon\,dx \geq 0$.

Now, we estimate the right hand side of (4.6). First, we notice that using the Taylor formula, we have

$$\begin{aligned}\mathcal{L}w(x,s) = (J * w - w)(x,s) &= \int J(y)[w(x-y,s) - w(x,s)]\,dy \\ &= \int J(y) y w_x(x,s)\,dy + \int J(y)\frac{y^2}{2} w_{xx}(x+\theta y, s)\,dy,\end{aligned}$$

where $\int J(y) y w_x(x,s)\,dy = w_x(x,s)\int J(y) y\,dy = 0$ by the symmetry assumption from (2.4). Therefore, by assumption (2.5),

$$\begin{aligned}\left|\int (\mathcal{L}w - w_{xx})\ \operatorname{sgn} v^\varepsilon\,dx\right| &\leq \int J(y)\frac{y^2}{2}\int |w_{xx}(x+\theta y, s)|\,dx\,dy + \|w_{xx}\|_1 \\ &\leq C\|w_{xx}\|_1.\end{aligned}$$

Applying these estimates to (4.6) we obtain

$$\frac{d}{dt}\|v^\varepsilon(t)\|_1 \leq C\|w_{xx}(t)\|_1. \tag{4.7}$$

Now, by Lemma 4.3, we have $\|w_{xx}(t)\|_1 \leq Ct^{-1}$ for all $t > 0$, which combined with (4.7) and integration completes the proof of Lemma 4.4. ■

Now, we are in a position to prove the convergence of regularized solutions to a rarefaction wave.

Proof of Theorem 4.1. Part I. Decay estimates. In the case $p = 1$, we use the equality (3.4) from Theorem 3.2. Since $u^\varepsilon_x \geq 0$, we have

$$\|\partial_x u^\varepsilon(t)\|_1 = \|\partial_x u_0\|_1 \quad \text{for all } t \geq 0. \tag{4.8}$$

In order to show (4.1) for $p \in (1,\infty)$, we multiply (3.13) by $(u^\varepsilon_x)^{p-1}$ and integrate the resulting equation over $\mathbb{R}$ to obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int (u^\varepsilon_x)^p\,dx &= \varepsilon\int u^\varepsilon_{xx}(u^\varepsilon_x)^{p-1}\,dx + \int (u^\varepsilon_x)^{p-1}\mathcal{L}u^\varepsilon_x\,dx \\ &\quad -\int ((u^\varepsilon_x)^2 + u^\varepsilon u^\varepsilon_x)(u^\varepsilon_x)^{p-1}\,dx.\end{aligned} \tag{4.9}$$

The first integral on the right-hand side of (4.9) is equal to $(\varepsilon/p)\int[(u_x^\varepsilon)^p]_x\,dx$. Thus, since $u_x^\varepsilon \in L^1(\mathbb{R})$, this term equals zero. The second integral on the right-hand side of (4.9) is nonpositive by (4.5) and (4.3) as well as by the assumptions (2.4) on the kernel of $\mathcal{L}$. Thus, since u_x^ε is integrable and nonnegative, integrating by parts in the third integral of the right-hand side of (4.9),

$$\int((u_x^\varepsilon)^2 + u^\varepsilon u_x^\varepsilon)(u_x^\varepsilon)^{p-1}\,dx = \int(u_x^\varepsilon)^{p+1}\,dx + \int u^\varepsilon\left(\frac{(u_x^\varepsilon)^p}{p}\right)_x dx$$
$$= \left(1-\frac{1}{p}\right)\int(u_x^\varepsilon)^{p+1},$$

we arrive at

$$\frac{1}{p}\frac{d}{dt}\|u_x^\varepsilon(t)\|_p^p \le -\left(1-\frac{1}{p}\right)\|u_x^\varepsilon(t)\|_{p+1}^{p+1}. \tag{4.10}$$

Combining this with the interpolation inequality

$$\|u_x^\varepsilon(t)\|_p^{p^2/(p-1)} \le \|u_x^\varepsilon(t)\|_{p+1}^{p+1}\|u_x^\varepsilon(t)\|_1^{1/(p-1)}$$

and with the conservation of L^1-norm in (4.8) we obtain the differential inequality

$$\frac{d}{dt}\|u_x^\varepsilon(t)\|_p^p \le -(p-1)(\|u_x^\varepsilon(t)\|_p^p)^{p/(p-1)}\|u_{0,x}(t)\|_1^{-1/(p-1)}. \tag{4.11}$$

Now the decay estimate (4.1) results from (4.11) by standard calculations.

The case $p=\infty$ of (4.1) follows by letting $p\to\infty$.

