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QUASITRIANGULAR HOM-HOPF ALGEBRAS

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Abstract. A twisted generalization of quasitriangular Hopf algebras called quasitriangular Hom-Hopf algebras is introduced. We characterize these algebras in terms of certain morphisms. We also give their equivalent description via a braided monoidal category $\widetilde{\mathcal{H}}({}_H\mathcal{M})$. Finally, we study the twisting structure of quasitriangular Hom-Hopf algebras by conjugation with Hom-2-cocycles.

1. Introduction. The theory of quantum groups, having its source in theoretical physics, is a new branch of mathematics developed in the last few years. It was introduced by Soviet mathematical physicists when investigating quantum integrable systems in quantum mechanics. In a sense, the investigation of quantum groups is just the investigation of Hopf algebras. By quantizing and other means, quasitriangular Hopf algebras are obtained. As is well known, quasitriangular Hopf algebras are crucial to the theory of quantum groups and R-matrices. For example, quasitriangular Hopf algebras have a very close relation to the quantum Yang–Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12},$$

which is one of the most interesting subjects in quantum mechanics. Thus the investigation of quasitriangular Hopf algebras is of interest not only in mathematics but also in physics. As a generalization of quasitriangular Hopf algebras, in this paper we mainly study quasitriangular Hom-Hopf algebras.

Hom-structures were first defined for Lie algebras. As deformations of Lie algebras, Hom-Lie algebras are determined by an endomorphism. That is, the Jacobi identity is replaced by the so-called Hom-Jacobi identity

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0,$$

where α is an endomorphism of Lie algebras. Earlier precursors of Hom-Lie algebras can be found in [12, 13, 10]. Recently, Hom-Lie structures have been further studied by many scholars [18, 24, 25, 27, 1, 2, 11]; these structures include Hom-Lie bialgebras, quasi-Hom-Lie algebras, Hom-Lie superalgebras,

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Hom-Lie color algebras, Hom-Lie admissible Hom-algebras, Hom-Nambu–Lie algebras and so on. In [4], we have studied the construction of Hom-Lie bialgebras from Hom-Lie algebras and Hom-Lie coalgebras.

Similar ideas were applied to other algebra structures. The concepts of Hom-algebras, Hom-coalgebras, Hom-Hopf algebras, Hom-alternative algebras, Hom-Jordan algebras, Hom-Poisson algebras, Hom-Leibniz algebras, infinitesimal Hom-bialgebras and Hom-power associative algebras were introduced in [9, 17, 19, 16, 26, 28, 3, 7, 8, 23].

Further, some modules and comodules on these Hom-algebras structures such as Hom-modules, Hom-comodules, Hom-Hopf modules and Hom-module algebras were considered, and the fundamental structure of Hom-Hopf modules was investigated in [3]. Moreover, the antipodes and Drinfel'd doubles of Hom-Hopf algebras were considered in [23]. And cobraided Hom-smash product Hopf algebras were studied in [14].

Motivated by Hom-Lie algebras and other Hom-type algebras, Yau invented the concept of quasitriangular Hom-Hopf algebras [28], which can be considered as generalized quantum groups. He studied properties of these algebras, and proved that each quasitriangular Hom-Hopf algebra $(H, \alpha, \mathcal{R})$ comes with a solution of the quantum Hom-Yang–Baxter equations:

$$(\mathcal{R}_{12}\mathcal{R}_{13})\mathcal{R}_{23} = \mathcal{R}_{23}(\mathcal{R}_{13}\mathcal{R}_{12}) \quad \text{and} \quad \mathcal{R}_{12}(\mathcal{R}_{13}\mathcal{R}_{23}) = (\mathcal{R}_{23}\mathcal{R}_{13})\mathcal{R}_{12},$$

where $\mathcal{R}_{12} = \mathcal{R} \otimes c$, $\mathcal{R}_{23} = c \otimes \mathcal{R}$, $\mathcal{R}_{13} = (\tau \otimes \mathrm{id})\mathcal{R}_{23}$ and c is a weak unit. Further, solutions of the Hom-Yang–Baxter equations were obtained from Hom-modules of suitable quasitriangular Hom-bialgebras. In addition, Chen and Zhang [5] proved an FRT type theorem for the quantum Hom-Yang–Baxter equations.

This paper has two purposes. One is to characterize quasitriangular Hom-Hopf algebras via the category $\widetilde{\mathcal{H}}({}_H\mathcal{M})$ of Hom-modules which is a braided monoidal category. The other is to investigate the twisting of quasitriangular Hom-Hopf algebras, that is, construction of a new quasitriangular Hom-Hopf algebra from an old one by the twisting process through a Hom-2-cocycle.

This paper is organized as follows. In Section 2, we recall the basic definitions concerning Hom-structures, including those of Hom-algebras, Hom-coalgebras, Hom-bialgebras, Hom-Hopf algebras, Hom-modules and Hom-comodules. In Section 3, we consider some properties of quasitriangular Hom-Hopf algebras and give an equivalent description of quasitriangular Hom-Hopf algebras via the braided monoidal category $\widetilde{\mathcal{H}}({}_H\mathcal{M})$. Moreover, we characterize the quasitriangular Hom-Hopf algebras in terms of certain morphisms. In Section 4, we introduce two sub-Hom-Hopf algebras of a quasitriangular Hom-Hopf algebra and study their relationship. In Section 5, we give a construction of a new quasitriangular Hom-Hopf algebra by using a Hom-2-cocycle. Further, we apply these main results to a quasitriangular Hom-Hopf algebra built on a 3-dimensional group algebra.

Throughout this paper we freely use the Hopf algebras and coalgebras terminology introduced in [6], [22] and [21].

2. Preliminaries. Throughout, all vector spaces and tensor spaces we consider are over a fixed field k. For a coalgebra, we write its comultiplication $\Delta(x) = x_1 \otimes x_2$ and denote the comodule structure $\rho(m) = m_{(0)} \otimes m_{(1)}$ with the summation symbols omitted according to [22]. Moreover, τ denotes the flip map.

Let $\mathcal{M}_k = (\mathcal{M}_k, \otimes, k, a, l, r)$ be the k-module category. The monoidal category $\mathcal{H}(\mathcal{M}_k)$ defined in [3] is as follows: objects are couples (M, μ), with $M \in \mathcal{M}_k$ and $\mu \in \mathrm{Aut}_k(M)$, a morphism $f : (M, \mu) \to (N, \nu)$ is a morphism $f : M \to N$ in $\mathcal{M}_k$ such that $\nu \circ f = f \circ \mu$, and the tensor product of (M, μ) and (N, ν) is given by

$$(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu).$$

Roughly speaking, Hom-structures are objects in the monoidal category $\widetilde{\mathcal{H}}(\mathcal{M}_k) = (\mathcal{H}(\mathcal{M}_k), \otimes, (k, \mathrm{id}), \widetilde{a}, \widetilde{l}, \widetilde{r})$ defined in [3], where the associativity constraint $\widetilde{a}$ is given by the fomula

$$\widetilde{a}_{M,N,L} = a_{M,N,L} \circ ((\mu \otimes \mathrm{id}) \otimes \varsigma^{-1}) = (\mu \otimes (\mathrm{id} \otimes \varsigma^{-1})) \circ a_{M,N,L}$$

for $(M, \mu), (N, \nu), (L, \varsigma) \in \mathcal{H}(\mathcal{M}_k)$, and the unit constraints $\widetilde{l}, \widetilde{r}$ are given by

$$\widetilde{l}_M = \mu \circ l_M = l_M \circ (\mathrm{id} \otimes \mu), \quad \widetilde{r}_M = \mu \circ r_M = r_M \circ (\mu \otimes \mathrm{id}).$$

The category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is called the *Hom-category* associated to the monoidal category $\mathcal{M}_k$, where a k-submodule $N \subseteq M$ is called a *subobject* of (M, μ) if μ restricts to an automorphism of N, that is, $(N, \mu|_N) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$. Since the category $\mathcal{M}_k$ has left duality, so does $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. Now let M^* be the left dual of $M \in \mathcal{M}_k$, and let $b_M : k \to M \otimes M^*$ and $d_M : M^* \otimes M \to k$ be the coevaluation and evaluation maps. Then the left dual of $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ is $(M^*, (\mu^*)^{-1})$, and the coevaluation and evaluation maps are given by

$$\widetilde{b}_M = (\mu \otimes \mu^*)^{-1} \circ b_M, \quad \widetilde{d}_M = d_M \circ (\mu^* \otimes \mu).$$

We now recall some useful definitions given in [3].

DEFINITION 2.1. A *Hom-algebra* is a vector space A together with an element $1_A \in A$ and linear maps

$$m : A \otimes A \to A,\ a \otimes b \mapsto ab, \quad \alpha \in \mathrm{Aut}(A),$$

such that

$$\alpha(a)(bc) = (ab)\alpha(c), \qquad \alpha(ab) = \alpha(a)\alpha(b),$$
$$a1_A = 1_A a = \alpha(a), \quad \alpha(1_A) = 1_A,$$

for all $a, b, c \in A$. In the following, we denote this Hom-algebra by (A, α).

In the language of Hopf algebras, m is multiplication, α is the twisting automorphism and 1_A is the unit. Let (A,α) and (A',α') be two Hom-algebras. A *Hom-algebra morphism* $f:(A,\alpha)\to(A',\alpha')$ is a linear map such that $f\circ\alpha=\alpha'\circ f$, $f(ab)=f(a)f(b)$ and $f(1_A)=1_{A'}$.

DEFINITION 2.2. A *Hom-coalgebra* is an object (C,γ) in the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with linear maps $\Delta:C\to C\otimes C$, $\Delta(c)=c_1\otimes c_2$ and $\varepsilon:C\to k$ such that

$$\begin{aligned}\gamma^{-1}(c_1)\otimes\Delta(c_2)&=\Delta(c_1)\otimes\gamma^{-1}(c_2),\\ \Delta(\gamma(c))&=\gamma(c_1)\otimes\gamma(c_2),\\ c_1\varepsilon(c_2)=\gamma^{-1}(c)&=\varepsilon(c_1)c_2,\quad \varepsilon(\gamma(c))=\varepsilon(c),\end{aligned}\tag{2.1}$$

for all $c\in C$.

Note that (2.1) is equivalent to $c_1\otimes c_{21}\otimes\gamma(c_{22})=\gamma(c_{11})\otimes c_{12}\otimes c_2$, which is often used in the rest of the paper. Let (C,γ) and (C',γ') be two Hom-coalgebras. A *Hom-coalgebra morphism* $f:(C,\gamma)\to(C',\gamma')$ is a linear map such that $f\circ\gamma=\gamma'\circ f$, $\Delta\circ f=(f\otimes f)\circ\Delta$ and $\varepsilon'\circ f=\varepsilon$.