Part II. Convergence towards a rarefaction wave. First, we recall that by Lemma 4.3 the large time asymptotics of $w(t)$ is described in $L^p(\mathbb{R})$ by the rarefaction wave $w^R(t)$, and the rate of this convergence is $t^{-1/2(1-1/p)}$. Thus, it is enough to estimate the L^p-norm of the difference of the solution u^ε of problem (3.1)–(3.2) and of the smooth approximation of the rarefaction wave satisfying (2.2)–(2.3). To this end, using the Gagliardo–Nirenberg–Sobolev inequality

$$\|v\|_p \le C\|v_x\|_\infty^a\|v\|_1^{1-a},$$

valid for every $1<p\le\infty$ and for $a=\frac{1}{2}(1-1/p)$, together with (4.1) and Lemma 4.3, we have

$$\|u^\varepsilon(t)-w(t)\|_p \le C(\|u_x^\varepsilon(t)\|_\infty + \|w_x(t)\|_\infty)^a\|u^\varepsilon(t)-w(t)\|_1^{1-a}$$
$$\le Ct^{-a}\|u^\varepsilon(t)-w(t)\|_1^{1-a}.$$

Finally, the logarithmic estimate of the L^1-norm from Lemma 4.4 completes the proof. ■

5. Passage to the limit $\varepsilon \to 0$. Here, we prove a result on the convergence as $\varepsilon \to 0$ of solutions u^ε for the regularized problem (3.1)–(3.2) towards a weak solution to problem (1.1)–(1.3).

THEOREM 5.1. *Let the assumptions on the initial data u_0 and the kernel J from (2.4)–(2.6) hold true, and let $u^\varepsilon = u^\varepsilon(x,t)$ be a solution to problem (3.1)–(3.2) with $\varepsilon > 0$. Then there exists a sequence $\varepsilon_n \to 0$ such that $u^{\varepsilon_n} \to u$ in $C([t_1,t_2], L^1_{\mathrm{loc}}(\mathbb{R}))$ for every $t_2 > t_1 > 0$, as well as $u^{\varepsilon_n} \to u$ a.e. in $\mathbb{R} \times (0,\infty)$, where u is a weak solution of problem* (1.1)–(1.3).

In the proof of this theorem, the following version of the Aubin–Lions–Simon compactness theorem will be used.

THEOREM 5.2. *Let $T > 0$, $1 < p \le \infty$, and $1 \le q \le \infty$. Assume that $Y \subset X \subset Z$ are Banach spaces such that Y is compactly embedded in X, and X is continuously embedded in Z. If A is a bounded subset of $W^{1,p}([0,T],Z)$ and of $L^q([0,T],Y)$, then A is relatively compact in $L^q([0,T],X)$, and also in $C([0,T],X)$ if $q = \infty$.*

The proof of Theorem 5.2 can be found in [S].

Proof of Theorem 5.1. First, we show the relative compactness of the family $\mathcal{F} = \{u^\varepsilon : \varepsilon \in (0,1]\}$ in the space $C((0,\infty), L^1_{\mathrm{loc}}(\mathbb{R}))$, and next we pass to the limit $\varepsilon \to 0$, using the Lebesgue dominated convergence theorem.

Step 1. We check the assumptions of the Aubin–Lions–Simon theorem in the case $p = q = \infty$, $Y = W^{1,1}(K)$, $A = \mathbf{1}_{K\times[t_1,t_2]}\mathcal{F}$, $X = L^1(K)$ and $Z = (C^2_K)^*$, with arbitrary $t_2 > t_1 > 0$, where $K \subset \mathbb{R}$ is a compact set and $(C^2_K)^*$ is the topological dual to the space of C^2 functions with compact support in K (with its natural norm). First, we notice that $L^1(K)$ is obviously continuously embedded in $(C^2_K)^*$, and by the Rellich–Kondrashov theorem, $W^{1,1}(K)$ is compactly embedded in $L^1(K)$. By (3.3),

$$\Big|\int_K u^\varepsilon(t)\varphi\,dx\Big| \le \|\varphi\|_{C^2_K}\|u_0\|_\infty|K|$$

for every $\varphi \in C^\infty_c(\mathbb{R})$. Hence, the family $\mathcal{F}$ is bounded in $L^\infty([t_1,t_2],(C^2_K)^*)$.

Now, we check that $\{u^\varepsilon_t\}$ is bounded in $L^\infty([t_1,t_2],(C^2_K)^*)$. To this end, we multiply (3.1) by $\varphi \in C^\infty_c(\mathbb{R})$ and integrate over $\mathbb{R}$. Integration by parts yields

$$\Big|\int_K u^\varepsilon_t(t)\varphi\,dx\Big| \le \|\varphi\|_{C^2_K}\Big(\varepsilon\int_K |u^\varepsilon|\,dx + \int_K |\mathcal{L}u^\varepsilon|\,dx + \int_K (u^\varepsilon)^2\,dx\Big). \tag{5.1}$$

By assumption on J in (2.4), the Young inequality, and (3.3), the right-hand side of (5.1) can be estimated by $\|\varphi\|_{C^2_K}\|u_0\|_\infty|K|\big(\|J\|_1 + 1 + \varepsilon + \frac{1}{2}\|u_0\|_\infty\big)$.

Now, again by (3.3) we have $\int_K |u^\varepsilon(t)|\,dx \le \|u_0\|_\infty|K|$. Moreover, from the decay estimate (4.1) for $p = \infty$ we obtain $\int_K |u^\varepsilon_x(t)|\,dx \le (1/t_1)|K|$.

All these estimates imply that $\mathcal{F}$ is bounded in $L^\infty([t_1,t_2],W^{1,1}(K))$. Thus, the Aubin–Lions–Simon theorem ensures that $\mathcal{F}$ is relatively compact in $C([t_1,t_2],L^1(K))$ for all $t_2>t_1>0$ and all compact sets $K\subset\mathbb{R}$.

Step 2. We deduce from Step 1 and from the Cantor diagonal argument that there exists a sequence $\varepsilon_n\to 0$ and a function $u\in C((0,\infty),L^1_{\mathrm{loc}}(\mathbb{R}))$ such that u^{ε_n} converges as $n\to\infty$ to u in $C([t_1,t_2],L^1(K))$ for all $t_2>t_1>0$ and all compact $K\subset\mathbb{R}$. Up to another subsequence, we can also assume that $u^{\varepsilon_n}\to u$ a.e. on $\mathbb{R}\times(0,\infty)$. This convergence and inequality (3.3) imply that $u\in L^\infty(\mathbb{R}\times(0,\infty))$.

Now, we prove that u is a weak solution of problem (1.1)–(1.3). To this end, we multiply (1.1) by $\varphi\in C_c^\infty(\mathbb{R}\times[0,\infty))$, and integrating the resulting equation over $\mathbb{R}\times[0,\infty)$ and integrating by parts we obtain

$$
\begin{aligned}
(5.2)\qquad & -\int_{\mathbb{R}}\int_0^\infty u^{\varepsilon_n}\varphi_t\,dt\,dx-\int_{\mathbb{R}}u_0(x)\varphi(x,0)\,dx\\
&=\varepsilon\int_{\mathbb{R}}\int_0^\infty u^{\varepsilon_n}\varphi_{xx}\,dt\,dx+\int_{\mathbb{R}}\int_0^\infty u^{\varepsilon_n}\mathcal{L}\varphi\,dt\,dx+\frac12\int_{\mathbb{R}}\int_0^\infty (u^{\varepsilon_n})^2\varphi_x\,dt\,dx.
\end{aligned}
$$

Thus, since $u^{\varepsilon_n}\to u$ a.e. as $n\to\infty$, since the sequence $\{u^{\varepsilon_n}\}$ is bounded in L^∞-norm by $\|u_0\|_\infty$, and since $\mathcal{L}\varphi$ is integrable, the Lebesgue dominated convergence theorem allows us to pass to the limit in (5.2). This completes the proof of Theorem 5.1. ∎

Proof of Theorem 2.1. Denote by u^{ε_n} the solution of the regularized problem (3.1)–(3.2) and by u a weak solution of problem (1.1)–(1.3). By Theorem 5.1, we know that $u^{\varepsilon_n}\to u$ a.e. on $\mathbb{R}\times(0,\infty)$ for a sequence $\varepsilon_n\to 0$. Therefore, by the Fatou lemma and Theorem 4.1, for each $R>0$ and $p\in[1,\infty]$ and for all $t>0$ we have

$$
\begin{aligned}
\|u(t)-w^R(t)\|_{L^p(-R,R)} &\le \liminf_{\varepsilon_n\to 0}\|u^{\varepsilon_n}(t)-w^R(t)\|_{L^p(-R,R)}\\
&\le Ct^{-(1-1/p)/2}[\log(2+t)]^{(1+1/p)/2}.
\end{aligned}
$$

Since $R>0$ is arbitrary and the right-hand side does not depend on R, we complete the proof of (2.7) by letting $R\to\infty$.

Since solutions of the regularized problem have the L^1-contraction property stated in Theorem 3.2, by an analogous passage to the limit $\varepsilon_n\to 0$ as described above, we obtain an L^1-contraction inequality for weak solutions to the nonlocal problem (1.1)–(1.3). Hence the weak solution to (1.1)–(1.3) is unique. ∎

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