DEFINITION 2.3. A *Hom-bialgebra* $H=(H,\alpha,m,\eta,\Delta,\varepsilon)$ is a bialgebra in the monoidal category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. This means that (H,α,m,η) is a Hom-algebra and $(H,\alpha,\Delta,\varepsilon)$ is a Hom-coalgebra such that Δ and ε are Hom-algebra maps, that is, for any $h,g\in H$,

$$\begin{aligned}\Delta(hg)&=\Delta(h)\Delta(g),\quad \Delta(1_H)=1_H\otimes 1_H,\\ \varepsilon(hg)&=\varepsilon(h)\varepsilon(g),\quad \varepsilon(1_H)=1_k.\end{aligned}$$

A Hom-bialgebra (H,α) is a *Hom-Hopf algebra* if there exists a morphism (called *antipode*) $S:H\to H$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ (i.e. $S\circ\alpha=\alpha\circ S$) such that

$$S*\mathrm{id}=\eta\circ\varepsilon=\mathrm{id}*S.$$

In fact, a Hom-Hopf algebra is a Hopf algebra in the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. Further, the antipode of Hom-Hopf algebras has almost all the properties of the antipode of Hopf algebras, such as

$$\begin{aligned}S(hg)&=S(g)S(h),\quad S(1_H)=1_H,\\ \Delta(S(h))&=S(h_2)\otimes S(h_1),\quad \varepsilon\circ S=\varepsilon.\end{aligned}$$

That is, S is a Hom-anti-(co)algebra homomorphism. Since α is bijective and commutes with the antipode S, we can also see that its inverse α^{-1} commutes with S, that is, $S\circ\alpha^{-1}=\alpha^{-1}\circ S$. For a finite-dimensional Hom-Hopf algebra $(H,\alpha,m,\eta,\Delta,\varepsilon,S)$, the dual $(H^*,(\alpha^*)^{-1})$ is also a Hom-Hopf algebra with the following structures: for all $g, h \in H$ and $\phi, \varphi \in H^*$,

$$\langle \phi\varphi, h\rangle = \langle \phi, h_1\rangle\langle \varphi, h_2\rangle, \quad 1_{H^*} = \varepsilon,$$
$$\langle \Delta(\phi), g \otimes h\rangle = \langle \phi, gh\rangle, \quad \varepsilon_{H^*} = \eta,$$
$$(\alpha^*)^{-1}(\phi) = \phi \circ \alpha^{-1}, \qquad S(\phi) = \phi \circ S^{-1}.$$

Now we recall actions and coactions of Hom-algebras and Hom-coalgebras respectively.

Definition 2.4. Let (A, α) be a Hom-algebra. A *right* (A, α)*-Hom-module* consists of (M, μ) in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a morphism $\psi : M \otimes A \to M$, $\psi(m \otimes a) = m \cdot a$, such that

$$(m \cdot a) \cdot \alpha(b) = \mu(m) \cdot (ab),$$
$$\mu(m \cdot a) = \mu(m) \cdot \alpha(a), \quad m \cdot 1_A = \mu(m),$$

for all $a, b \in A$ and $m \in M$.

Similarly, we can define a left (A, α)-Hom-module. A Hom-algebra (A, α) can be considered as a Hom-module over itself under multiplication. Let (M, μ) and (N, ν) be two left (A, α)-Hom-modules. A map $f : M \to N$ is called a *left* (A, α)*-Hom-module morphism* if $f(a \cdot m) = a \cdot f(m)$ and $f \circ \mu = \nu \circ f$. We denote by $\widetilde{\mathcal{H}}({}_A\mathcal{M})$ the category of left (A, α)-Hom modules. If $(M, \mu), (N, \nu) \in \widetilde{\mathcal{H}}({}_A\mathcal{M})$, then $(M \otimes N, \mu \otimes \nu) \in \widetilde{\mathcal{H}}({}_A\mathcal{M})$ via the left H-action

$$h \cdot (m \otimes n) = h_1 \cdot m \otimes h_2 \cdot n.$$

Let $(M, \mu) \in \widetilde{\mathcal{H}}({}_A\mathcal{M})$ and $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_A)$. Then (M, μ) is called a *Hom-bimodule* if the following compatibility condition holds:

$$\alpha(h) \cdot (m \cdot g) = (h \cdot m) \cdot \alpha(g)$$

for any $m \in M$ and $h, g \in H$.

Dually, we can define Hom-comodules. Let $C = (C, \gamma)$ be a Hom-coalgebra. A *right* (C, γ)*-Hom-comodule* is an object (M, μ) in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a k-linear map $\rho_M : M \to M \otimes C$, $\rho_M(m) = m_{(0)} \otimes m_{(1)}$, such that

$$\mu^{-1}(m_{(0)}) \otimes \Delta_C(m_{(1)}) = m_{(0)(0)} \otimes (m_{(0)(1)} \otimes \gamma^{-1}(m_{(1)})),$$
$$\rho_M(\mu(m)) = \mu(m_{(0)}) \otimes \gamma(m_{(1)}), \quad m_{(0)}\varepsilon(m_{(1)}) = \mu^{-1}(m),$$

for all $m \in M$.

(C, γ) is a Hom-comodule over itself via comultiplication. Let (M, μ), (N, ν) be two right (C, γ)-Hom comodules. A map $g : M \to N$ is called a *right* (C, γ)*-Hom comodule morphism* if $g \circ \mu = \nu \circ g$ and $g(m_{(0)}) \otimes m_{(1)} = g(m)_{(0)} \otimes g(m)_{(1)}$. We denote by $\widetilde{\mathcal{H}}(\mathcal{M}^C)$ the category of right (C, γ)-Hom-comodules. If $(M, \mu), (N, \nu) \in \widetilde{\mathcal{H}}(\mathcal{M}^C)$, then $(M \otimes N, \mu \otimes \nu) \in \widetilde{\mathcal{H}}(\mathcal{M}^C)$ with the Hom-comodule structure

$$\rho(m\otimes n)=m_{(0)}\otimes n_{(0)}\otimes m_{(1)}n_{(1)}.$$

Note that all the above definitions coincide with the usual definitions of algebras, coalgebras, bialgebras, modules and comodules respectively when the automorphisms are id.

3. Quasitriangular Hom-Hopf algebras. In this section, we consider some properties of quasitriangular Hom-Hopf algebras, and give an equivalent description of quasitriangular Hom-Hopf algebras via the braided monoidal category $\widetilde{\mathcal{H}}({}_H\mathcal{M})$.

DEFINITION 3.1. A *quasitriangular Hom-Hopf algebra* is a Hom-Hopf algebra together with an invertible element $\mathcal{R}\in H\otimes H$ obeying

$$(3.1)\qquad \begin{aligned}(\alpha\otimes\alpha\otimes \mathrm{id})(\Delta\otimes \mathrm{id})(\mathcal{R})&=\mathcal{R}_{13}\mathcal{R}_{23},\\ (\mathrm{id}\otimes\alpha\otimes\alpha)(\mathrm{id}\otimes\Delta)(\mathcal{R})&=\mathcal{R}_{13}\mathcal{R}_{12},\end{aligned}$$

$$(3.2)\qquad \mathcal{R}\Delta(h)=\tau\circ\Delta(h)(\alpha\otimes\alpha)(\mathcal{R}),\quad \forall h\in H,$$

where $\mathcal{R}$ is called the *Hom-quasitriangular structure*.

Write $\mathcal{R}=\mathcal{R}^{(1)}\otimes\mathcal{R}^{(2)}$; this notation is used in $\mathcal{R}_{ij}=1_H\otimes\cdots\otimes\mathcal{R}^{(1)}\otimes 1_H\otimes\cdots\otimes\mathcal{R}^{(2)}\otimes\cdots\otimes 1_H$, the element of $H\otimes\cdots\otimes H$ which is $\mathcal{R}$ in the ith and jth factors. For convenience, we omit the summation symbols in the following and briefly denote the triple $(H,\alpha,\mathcal{R})$ as a quasitriangular Hom-bialgebra or Hom-Hopf algebra. Then (3.1) is equivalent to

$$(3.3)\qquad \mathcal{R}_1^{(1)}\otimes\mathcal{R}_2^{(1)}\otimes\mathcal{R}^{(2)}=\mathcal{R}^{(1)}\otimes\mathcal{R}'^{(1)}\otimes\mathcal{R}^{(2)}\mathcal{R}'^{(2)},$$

$$(3.4)\qquad \mathcal{R}^{(1)}\otimes\mathcal{R}_1^{(2)}\otimes\mathcal{R}_2^{(2)}=\mathcal{R}^{(1)}\mathcal{R}'^{(1)}\otimes\mathcal{R}'^{(2)}\otimes\mathcal{R}^{(2)},$$

which is the same as the structure of the usual quasitriangular Hopf algebras. And (3.2) can be rewritten as

$$\mathcal{R}^{(1)}h_1\otimes\mathcal{R}^{(2)}h_2=h_2\alpha(\mathcal{R}^{(1)})\otimes h_1\alpha(\mathcal{R}^{(2)}),$$

which is equivalent to $\alpha(\mathcal{R}^{(1)})\otimes\alpha(\mathcal{R}^{(2)})=\mathcal{R}^{(1)}\otimes\mathcal{R}^{(2)}$ (i.e., $\mathcal{R}$ is α-invariant) and

$$(3.5)\qquad \mathcal{R}^{(1)}h_1\otimes\mathcal{R}^{(2)}h_2=h_2\mathcal{R}^{(1)}\otimes h_1\mathcal{R}^{(2)}.$$

Note that our definition of quasitriangular Hom-Hopf algebras is different from Yau's [28].

If $\mathcal{R}$ is a Hom-quasitriangular structure for a quasitriangular Hom-bialgebra (H,α), then so is $\tau(\mathcal{R}^{-1})$. Also, by a direct checking, $\tau(\mathcal{R})$ and $\mathcal{R}^{-1}$ are Hom-quasitriangular structures on (H^{op},α) or $(H^{\mathrm{cop}},\alpha)$ which are defined as in the ordinary Hom-bialgebra with the Hom-anti-algebra structure or Hom-anti-coalgebra structure.

Although our definition of quasitriangular Hom-Hopf algebras is different from Yau's, we can get more properties under our definition.

PROPOSITION 3.2. *Let $(H, \alpha, S, \mathcal{R})$ be a quasitriangular Hom-Hopf algebra. Then*

(i) $(\varepsilon \otimes \mathrm{id})(\mathcal{R}) = (\mathrm{id} \otimes \varepsilon)(\mathcal{R}) = 1_H$,
(ii) $\mathcal{R}^{-1} = (S \otimes \mathrm{id})(\mathcal{R}) = (\mathrm{id} \otimes S)(\mathcal{R})$,
(iii) $\mathcal{R}$ *obeys the quantum Hom-Yang–Baxter equation (QHYBE)*

$$r^{(1)}R^{(1)} \otimes \mathcal{R}^{(1)}R^{(2)} \otimes \mathcal{R}^{(2)}r^{(2)} = R^{(1)}r^{(1)} \otimes R^{(2)}\mathcal{R}^{(1)} \otimes r^{(2)}\mathcal{R}^{(2)}. \tag{3.6}$$

Proof. (i) Applying $\varepsilon \otimes \varepsilon \otimes \mathrm{id}$ to (3.3), we get $(\varepsilon \otimes \mathrm{id})(\mathcal{R}) = 1_H$, since $\mathcal{R}$ is invertible. Similarly, applying $\mathrm{id} \otimes \varepsilon \otimes \varepsilon$ to (3.4) yields $(\mathrm{id} \otimes \varepsilon)(\mathcal{R}) = 1_H$.

(ii) Since

$$\begin{aligned}
\mathcal{R}(S \otimes \mathrm{id})(\mathcal{R}) &= \mathcal{R}^{(1)}S(\mathcal{R}'^{(1)}) \otimes \mathcal{R}^{(2)}\mathcal{R}'^{(2)} \\
&= (m \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes S \otimes \mathrm{id})(\mathcal{R}^{(1)} \otimes \mathcal{R}'^{(1)} \otimes \mathcal{R}^{(2)}\mathcal{R}'^{(2)}) \\
&\overset{(3.3)}{=} (m \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes S \otimes \mathrm{id})(\mathcal{R}_1^{(1)} \otimes \mathcal{R}_2^{(1)} \otimes \mathcal{R}^{(2)}) \\
&= \mathcal{R}_1^{(1)}S(\mathcal{R}_2^{(1)}) \otimes \mathcal{R}^{(2)} = 1_H \otimes 1_H
\end{aligned}$$

and $\mathcal{R}(\mathrm{id} \otimes S)(\mathcal{R}) = 1_H \otimes 1_H$, we have $\mathcal{R}^{-1} = (S \otimes \mathrm{id})(\mathcal{R})$. Similarly, $\mathcal{R}^{-1} = (\mathrm{id} \otimes S)(\mathcal{R})$.

(iii) Starting from the right side of the QHYBE, we have

$$\begin{aligned}
R^{(1)}r^{(1)} \otimes R^{(2)}\mathcal{R}^{(1)} \otimes r^{(2)}\mathcal{R}^{(2)} &= R^{(1)}\mathcal{R}_1^{(1)} \otimes R^{(2)}\mathcal{R}_2^{(1)} \otimes \mathcal{R}^{(2)} \\
= \mathcal{R}_2^{(1)}R^{(1)} \otimes \mathcal{R}_1^{(1)}R^{(2)} \otimes \mathcal{R}^{(2)} &= r^{(1)}R^{(1)} \otimes \mathcal{R}^{(1)}R^{(2)} \otimes \mathcal{R}^{(2)}r^{(2)}. \quad \blacksquare
\end{aligned}$$

If $(H, m, \eta, \Delta, \varepsilon, S, \mathcal{R})$ is a quasitriangular Hopf algebra and $\alpha : H \to H$ is a Hopf algebra automorphism, then $H_\alpha = (H, m_\alpha = \alpha \circ m, \eta, \Delta_\alpha = \Delta \circ \alpha^{-1}, \varepsilon, S, \mathcal{R})$ is a quasitriangular Hom-Hopf algebra. We know that H_α is a Hom-Hopf algebra from [28, Theorem 3.1]. It is easy to check that $\mathcal{R}$ is still a Hom-quasitriangular structure for H_α, so H_α is a quasitriangular Hom-Hopf algebra. This conclusion provides a practical method for finding examples of quasitriangular Hom-Hopf algebras.

EXAMPLE 3.3. Given a finite abelian group with identity e, the group algebra kG of G over k is a finite-dimensional commutative and cocommutative Hopf algebra. The multiplication m is given by

$$\Big(\sum_{x \in G} a_x x\Big)\Big(\sum_{y \in G} b_y y\Big) = \sum_{z \in G}\Big(\sum_{xy = z} a_x b_y\Big) z$$

for $a_x, b_y \in k$, and the unit is e. The comultiplication, counit and antipode are

$$\Delta(x) = x \otimes x, \quad \varepsilon(x) = 1, \quad S(x) = x^{-1},$$

for any $x \in G$. A quasitriangular structure for kG is equivalent to a function $\mathcal{R}$ on $G \otimes G$ such that

$$\sum_{cd=y} \mathcal{R}(x,c)\mathcal{R}(z,d) = \delta_{x,z}\mathcal{R}(x,y), \qquad \sum_{cd=x} \mathcal{R}(c,y)\mathcal{R}(d,z) = \delta_{y,z}\mathcal{R}(x,y),$$

$$\sum_{y} \mathcal{R}(x,y) = \delta_{x,e} = \sum_{y} \mathcal{R}(y,x),$$

for all $x, y, z \in G$, where $\delta_{x,y}$ denotes the Kronecker delta [15]. Thus we have a quasitriangular Hopf algebra $(kG, m, e, \Delta, \varepsilon, S, \mathcal{R})$ for some fixed $\mathcal{R}$.

Let a k-linear map $\alpha : G \to G$ be a group homomorphism. We can find easily that it is a Hopf algebra morphism on kG. Thus we have a quasitriangular Hom-Hopf algebra $kG_\alpha = (kG, m_\alpha, e, \Delta_\alpha, \varepsilon, S, \mathcal{R})$. In particular, consider the k-linear group homomorphism $\alpha : x \mapsto x^{-1}$, $x \in G$. Since kG is a commutative cocommutative Hopf algebra, we can check that α is a Hopf algebra automorphism. So we obtain a quasitriangular Hom-Hopf algebra kG_α with $\alpha^{-1} = \alpha$, and the twisted multiplication and comultiplication are given by

$$m_\alpha(x \otimes y) = x^{-1}y^{-1}, \qquad \Delta_\alpha(x) = x^{-1} \otimes x^{-1},$$

for any $x, y \in G$.

We just take the abelian group $G = \{1, g, g^2\}$ of three elements for example, where $g^{-1} = g^2$, $g^3 = 1$ and $(g^2)^{-1} = g$. Just as above, there is a quasitriangular Hopf algebra structure on the group algebra kG. And the Hom-quasitriangular structure is as follows:

$$\mathcal{R} = \frac{1}{3} \sum_{a,b=0}^{2} e^{-2\pi i ab/3} g^a \otimes g^b.$$

In fact,

$$\begin{aligned}
\mathcal{R}_{(13)}\mathcal{R}_{(23)} &= \frac{1}{9} \sum e^{-2\pi i(ab+cd)/3} g^a \otimes g^c \otimes g^{b+d} \\
&= \frac{1}{9} \sum e^{-2\pi i b(a-c)/3} e^{-2\pi i c b'/3} g^a \otimes g^c \otimes g^{b'} \\
&= \frac{1}{3} \sum e^{-2\pi i a b'/3} g^a \otimes g^a \otimes g^{b'},
\end{aligned}$$

where $b' = b + d$, and in the third step we use $\frac{1}{n}\sum_{b=0}^{n-1} e^{2\pi i ab/n} = \delta_{a,0}$ (i.e. 1 if $a = 0$ and 0 otherwise). This equals $(\Delta \otimes \mathrm{id})(\mathcal{R})$, as required. The other equalities of quasitriangular Hopf algebra can be obtained similarly. Thus we have a quasitriangular Hopf algebra $(kG, m, \Delta, \mathcal{R})$. Taking the k-linear group homomorphism $\alpha : x \mapsto x^{-1}$, $x \in G$, we obtain the quasitriangular Hom-Hopf algebra $kG_\alpha = (kG, m_\alpha, \Delta_\alpha, \mathcal{R})$ with multiplication and comultiplication given as follows:

$$g1 = g^2, \quad g^21 = g, \quad gg = g, \quad gg^2 = 1, \quad g^2g^2 = g^2,$$
$$\Delta_\alpha(1) = 1 \otimes 1, \quad \Delta_\alpha(g) = g^2 \otimes g^2, \quad \Delta_\alpha(g^2) = g \otimes g.$$

In the following, we provide a characterization of the axioms of quasitriangular Hom-bialgebras.

PROPOSITION 3.4. *Let (H, α) be a finite-dimensional Hom-bialgebra and let $\mathcal{R}$ in $H \otimes H$ induce a linear map $f_1 : H^* \to H$ by*

$$f_1(\phi) = \mathcal{R}^{(1)}\langle \phi, \mathcal{R}^{(2)}\rangle \quad \textit{for all } \phi \in H^*,$$

where $(H^*, (\alpha^*)^{-1})$ is the dual of (H, α). Then $\mathcal{R}$ is α-invariant and axiom (3.1) *holds if and only if f_1 is a Hom-coalgebra and Hom-anti-algebra map* (*i.e. there is a bialgebra map $H^{*\mathrm{op}} \to H$*).

Likewise, $\mathcal{R}$ is α-invariant and axiom (3.1) *holds if and only if the map*

$$f_2 : H^* \to H; \quad \phi \mapsto \langle \phi, \mathcal{R}^{(1)}\rangle \mathcal{R}^{(2)}$$

is a Hom-algebra and Hom-anti-coalgebra map.

Proof. For all $\phi \in H^*$, we have

$$f_1 \circ (\alpha^*)^{-1}(\phi) = \mathcal{R}^{(1)}\langle \phi, \alpha^{-1}(\mathcal{R}^{(2)})\rangle,$$
$$\alpha \circ f_1(\phi) = \alpha(\mathcal{R}^{(1)})\langle \phi, \mathcal{R}^{(2)}\rangle.$$

So $f_1 \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ if and only if $(\alpha \otimes \alpha)(\mathcal{R}) = \mathcal{R}$. Moreover, f_1 preserves the unit and counit by Proposition 3.2(i).

For any $\phi, \psi \in H^*$, we have

$$\begin{aligned} f_1(\phi\psi) &= \mathcal{R}^{(1)}\langle \phi\psi, \mathcal{R}^{(2)}\rangle = \mathcal{R}^{(1)}\langle \phi, \mathcal{R}^{(2)}_1\rangle\langle \psi, \mathcal{R}^{(2)}_2\rangle, \\ f_1(\psi) f_1(\phi) &= \mathcal{R}^{(1)}\mathcal{R}'^{(1)}\langle \psi, \mathcal{R}^{(2)}\rangle\langle \phi, \mathcal{R}'^{(2)}\rangle \\ &= \mathcal{R}^{(1)}\mathcal{R}'^{(1)}\langle \phi, \mathcal{R}'^{(2)}\rangle\langle \psi, \mathcal{R}^{(2)}\rangle, \end{aligned}$$

so the above two expressions are equal if and only if (3.4) holds if and only if f_1 a Hom-anti-algebra map.

In addition,

$$\begin{aligned} \Delta(f_1(\phi)) &= \Delta(\mathcal{R}^{(1)})\langle \phi, \mathcal{R}^{(2)}\rangle = \mathcal{R}^{(1)}_1 \otimes \mathcal{R}^{(1)}_2\langle \phi, \mathcal{R}^{(2)}\rangle, \\ (f_1 \otimes f_1)(\Delta(\phi)) &= f_1(\phi_1) \otimes f_1(\phi_2) = \mathcal{R}^{(1)}\langle \phi_1, \mathcal{R}^{(2)}\rangle \otimes \mathcal{R}'^{(1)}\langle \phi_2, \mathcal{R}'^{(2)}\rangle \\ &= \mathcal{R}^{(1)} \otimes \mathcal{R}'^{(1)}\langle \phi, \mathcal{R}^{(2)}\mathcal{R}'^{(2)}\rangle, \end{aligned}$$

hence f_1 is a Hom-coalgebra map if and only if $\mathcal{R}$ satisfies axiom (3.3). Thus $\mathcal{R}$ is α-invariant and satisfies axiom (3.1) if and only if f_1 is a Hom-coalgebra and Hom-anti-algebra map.

Similarly, $\mathcal{R}$ is α-invariant and satisfies axiom (3.1) if and only if f_2 is a Hom-algebra and Hom-anti-coalgebra map. ■

THEOREM 3.5. *Let (H,α) be a Hom-Hopf algebra. Then there is a Hom-quasitriangular structure on (H,α) if and only if the category $\widetilde{\mathcal{H}}({}_H\mathcal{M})$ of left H-Hom-modules is a braided monoidal category.*

Proof. Assume that $\mathcal{R}=\mathcal{R}^{(1)}\otimes\mathcal{R}^{(2)}$ is a Hom-quasitriangular structure on Hom-Hopf algebras (H,α). For all $(M,\mu),(N,\nu)\in\widetilde{\mathcal{H}}({}_H\mathcal{M})$, define a map $\mathrm{Br}_{M,N}: M\otimes N\to N\otimes M$ by

$$m\otimes n\mapsto \mathcal{R}^{(2)}\cdot\nu^{-1}(n)\otimes\mathcal{R}^{(1)}\cdot\mu^{-1}(m)$$

for all $m\in M$ and $n\in N$. For any morphism $f:(M,\mu)\to(M',\mu')$ and $g:(N,\nu)\to(N',\nu')$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$,

$$(g\otimes f)\circ\mathrm{Br}_{M,N}=\mathrm{Br}_{M',N'}\circ(f\otimes g),$$

so the naturality of Br holds.

Next we need to check that Br is a morphism of left H-Hom-modules. In fact, for any $m\in M$ and $n\in N$,

$$\begin{aligned}
h\cdot\mathrm{Br}_{M,N}(m\otimes n)&=h\cdot(\mathcal{R}^{(2)}\cdot\nu^{-1}(n)\otimes\mathcal{R}^{(1)}\cdot\mu^{-1}(m))\\
&=h_1\cdot(\mathcal{R}^{(2)}\cdot\nu^{-1}(n))\otimes h_2\cdot(\mathcal{R}^{(1)}\cdot\mu^{-1}(m))\\
&=\alpha^{-1}(h_1\mathcal{R}^{(2)})\cdot n\otimes\alpha^{-1}(h_2\mathcal{R}^{(1)})\cdot m\\
&\overset{(3.5)}{=}\alpha^{-1}(\mathcal{R}^{(2)}h_2)\cdot n\otimes\alpha^{-1}(\mathcal{R}^{(1)}h_1)\cdot m\\
&=\mathcal{R}^{(2)}\cdot(\alpha^{-1}(h_2)\cdot\nu^{-1}(n))\otimes\mathcal{R}^{(1)}\cdot(\alpha^{-1}(h_1)\cdot\mu^{-1}(m))\\
&=\mathcal{R}^{(2)}\cdot\nu^{-1}(h_2\cdot n)\otimes\mathcal{R}^{(1)}\cdot\mu^{-1}(h_1\cdot m)\\
&=\mathrm{Br}_{M,N}(h_1\cdot m\otimes h_2\cdot n)=\mathrm{Br}_{M,N}(h\cdot(m\otimes n)).
\end{aligned}$$

In the following, we will prove the two hexagon axioms. For all $m\in M$, $n\in N$ and $l\in(L,\varsigma)\in\widetilde{\mathcal{H}}({}_H\mathcal{M})$, we have

$$\begin{aligned}
\widetilde{a}^{-1}\circ\mathrm{Br}\circ\widetilde{a}^{-1}(m\otimes(n\otimes l))&=\widetilde{a}^{-1}\circ\mathrm{Br}((\mu^{-1}(m)\otimes n)\otimes\varsigma(l))\\
&=\widetilde{a}^{-1}(\mathcal{R}^{(2)}\cdot l\otimes\mathcal{R}^{(1)}\cdot(\mu^{-2}(m)\otimes\nu^{-1}(n)))\\
&=\widetilde{a}^{-1}(\mathcal{R}^{(2)}\cdot l\otimes(\mathcal{R}^{(1)}_1\cdot\mu^{-2}(m)\otimes\mathcal{R}^{(1)}_2\cdot\nu^{-1}(n)))\\
&=(\varsigma^{-1}(\mathcal{R}^{(2)}\cdot l)\otimes\mathcal{R}^{(1)}_1\cdot\mu^{-2}(m))\otimes\nu(\mathcal{R}^{(1)}_2\cdot\nu^{-1}(n))\\
&=(\alpha^{-1}(\mathcal{R}^{(2)})\cdot\varsigma^{-1}(l)\otimes\mathcal{R}^{(1)}_1\cdot\mu^{-2}(m))\otimes\alpha(\mathcal{R}^{(1)}_2)\cdot n,
\end{aligned}$$

and

$$\begin{aligned}
&(\mathrm{Br}\otimes\mathrm{id})\circ\widetilde{a}^{-1}\circ(\mathrm{id}\otimes\mathrm{Br})(m\otimes(n\otimes l))\\
&\quad=(\mathrm{Br}\otimes\mathrm{id})\circ\widetilde{a}^{-1}(m\otimes(\mathcal{R}^{(2)}\cdot\varsigma^{-1}(l)\otimes\mathcal{R}^{(1)}\cdot\nu^{-1}(n)))\\
&\quad=(\mathrm{Br}\otimes\mathrm{id})((\mu^{-1}(m)\otimes\mathcal{R}^{(2)}\cdot\varsigma^{-1}(l))\otimes\nu(\mathcal{R}^{(1)}\cdot\nu^{-1}(n)))\\
&\quad=(\mathcal{R}'^{(2)}\cdot\varsigma^{-1}(\mathcal{R}^{(2)}\cdot\varsigma^{-1}(l))\otimes\mathcal{R}'^{(1)}\cdot\mu^{-2}(m))\otimes\alpha(\mathcal{R}^{(1)})\cdot n\\
&\quad=(\alpha^{-1}(\mathcal{R}'^{(2)}\mathcal{R}^{(2)})\cdot\varsigma^{-1}(l)\otimes\mathcal{R}'^{(1)}\cdot\mu^{-2}(m))\otimes\alpha(\mathcal{R}^{(1)})\cdot n.
\end{aligned}$$

The above two expressions are equal because of (3.3). Similarly, axiom (3.4) implies $\widetilde{a} \circ \mathrm{Br} \circ \widetilde{a} = (\mathrm{id} \otimes \mathrm{Br}) \circ \widetilde{a} \circ (\mathrm{Br} \otimes \mathrm{id})$. Thus $\widetilde{\mathcal{H}}({}_H\mathcal{M})$ is a braided monoidal category.

Conversely, if Br is the braiding structure of the braided monoidal category $\widetilde{\mathcal{H}}({}_H\mathcal{M})$, set

$$\mathcal{R} = \tau \circ \mathrm{Br}(1_H \otimes 1_H) \in H \otimes H.$$

Then it is easy to deduce that $\mathcal{R}$ is α-invariant and invertible from the fact that Br is an isomorphism. Further, the above proof shows that the hexagon axioms are equivalent to (3.1), and Br is a morphism of left H-Hom-modules if and only if $\mathcal{R}^{(1)}h_1 \otimes \mathcal{R}^{(2)}h_2 = h_2\mathcal{R}^{(1)} \otimes h_1\mathcal{R}^{(2)}$. So $\mathcal{R}$ is a Hom-quasitriangular structure of (H, α). ■

4. Minimal quasitriangular Hom-Hopf algebras. In this section, we introduce two sub-Hom-Hopf algebras of a quasitriangular Hom-Hopf algebra, and study their relationship.

A *sub-quasitriangular Hom-Hopf algebra* of a quasitriangular Hom-Hopf algebra $(H, \mathcal{R}, \alpha)$ is a quasitriangular Hom-Hopf algebra $(H', \mathcal{R}', \alpha|_{H'})$ such that $(H', \alpha|_{H'})$ is a sub-Hom-Hopf algebra of (H, α).

DEFINITION 4.1. A quasitriangular Hom-Hopf algebra is defined to be *minimal* if it has no proper sub-quasitriangular Hom-Hopf algebras.

Minimal quasitriangular Hom-Hopf algebras generalize the notion of minimal quasitriangular Hopf algebras [20].

Let $(M, \mu), (N, \nu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ and $R \in M \otimes N$. Define the subspaces $(R_l, \mu_l) \subseteq (M, \mu)$ and $(R_r, \nu_r) \subseteq (N, \nu)$ by

$$R_l = \{(\mathrm{id} \otimes \alpha)(R) \mid \alpha \in N^*\}, \qquad R_r = \{(\beta \otimes \mathrm{id})(R) \mid \beta \in M^*\}, \tag{4.1}$$

where μ_l and ν_r are the automorphisms μ and ν restricted to R_l and R_r respectively. Assume that $R \neq 0$ and write $R = \sum_{i=1}^{p} m_i \otimes n_i \in M \otimes N$, where p is as small as possible. Then $m_1, \dots, m_p$ is a basis for R_l and $n_1, \dots, n_p$ is a basis for R_r. In particular, $\dim R_l = \dim R_r$. This common dimension is called the *rank* of R and is denoted by $\mathrm{rank}(R)$.

PROPOSITION 4.2. *Let $(H, \alpha, \mathcal{R}, S)$ be a quasitriangular Hom-Hopf algebra. Set $A = (\mathcal{R}_l, \alpha_l)$ and $B = (\mathcal{R}_r, \alpha_r)$. Then*

(i) *A, B are finite-dimensional sub-Hom-Hopf algebras of H.*
(ii) $\dim A = \dim B = \mathrm{rank}(\mathcal{R})$.
(iii) *The map $f : A^{*\mathrm{cop}} \to B$ defined by $f(\phi) = (\phi \otimes \mathrm{id})(\mathcal{R})$ for $\phi \in A^*$ is a Hom-Hopf algebra isomorphism.*

Proof. (i) From the definition of $(\mathcal{R}_l, \alpha_l)$,

$$A = \{\mathcal{R}^{(1)}\langle \phi, \mathcal{R}^{(2)}\rangle \mid \phi \in H^*\} = \mathrm{Im}\, f_1,$$

where f_1 is the map defined in Proposition 3.4. We know that f_1 is a morphism of Hom-coalgebras and Hom-anti-algebras. Hence

$$\Delta \circ f_1(\phi) = (f_1 \otimes f_1) \circ \Delta(\phi) \subseteq \operatorname{Im} f_1 \otimes \operatorname{Im} f_1 = A \otimes A,$$
$$f_1(\phi) f_1(\psi) = f_1(\psi\phi) \subseteq \operatorname{Im} f_1 = A,$$

for all $\phi, \psi \in H^*$. In addition, $\alpha(A) \subseteq A$, since

$$\begin{aligned}\alpha(\mathcal{R}^{(1)})\langle \phi, \mathcal{R}^{(2)}\rangle &= \alpha(\mathcal{R}^{(1)})\langle (\alpha^*)^{-1}(\phi), \alpha(\mathcal{R}^{(2)})\rangle \\ &= \mathcal{R}^{(1)}\langle (\alpha^*)^{-1}(\phi), \mathcal{R}^{(2)}\rangle \in A.\end{aligned}$$

It follows that A is both a sub-Hom-coalgebra and a sub-Hom-algebra of H. Thus A is a sub-Hom-bialgebra. Further, from $(S \otimes S)(\mathcal{R}) = \mathcal{R}$, we have $S(A) \subseteq A$. Thus A is a sub-Hom-Hopf algebra of H. Similarly, B is also a finite-dimensional sub-Hom-Hopf algebra of H.

(ii) This is obvious from the definition.

(iii) In fact, $f = f_2|_{A^{*\mathrm{cop}}}$, where f_2 is given in Proposition 3.4. We know that f_2 is a map of Hom-algebras and Hom-anti-coalgebras, so f is a Hom-bialgebra map. It follows from (i) and (ii) that f is a Hom-Hopf algebra isomorphism. ■

LEMMA 4.3. *Let (C, Δ, γ) be a Hom-coalgebra. Then the dual $(C^*, (\gamma^*)^{-1})$ of (C, Δ, γ) determines a $(C^*, (\gamma^*)^{-1})$-Hom-bimodule on (C, γ), defined as follows:*

$$\phi \rightharpoonup c = \gamma^2(c_1)\langle \phi, c_2\rangle, \qquad c \leftharpoonup \phi = \langle \phi, c_1\rangle \gamma^2(c_2), \tag{4.2}$$

where $\phi \in C^*$ and $c \in C$.

In addition, $\langle \phi\psi, c\rangle = \langle (\gamma^*)^{-2}(\phi),\ \psi \rightharpoonup c\rangle = \langle (\gamma^*)^{-2}(\psi), c \leftharpoonup \phi\rangle$ for any $\phi, \psi \in C^*$ and $c \in C$.

Proof. It is easy to show that the dual $(C^*, (\gamma^*)^{-1})$ of (C, Δ, γ) is a Hom-algebra with the following multiplication and automorphism:

$$\langle \phi\psi, c\rangle = \langle \phi, c_1\rangle\langle \psi, c_2\rangle, \qquad (\gamma^*)^{-1}(\phi) = \phi \circ \gamma^{-1},$$

for $\phi, \psi \in C^*$ and $c \in C$.

Firstly, the actions "$\rightharpoonup$" and "$\leftharpoonup$" define a left and a right C^*-Hom-module structure on C. In fact, by the Hom-coassociativity, for any $\phi, \psi \in C^*$ and $c \in C$,

$$\begin{aligned}(\phi\psi) \rightharpoonup \gamma(c) &= \gamma^3(c_1)\langle \phi\psi, \gamma(c_2)\rangle = \gamma^3(c_1)\langle \phi, \gamma(c_{21})\rangle\langle \psi, \gamma(c_{22})\rangle \\ &= \gamma^4(c_{11})\langle \phi, \gamma(c_{12})\rangle\langle \psi, c_2\rangle = \gamma^4(c_{11})\langle (\gamma^*)^{-1}(\phi), \gamma^2(c_{12})\rangle\langle \psi, c_2\rangle \\ &= (\gamma^*)^{-1}(\phi) \rightharpoonup (\gamma^2(c_1)\langle \psi, c_2\rangle) = (\gamma^*)^{-1}(\phi) \rightharpoonup (\psi \rightharpoonup c),\end{aligned}$$

$$\gamma(\phi \rightharpoonup c) = \gamma^3(c_1)\langle\phi, c_2\rangle = \gamma^3(c_1)\langle(\gamma^*)^{-1}(\phi), \gamma(c_2)\rangle$$
$$= (\gamma^*)^{-1}(\phi) \rightharpoonup \gamma(c),$$
$$\varepsilon \rightharpoonup c = \gamma^2(c_1)\langle\varepsilon, c_2\rangle = \gamma^2(\gamma^{-1}(c)) = \gamma(c).$$

So, C is a left C^*-Hom-module. Similarly, it is also a right C^*-Hom-module.

Next, the compatibility condition of Hom-bimodule holds: for any $c \in C$ and $\phi \in C^*$,

$$\begin{aligned}(\gamma^*)^{-1}(\phi) \rightharpoonup (c \leftharpoonup \psi) &= \langle\psi, c_1\rangle(\gamma^*)^{-1}(\phi) \rightharpoonup \gamma^2(c_2)\\ &= \langle\psi, c_1\rangle\langle(\gamma^*)^{-1}(\phi), \gamma^2(c_{22})\rangle\gamma^4(c_{21})\\ &= \langle\psi, \gamma(c_{11})\rangle\langle(\gamma^*)^{-1}(\phi), \gamma(c_2)\rangle\gamma^4(c_{12})\\ &= \langle(\gamma^*)^{-1}(\psi), \gamma^2(c_{11})\rangle\langle\phi, c_2 > \gamma^4(c_{12})\\ &= (\phi \rightharpoonup c) \leftharpoonup (\gamma^*)^{-1}(\psi).\end{aligned}$$

Finally, for any $\phi, \psi \in C^*$ and $c \in C$,

$$\langle\phi\psi, c\rangle = \langle(\gamma^*)^{-2}(\phi), \psi \rightharpoonup c\rangle = \langle(\gamma^*)^{-2}(\psi), c \leftharpoonup \phi\rangle$$

by the definition of the actions. ∎

Similarly, we can define a Hom-bimodule structure on the dual space A^* of some Hom-algebra (A, m, α); the proof is analogous to that of the above lemma, using the Hom-associativity of the Hom-algebra.

LEMMA 4.4. *Let (A, m, α) be a finite-dimensional Hom-algebra and $(A^*, (\alpha^*)^{-1})$ be its dual with comultiplication $\Delta(\phi) = \phi_1 \otimes \phi_2$. Then the Hom-algebra A induces an (A, α)-Hom-bimodule structure on $(A^*, (\alpha^*)^{-1})$ by the transpose action: for any $a, b \in A$ and $\phi \in A^*$,*

$$a \triangleright \phi = (\alpha^*)^{-2}(\phi_1)\langle\phi_2, a\rangle, \qquad \phi \triangleleft a = \langle\phi_1, a\rangle(\alpha^*)^{-2}(\phi_2),$$

that is,

$$\langle a \triangleright \phi, b\rangle = \langle\phi, \alpha^{-2}(b)a\rangle, \qquad \langle\phi \triangleleft a, b\rangle = \langle\phi, a\alpha^{-2}(b)\rangle. \tag{4.3}$$

Proof. This is straightforward. ∎

From the above lemma, we have the following conclusion.

PROPOSITION 4.5. *Let $(H, \mathcal{R}, \alpha)$ be a quasitriangular Hom-Hopf algebra with a bijective antipode S, and $A = \mathcal{R}_l$, $B = \mathcal{R}_r$. Then, for any sub-Hom-coalgebra $(C, \alpha|_C)$ of H, we have $AC = CA$ and $BC = CB$. In particular, $AB = BA$.*

Proof. As $(H, \mathcal{R}, \alpha)$ is a quasitriangular Hom-Hopf algebra with a bijective antipode S, axiom (3.5) can be written as the following formulas:

$$\mathcal{R}^{(1)}h \otimes \mathcal{R}^{(2)} = \alpha^2(h_{12})\mathcal{R}^{(1)} \otimes (h_{11}\alpha^{-2}(\mathcal{R}^{(2)}))S(h_2), \tag{4.4}$$

$$h\mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} = \mathcal{R}^{(1)}\alpha^2(h_{21}) \otimes S(h_1)(\alpha^{-2}(\mathcal{R}^{(2)})h_{22}), \tag{4.5}$$

$$\mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} h = (h_{22}\alpha^{-2}(\mathcal{R}^{(1)}))S^{-1}(h_1) \otimes \alpha^2(h_{21})\mathcal{R}^{(2)}, \tag{4.6}$$

$$\mathcal{R}^{(1)} \otimes h\mathcal{R}^{(2)} = S^{-1}(h_2)(\alpha^{-2}(\mathcal{R}^{(1)})h_{11}) \otimes \mathcal{R}^{(1)}\alpha^2(h_{12}), \tag{4.7}$$

for all $h \in H$. Here we just prove (4.4) for example. From (3.5), that is,

$$h_2\mathcal{R}^{(1)} \otimes h_1\mathcal{R}^{(2)} = \mathcal{R}^{(1)}h_1 \otimes \mathcal{R}^{(2)}h_2,$$

we have

$$\begin{aligned} h_{12}\mathcal{R}^{(1)} \otimes (h_{11}\mathcal{R}^{(2)})S(h_2) &= \mathcal{R}^{(1)}h_{11} \otimes (\mathcal{R}^{(2)}h_{12})S(h_2) \\ &= \mathcal{R}^{(1)}\alpha^{-1}(h_1) \otimes (\mathcal{R}^{(2)}h_{21})\alpha(S(h_{22})) \\ &= \mathcal{R}^{(1)}\alpha^{-1}(h_1) \otimes \alpha(\mathcal{R}^{(2)})(h_{21}S(h_{22})) \\ &= \mathcal{R}^{(1)}\alpha^{-1}(h_1) \otimes \alpha(\mathcal{R}^{(2)})\varepsilon(h_2)1_H \\ &= \mathcal{R}^{(1)}\alpha^{-2}(h) \otimes \alpha^2(\mathcal{R}^{(2)}), \end{aligned}$$

which is equivalent to $\mathcal{R}^{(1)}h \otimes \mathcal{R}^{(2)} = \alpha^2(h_{12})\mathcal{R}^{(1)} \otimes (h_{11}\alpha^{-2}(\mathcal{R}^{(2)}))S(h_2)$ by the α-invariance of $\mathcal{R}$. The other formulas are obtained in a similar way.

If $a \in A$, then $a = \mathcal{R}^{(1)}\langle \phi, \mathcal{R}^{(2)}\rangle$ for some $\phi \in H^*$. For $c \in C$,

$$\begin{aligned} ac &= \mathcal{R}^{(1)}c\langle \phi, \mathcal{R}^{(2)}\rangle \overset{(4.4)}{=} \alpha^2(c_{12})\mathcal{R}^{(1)}\langle \phi, (c_{11}\alpha^{-2}(\mathcal{R}^{(2)}))S(c_2)\rangle \\ &\overset{(4.3)}{=} \alpha^2(c_{12})\mathcal{R}^{(1)}\langle S(c_2) \triangleright \phi, \alpha^2(c_{11})\mathcal{R}^{(2)}\rangle \\ &\overset{(4.3)}{=} \alpha^2(c_{12})\mathcal{R}^{(1)}\langle (S(c_2) \triangleright \phi) \triangleleft \alpha^2(c_{11}), \alpha^2(\mathcal{R}^{(2)})\rangle \\ &= \alpha^2(c_{12})\mathcal{R}^{(1)}\langle (\alpha^*)^2((S(c_2) \triangleright \phi) \triangleleft \alpha^2(c_{11})), \mathcal{R}^{(2)}\rangle \in CA. \end{aligned}$$

In addition,

$$\begin{aligned} ca &= c\mathcal{R}^{(1)}\langle \phi, \mathcal{R}^{(2)}\rangle \overset{(4.5)}{=} \mathcal{R}^{(1)}\alpha^2(c_{21})\langle \phi, S(c_1)(\alpha^{-2}(\mathcal{R}^{(2)})c_{22})\rangle \\ &\overset{(4.3)}{=} \mathcal{R}^{(1)}\alpha^2(c_{21})\langle \phi \triangleleft S(c_1), \mathcal{R}^{(2)}\alpha^2(c_{22})\rangle \\ &\overset{(4.3)}{=} \mathcal{R}^{(1)}\alpha^2(c_{21})\langle \alpha^2(c_{22}) \triangleright (\phi \triangleleft S(c_1)), \alpha^2(\mathcal{R}^{(2)})\rangle \\ &= \mathcal{R}^{(1)}\alpha^2(c_{21})\langle (\alpha^*)^2(\alpha^2(c_{22}) \triangleright (\phi \triangleleft S(c_1))), \mathcal{R}^{(2)}\rangle \in AC. \end{aligned}$$

So $AC = CA$. Similarly, by using (4.6) and (4.7), we have $BC = CB$. In particular, $AB = BA$. ∎

As a consequence of Proposition 4.5, we have the following result.

THEOREM 4.6. *Let $(H, \mathcal{R}, \alpha)$ be a quasitriangular Hom-Hopf algebra, $A = \mathcal{R}_l$, and $B = \mathcal{R}_r$. Suppose that $H_\mathcal{R}$ is the sub-Hom-Hopf algebra generated by $A + B$. Then $H_\mathcal{R} = AB$. In particular, $H_\mathcal{R}$ is a finite-dimensional minimal quasitriangular Hom-Hopf algebra.*

EXAMPLE 4.7. Let $kG_\alpha = (kG, m_\alpha, \Delta_\alpha, \mathcal{R})$ be the 3-dimensional quasitriangular Hom-Hopf algebra defined in Example 3.3, where $\mathcal{R} =$

$\frac{1}{3}\sum_{a,b=0}^{2} e^{-2\pi iab/3} g^a \otimes g^b$ and $\alpha(x) = x^{-1}$ for any $x \in kG$. By direct checking, we find that $\mathcal{R}$ is α-invariant, $\mathcal{R}_l = \mathcal{R}_r = kG$ and $\mathrm{rank}(\mathcal{R}) = 3$. Of course, $\mathcal{R}_l\mathcal{R}_r = \mathcal{R}_r\mathcal{R}_l = kG_{\mathcal{R}}$, which are all 3-dimensional.

5. Reconstruction of quasitriangular Hom-Hopf algebras. In this section, we provide a "twisting" construction of quasitriangular Hom-Hopf algebras by conjugation with a Hom-2-cocycle. As an application, we give a further study in the case of commutation.

DEFINITION 5.1. Let (H, α) be a Hom-bialgebra. A *Hom-2-cocycle* is an invertible and α-invariant element $\mathcal{X} = \mathcal{X}^{(1)} \otimes \mathcal{X}^{(2)} \in H \otimes H$ (i.e. $\alpha^{\otimes 2}(\mathcal{X}) = \mathcal{X}$) such that

$$\mathcal{X}_{12}(\Delta \otimes \alpha^{-1})(\mathcal{X}) = \mathcal{X}_{23}(\alpha^{-1} \otimes \Delta)(\mathcal{X}), \tag{5.1}$$

where $\mathcal{X}_{12} = \mathcal{X}^{(1)} \otimes \mathcal{X}^{(2)} \otimes 1_H$ and $\mathcal{X}_{23} = 1_H \otimes \mathcal{X}^{(1)} \otimes \mathcal{X}^{(2)}$. It is called *counital* if

$$(\varepsilon \otimes \mathrm{id})(\mathcal{X}) = 1_H = (\mathrm{id} \otimes \varepsilon)(\mathcal{X}). \tag{5.2}$$

EXAMPLE 5.2. (1) Let $(H, \alpha, \mathcal{R})$ be a quasitriangular Hom-Hopf algebra. Then the Hom-quasitriangular structure $\mathcal{R}$ can be regarded as a Hom-2-cocycle. First, $\mathcal{R}$ is invertible and α-invariant. Next,

$$\begin{aligned}
\mathcal{R}_{12}(\Delta \otimes \alpha^{-1})(\mathcal{R}) &= \mathcal{R}^{(1)} r_1^{(1)} \otimes \mathcal{R}^{(2)} r_2^{(1)} \otimes r^{(2)} \\
&\overset{(3.3)}{=} \mathcal{R}^{(1)} R^{(1)} \otimes \mathcal{R}^{(2)} r^{(1)} \otimes R^{(2)} r^{(2)} \\
&\overset{(3.6)}{=} \mathcal{R}^{(1)} R^{(1)} \otimes r^{(1)} R^{(2)} \otimes r^{(2)} \mathcal{R}^{(2)} \\
&\overset{(3.4)}{=} \mathcal{R}^{(1)} \otimes r^{(1)} \mathcal{R}_1^{(2)} \otimes r^{(2)} \mathcal{R}_2^{(2)} \\
&= \mathcal{R}_{23}(\alpha^{-1} \otimes \Delta)(\mathcal{R}),
\end{aligned}$$

which is just (5.1). Furthermore, $\mathcal{R}$ is counital because $(\varepsilon \otimes \mathrm{id})(\mathcal{R}) = (\mathrm{id} \otimes \varepsilon)(\mathcal{R}) = 1_H$.

Let $kG_\alpha = (kG, m_\alpha, \Delta_\alpha, \mathcal{R})$ be the quasitriangular Hom-Hopf algebra defined in Example 4.7. Since $\mathcal{R} = \frac{1}{3}\sum_{a,b=0}^{2} e^{-2\pi iab/3} g^a \otimes g^b$ is α-invariant, we have a Hom-2-cocycle $\mathcal{R}$ on kG_α. Moreover, $\mathcal{R}$ is counital by direct computation.

(2) Let G be a finite abelian group with unit e, and kG be the group algebra as defined in Example 3.3. We consider the dual k-linear space $(kG)^*$ of kG, which can be identified with the set of functions on G with values in k. There is a commutative and cocommutative Hopf algebra structure on $(kG)^*$ denoted by $k(G)$ and the structure maps are given as follows. The multiplication and unit are

$$\langle \phi\psi, x\rangle = \langle \phi, x\rangle\langle \psi, x\rangle, \qquad \langle \eta(\lambda), x\rangle = \lambda,$$

for $\phi, \psi \in k(G)$, $x \in G$ and $\lambda \in k$. The comultiplication, counit and antipode are

$$\langle \Delta(\phi), x \otimes y \rangle = \langle \phi, xy \rangle, \tag{5.3}$$
$$\varepsilon(\phi) = \phi(e), \quad \langle S(\phi), x \rangle = \langle \phi, x^{-1} \rangle,$$

for $\phi \in k(G)$ and $x, y \in G$. Then a quasitriangular structure on $k(G)$ means a function $\mathcal{R}$ on $G \otimes G$ obeying

$$\mathcal{R}(xy \otimes z) = \mathcal{R}(x \otimes z)\mathcal{R}(y \otimes z), \quad \mathcal{R}(x \otimes yz) = \mathcal{R}(x \otimes y)\mathcal{R}(x \otimes z),$$
$$\mathcal{R}(x \otimes e) = 1 = \mathcal{R}(e \otimes x),$$

for all $x, y, z \in G$. This means that a quasitriangular structure for $k(G)$ is precisely a bicharacter of G.

If a k-linear map $\alpha : G \to G$ is a group homomorphism, then there exists an inductive Hopf algebra morphism $(\alpha^*)^{-1} : k(G) \to k(G)$ given by $(\alpha^*)^{-1}(\phi) = \phi \circ \alpha^{-1}$. So we have the quasitriangular Hom-Hopf algebra $k(G)_{(\alpha^*)^{-1}} = (k(G), m_{(\alpha^*)^{-1}}, e, \Delta_{\alpha^*}, \varepsilon, S, \mathcal{R})$, for some fixed quasitriangular structure $\mathcal{R}$. In particular, when α is given by $\alpha(x) = x^{-1}$, i.e., $\langle (\alpha^*)^{-1}(\phi), x \rangle = \langle \phi, x^{-1} \rangle$ for $x \in G$ and $\phi \in k(G)$, we obtain a quasitriangular Hom-Hopf algebra $k(G)_{(\alpha^*)^{-1}}$ satisfying $\alpha^{*2} = \mathrm{id}$ with multiplication and comultiplication

$$\langle m_{(\alpha^*)^{-1}}(\phi \otimes \psi), x \rangle = \langle \phi, x^{-1} \rangle \langle \psi, x^{-1} \rangle,$$
$$\langle \Delta_{\alpha^*}(\phi), x \otimes y \rangle = \langle \phi, x^{-1}y^{-1} \rangle,$$

for any $\phi, \psi \in k(G)$ and $x, y \in G$.

Furthermore, there is a counital Hom-2-cocycle on $k(G)_{(\alpha^*)^{-1}}$, meaning a non-zero function $\mathcal{X}$ on $G \otimes G$ such that

$$\mathcal{X}(\alpha(x) \otimes \alpha(y)) = \mathcal{X}(x \otimes y), \tag{5.4}$$
$$\mathcal{X}(y \otimes z)\mathcal{X}(x \otimes yz) = \mathcal{X}(x \otimes y)\mathcal{X}(xy \otimes z), \tag{5.5}$$
$$\mathcal{X}(e \otimes x) = 1 = \mathcal{X}(x \otimes e), \tag{5.6}$$

for all $x, y, z \in G$. In fact, the equality (5.4) is just the α-invariance of $\mathcal{X}$. Moreover, since comultiplication is defined in (5.3), (5.1) corresponds to (5.5), and (5.2) is equivalent to (5.6).

THEOREM 5.3. *Let $(H, m, \eta, \Delta, \varepsilon, \alpha, \mathcal{R})$ be a quasitriangular Hom-bialgebra and $\mathcal{X}$ be a counital Hom-2-cocycle. Then there is a quasitriangular Hom-bialgebra $H_{\mathcal{X}} = (H, m, \eta, \Delta_{\mathcal{X}}, \varepsilon, \alpha, \mathcal{R}_{\mathcal{X}})$, where*

$$\Delta_{\mathcal{X}}(h) = (\mathcal{X}\Delta(h))\mathcal{X}^{-1}, \quad \mathcal{R}_{\mathcal{X}} = (\mathcal{X}^{21}\mathcal{R})\mathcal{X}^{-1},$$

for all $h \in H_{\mathcal{X}}$, in which $\mathcal{X}^{21} = \tau(\mathcal{X}) = \mathcal{X}^{(2)} \otimes \mathcal{X}^{(1)}$.

Furthermore, if $(H, S, \alpha, \mathcal{R})$ is a quasitriangular Hom-Hopf algebra with $\alpha^2 = \mathrm{id}$, then $H_{\mathcal{X}}$ is a quasitriangular Hom-Hopf algebra with antipode $S_{\mathcal{X}}(h) = (US(h))U^{-1}$, where $U = \mathcal{X}^{(1)}S(\mathcal{X}^{(2)})$.

Proof. We prove the theorem in three steps.

STEP 1. It is easy to see that

$$\Delta_{\mathcal{X}} \circ \alpha = \alpha^{\otimes 2} \circ \Delta_{\mathcal{X}}, \qquad (\varepsilon \otimes \mathrm{id}) \circ \Delta_{\mathcal{X}}(h) = (\mathrm{id} \otimes \varepsilon) \circ \Delta_{\mathcal{X}}(h) = \alpha^{-1}(h),$$

for any $h \in H_{\mathcal{X}}$. Then we check that $\Delta_{\mathcal{X}}$ is Hom-coassociative: on one hand,

$$\begin{aligned}(\Delta_{\mathcal{X}} \otimes \alpha^{-1}) \circ \Delta_{\mathcal{X}}(h) &= (\Delta_{\mathcal{X}} \otimes \alpha^{-1})((\mathcal{X}\Delta(h))\mathcal{X}^{-1}) \\ &= (\mathcal{X}_{12}(\Delta \otimes \alpha^{-1})((\mathcal{X}\Delta(h))\mathcal{X}^{-1}))\mathcal{X}_{12}^{-1} \\ &= (\mathcal{X}_{12}(\Delta \otimes \alpha^{-1})(\mathcal{X}(\Delta(h)\mathcal{X}^{-1})))\mathcal{X}_{12}^{-1} \\ &= ((\mathcal{X}_{12}(\Delta \otimes \alpha^{-1})(\mathcal{X}))\alpha^{\otimes 3} \circ (\Delta \otimes \alpha^{-1})(\Delta(h)\mathcal{X}^{-1}))\mathcal{X}_{12}^{-1} \\ &= (\mathcal{X}_{12}(\Delta \otimes \alpha^{-1})(\mathcal{X}))(\alpha^{\otimes 3} \circ (\Delta \otimes \alpha^{-1})(\Delta(h)\mathcal{X}^{-1})\mathcal{X}_{12}^{-1}) \\ &= (\mathcal{X}_{12}(\Delta \otimes \alpha^{-1})(\mathcal{X}))(\alpha^{\otimes 3} \circ \alpha^{\otimes 3} \circ (\Delta \otimes \alpha^{-1}) \\ &\qquad \circ \Delta(h)((\Delta \otimes \alpha^{-1})(\mathcal{X}^{-1})\mathcal{X}_{12}^{-1}));\end{aligned}$$

on the other hand,

$$\begin{aligned}(\alpha^{-1} \otimes \Delta_{\mathcal{X}}) \circ \Delta_{\mathcal{X}}(h) &= (\alpha^{-1} \otimes \Delta_{\mathcal{X}})((\mathcal{X}\Delta(h))\mathcal{X}^{-1}) \\ &= (\mathcal{X}_{23}(\alpha^{-1} \otimes \Delta)((\mathcal{X}\Delta(h))\mathcal{X}^{-1}))\mathcal{X}_{23}^{-1} \\ &= (\mathcal{X}_{23}(\alpha^{-1} \otimes \Delta)(\mathcal{X}(\Delta(h)\mathcal{X}^{-1})))\mathcal{X}_{23}^{-1} \\ &= ((\mathcal{X}_{23}(\alpha^{-1} \otimes \Delta)(\mathcal{X}))\alpha^{\otimes 3} \circ (\alpha^{-1} \otimes \Delta)(\Delta(h)\mathcal{X}^{-1}))\mathcal{X}_{23}^{-1} \\ &= (\mathcal{X}_{23}(\alpha^{-1} \otimes \Delta)(\mathcal{X}))(\alpha^{\otimes 3} \circ (\alpha^{-1} \otimes \Delta)(\Delta(h)\mathcal{X}^{-1})\mathcal{X}_{23}^{-1}) \\ &= (\mathcal{X}_{23}(\alpha^{-1} \otimes \Delta)(\mathcal{X}))(\alpha^{\otimes 3} \circ \alpha^{\otimes 3} \circ (\alpha^{-1} \otimes \Delta) \\ &\qquad \circ \Delta(h)((\alpha^{-1} \otimes \Delta)(\mathcal{X}^{-1})\mathcal{X}_{23}^{-1})).\end{aligned}$$

We see that (5.1) ensures that the above two expressions are equal, that is, $\Delta_{\mathcal{X}}$ is Hom-coassociative. Secondly, $\Delta_{\mathcal{X}}$ is a Hom-algebra map: since $\mathcal{X}$ is α-invariant, for any $h, g \in H$,

$$\begin{aligned}\Delta_{\mathcal{X}}(1_H) &= (\mathcal{X}\Delta(1))\mathcal{X}^{-1} = (\alpha \otimes \alpha)(\mathcal{X})\mathcal{X}^{-1} = \mathcal{X}\mathcal{X}^{-1} = 1_H \otimes 1_H, \\ \Delta_{\mathcal{X}}(h)\Delta_{\mathcal{X}}(g) &= ((\mathcal{X}\Delta(h))\mathcal{X}^{-1})((\mathcal{X}\Delta(g))\mathcal{X}^{-1}) \\ &= (((\mathcal{X}(\alpha^{-1} \otimes \alpha^{-1}) \circ \Delta(h))\mathcal{X}^{-1})\mathcal{X})(\alpha^{\otimes 2} \circ \Delta(g)\mathcal{X}^{-1}) \\ &= ((\mathcal{X}\Delta(h))(\mathcal{X}^{-1}\mathcal{X}))(\alpha^{\otimes 2} \circ \Delta(g)\mathcal{X}^{-1}) \\ &= (\mathcal{X}\alpha^{\otimes 2} \circ \Delta(h))(\alpha^{\otimes 2} \circ \Delta(g)\mathcal{X}^{-1}) \\ &= ((\mathcal{X}\Delta(h))\alpha^{\otimes 2} \circ \Delta(g))\mathcal{X}^{-1} \\ &= (\mathcal{X}(\Delta(h)\Delta(g)))\mathcal{X}^{-1} = \Delta_{\mathcal{X}}(hg).\end{aligned}$$

STEP 2. We will prove that $\mathcal{R}_{\mathcal{X}}$ is a Hom-quasitriangular structure for the Hom-bialgebra $H_{\mathcal{X}}$. It is obvious that $\mathcal{R}_{\mathcal{X}}$ is invertible and α-invariant, by the invertibility and invariance of $\mathcal{R}$ and $\mathcal{X}$. In the following computation, we ignore the Hom-associativity because of the invariance of $\mathcal{R}_{\mathcal{X}}, \mathcal{R}, \mathcal{X}$:

$$
\begin{aligned}
&(\alpha\otimes\alpha\otimes \mathrm{id})(\Delta_{\mathcal{X}}\otimes \mathrm{id})(\mathcal{R}_{\mathcal{X}})\\
&\quad= (\Delta_{\mathcal{X}}\otimes\alpha^{-1})(\mathcal{R}_{\mathcal{X}}) = \mathcal{X}_{12}(\Delta\otimes\alpha^{-1})(\mathcal{X}^{21}\mathcal{R}\mathcal{X}^{-1})\mathcal{X}_{12}^{-1}\\
&\quad= \mathcal{X}_{12}(\Delta\otimes\alpha^{-1})(\mathcal{X}^{21})(\Delta\otimes\alpha^{-1})(\mathcal{R})(\Delta\otimes\alpha^{-1})(\mathcal{X}^{-1})\mathcal{X}_{12}^{-1}\\
&\quad\overset{(3.1)}{=} \mathcal{X}_{12}(\Delta\otimes\alpha^{-1})(\mathcal{X}^{21})\mathcal{R}_{13}\mathcal{R}_{23}(\Delta\otimes\alpha^{-1})(\mathcal{X}^{-1})\mathcal{X}_{12}^{-1}\\
&\quad\overset{(5.1)}{=} \mathcal{X}_{12}(\Delta\otimes\alpha^{-1})(\mathcal{X}^{21})\mathcal{R}_{13}\mathcal{R}_{23}(\alpha^{-1}\otimes\Delta)(\mathcal{X}^{-1})\mathcal{X}_{23}^{-1}\\
&\quad\overset{(3.2)}{=} \mathcal{X}_{12}(\Delta\otimes\alpha^{-1})(\mathcal{X}^{21})\mathcal{R}_{13}(\alpha^{-1}\otimes\tau\circ\Delta)(\mathcal{X}^{-1})\mathcal{R}_{23}\mathcal{X}_{23}^{-1}\\
&\quad\overset{(a)}{=} \mathcal{X}_{31}(\mathcal{X}_2^{(1)}\otimes\alpha^{-1}(\mathcal{X}^{(2)})\otimes\mathcal{X}_1^{(1)})\mathcal{R}_{13}(\alpha^{-1}\otimes\tau\circ\Delta)(\mathcal{X}^{-1})\mathcal{R}_{23}\mathcal{X}_{23}^{-1}\\
&\quad\overset{(3.2)}{=} \mathcal{X}_{31}\mathcal{R}_{13}(\mathcal{X}_1^{(1)}\otimes\alpha^{-1}(\mathcal{X}^{(2)})\otimes\mathcal{X}_2^{(1)})(\alpha^{-1}\otimes\tau\circ\Delta)(\mathcal{X}^{-1})\mathcal{R}_{23}\mathcal{X}_{23}^{-1}\\
&\quad\overset{(b)}{=} \mathcal{X}_{31}\mathcal{R}_{13}\mathcal{X}_{13}^{-1}\mathcal{X}_{32}(\alpha^{-1}\otimes\tau\circ\Delta)(\mathcal{X})(\alpha^{-1}\otimes\tau\circ\Delta)(\mathcal{X}^{-1})\mathcal{R}_{23}\mathcal{X}_{23}^{-1}\\
&\quad= (\mathcal{R}_{\mathcal{X}})_{13}((\mathcal{X}_{32}\mathcal{R}_{23})\mathcal{X}_{23}^{-1}) = (\mathcal{R}_{\mathcal{X}})_{13}(\mathcal{R}_{\mathcal{X}})_{23}.
\end{aligned}
$$

In the equalities (a) and (b), we use the Hom-2-cocycle condition, in which we make a cyclic rotation of the factors in $H\otimes H\otimes H$ to get $\mathcal{X}_{12}(\Delta\otimes\alpha^{-1})(\mathcal{X}_{21}) = \mathcal{X}_{31}(\mathcal{X}_2^{(1)}\otimes\alpha^{-1}(\mathcal{X}^{(2)})\otimes\mathcal{X}_1^{(1)})$ and a further permutation in $H\otimes H\otimes H$ to obtain $\mathcal{X}_{31}(\mathcal{X}_1^{(1)}\otimes\alpha^{-1}(\mathcal{X}^{(2)})\otimes\mathcal{X}_2^{(1)}) = \mathcal{X}_{32}(\alpha^{-1}\otimes\tau\circ\Delta)(\mathcal{X})$, which is (b) $\mathcal{X}_1^{(1)}\otimes\alpha^{-1}(\mathcal{X}^{(2)})\otimes\mathcal{X}_2^{(1)} = \mathcal{X}_{31}^{-1}\mathcal{X}_{32}(\alpha^{-1}\otimes\tau\circ\Delta)(\mathcal{X})$. So, $(\alpha\otimes\alpha\otimes\mathrm{id})\circ(\Delta_{\mathcal{X}}\otimes\mathrm{id})(\mathcal{R}_{\mathcal{X}}) = (\mathcal{R}_{\mathcal{X}})_{13}(\mathcal{R}_{\mathcal{X}})_{23}$ as required. Similarly, we can get $(\mathrm{id}\otimes\alpha\otimes\alpha)\circ(\mathrm{id}\otimes\Delta_{\mathcal{X}})(\mathcal{R}_{\mathcal{X}}) = (\mathcal{R}_{\mathcal{X}})_{13}(\mathcal{R}_{\mathcal{X}})_{12}$, that is, $\mathcal{R}_{\mathcal{X}}$ obeys axiom (3.1).

Moreover, for any $h\in H_{\mathcal{X}}$,

$$
\begin{aligned}
\tau\circ\Delta_{\mathcal{X}}(h) &= (\mathcal{X}^{21}\tau\circ\Delta(h))\mathcal{X}^{-21} = ((\mathcal{X}^{21}(\mathcal{R}\Delta(h)))\mathcal{R}^{-1})\mathcal{X}^{-21}\\
&= (((\mathcal{X}^{21}\mathcal{R})\alpha^{\otimes 2}\circ\Delta(h))\mathcal{R}^{-1})\mathcal{X}^{-21} = (((\mathcal{R}_{\mathcal{X}}\mathcal{X})\alpha^{\otimes 2}\circ\Delta(h))\mathcal{R}^{-1})\mathcal{X}^{-21}\\
&= ((\mathcal{R}_{\mathcal{X}}(\mathcal{X}\Delta(h)))\mathcal{R}^{-1})\mathcal{X}^{-21} = (\mathcal{R}_{\mathcal{X}}(\mathcal{X}\alpha^{\otimes 2}\Delta(h)))(\mathcal{R}^{-1}\mathcal{X}^{-21})\\
&= (\mathcal{R}_{\mathcal{X}}(\mathcal{X}\alpha^{\otimes 2}\Delta(h)))(\mathcal{X}^{-1}\mathcal{R}_{\mathcal{X}}^{-1}) = ((\mathcal{R}_{\mathcal{X}}(\mathcal{X}\Delta(h)))\mathcal{X}^{-1})\mathcal{R}_{\mathcal{X}}^{-1}\\
&= (\mathcal{R}_{\mathcal{X}}((\mathcal{X}\Delta(h))\mathcal{X}^{-1}))\mathcal{R}_{\mathcal{X}}^{-1} = (\mathcal{R}_{\mathcal{X}}\Delta_{\mathcal{X}}(h))\mathcal{R}_{\mathcal{X}}^{-1},
\end{aligned}
$$

which is just axiom (3.5) for $\mathcal{R}_{\mathcal{X}}$.

Step 3. Finally, if H is a Hom-Hopf algebra with $\alpha^2 = \mathrm{id}$, we check that $S_{\mathcal{X}}$ is an antipode for $H_{\mathcal{X}}$. For this, we need to show that U is invertible. We define $U^{-1} = S(\mathcal{X}^{-(1)})\mathcal{X}^{-(2)}$, where $\mathcal{X}^{-1} = \mathcal{X}^{-(1)}\otimes\mathcal{X}^{-(2)}$ in our notation, and $\mathcal{X}'^{-1}$ is another copy of $\mathcal{X}^{-1}$ in the following computation:

$$
\begin{aligned}
UU^{-1} &= (\mathcal{X}^{(1)}S(\mathcal{X}^{(2)}))(S(\mathcal{X}^{-(1)})\mathcal{X}^{-(2)})\\
&= (\alpha^{-1}(\mathcal{X}^{(1)}S(\mathcal{X}^{(2)}))S(\mathcal{X}^{-(1)}))\alpha(\mathcal{X}^{-(2)})\\
&= (\mathcal{X}^{(1)}\alpha^{-1}\circ S(\mathcal{X}^{-(1)}\mathcal{X}^{(2)}))\alpha(\mathcal{X}^{-(2)})
\end{aligned}
$$

$$= (\alpha(\mathcal{X}^{(1)})S(\alpha^{-1}(\mathcal{X}^{-(1)})\mathcal{X}^{(2)}))\alpha(\mathcal{X}^{-(2)})$$
$$\overset{(c)}{=} ((\alpha^{-2}(\mathcal{X}'^{-(1)})\mathcal{X}^{(1)})S(\alpha^{-1}(\mathcal{X}^{-(1)})\mathcal{X}^{(2)}))(\alpha^{-1}(S(\mathcal{X}_1'^{-(2)})\mathcal{X}_2'^{-(2)})\mathcal{X}^{-(2)})$$
$$= ((\alpha^{-2}(\mathcal{X}'^{-(1)})\mathcal{X}^{(1)})S(\alpha^{-1}(\mathcal{X}^{-(1)})\mathcal{X}^{(2)}))(S(\mathcal{X}_1'^{-(2)})\alpha^{-1}(\mathcal{X}_2'^{-(2)}\mathcal{X}^{-(2)}))$$
$$= (\alpha^{-1}((\alpha^{-2}(\mathcal{X}'^{-(1)})\mathcal{X}^{(1)})S(\alpha^{-1}(\mathcal{X}^{-(1)})\mathcal{X}^{(2)}))S(\mathcal{X}_1'^{-(2)}))(\mathcal{X}_2'^{-(2)}\mathcal{X}^{-(2)})$$
$$= ((\alpha^{-2}(\mathcal{X}'^{-(1)})\mathcal{X}^{(1)})S(\alpha^{-2}(\mathcal{X}^{-(1)})\alpha^{-1}(\mathcal{X}^{(2)}))S(\alpha^{-1}(\mathcal{X}_1'^{-(2)})))(\mathcal{X}_2'^{-(2)}\mathcal{X}^{-(2)})$$
$$= ((\alpha^{-2}(\mathcal{X}'^{-(1)})\mathcal{X}^{(1)})S(\alpha^{-1}(\mathcal{X}_1'^{-(2)})(\alpha^{-2}(\mathcal{X}^{-(1)})\alpha^{-1}(\mathcal{X}^{(2)}))))(\mathcal{X}_2'^{-(2)}\mathcal{X}^{-(2)})$$
$$= ((\alpha^{-2}(\mathcal{X}'^{-(1)})\mathcal{X}^{(1)})S(\alpha^{-2}(\mathcal{X}_1'^{-(2)}\mathcal{X}^{-(1)})\mathcal{X}^{(2)}))(\mathcal{X}_2'^{-(2)}\mathcal{X}^{-(2)})$$
$$= ((\mathcal{X}'^{-(1)}\mathcal{X}^{(1)})S((\mathcal{X}_1'^{-(2)}\mathcal{X}^{-(1)})\mathcal{X}^{(2)}))\alpha^2(\mathcal{X}_2'^{-(2)}\mathcal{X}^{-(2)})$$
$$\overset{(d)}{=} (\mathcal{X}_1^{-(1)}S(\mathcal{X}_2^{-(1)}))\mathcal{X}^{-(2)} = \varepsilon(\mathcal{X}^{-(1)})\alpha(\mathcal{X}^{-(2)}) = 1_H.$$

For (c), we use the fact $\mathcal{X}^{(-1)}\varepsilon(\mathcal{X}^{(-2)}) = 1_H$ from (5.2). And for (d), we apply the Hom-2-cocycle condition (5.1) in the form

$$(\Delta\otimes\alpha^{-1})\mathcal{X}^{-1} = (((\alpha^{-1}\otimes\Delta)\mathcal{X}^{-1})\mathcal{X}_{23}^{-1})\mathcal{X}_{12},$$

that is,

$$\mathcal{X}_1^{-(1)}\otimes\mathcal{X}_2^{-(1)}\otimes\mathcal{X}^{-(2)} = \mathcal{X}'^{-(1)}\mathcal{X}^{(1)}\otimes(\mathcal{X}_1'^{-(2)}\mathcal{X}^{-(1)})\mathcal{X}^{(2)}\otimes\alpha^2(\mathcal{X}_2'^{-(2)}\mathcal{X}^{-(2)}).$$

Similarly, from another form of the Hom-2-cocycle condition

$$(\Delta\otimes\alpha^{-1})\mathcal{X} = \mathcal{X}_{12}^{-1}(\mathcal{X}_{23}(\alpha^{-1}\otimes\Delta)\mathcal{X}),$$

i.e.,

$$\mathcal{X}_1^{(1)}\otimes\mathcal{X}_2^{(1)}\otimes\mathcal{X}^{(2)} = \mathcal{X}^{-(1)}\mathcal{X}'^{(1)}\otimes\mathcal{X}^{-(2)}(\mathcal{X}^{(1)}\mathcal{X}_1'^{(2)})\otimes\alpha^2(\mathcal{X}^{(2)}\mathcal{X}_2'^{(2)}),$$

where $\mathcal{X}'$ is another copy of $\mathcal{X}$, we have $U^{-1}U = 1_H$. Further, from the α-invariance of $\mathcal{X}$, we know that U, U^{-1} are both α-invariant. So we have $\alpha\circ S_{\mathcal{X}}(h) = (US\circ\alpha(h))U^{-1} = S_{\mathcal{X}}\circ\alpha(h)$ for any $h\in H_{\mathcal{X}}$. By using the definitions, the properties of the antipode and $\alpha^2 = \mathrm{id}$, we have

$$m\circ(S_{\mathcal{X}}\otimes\mathrm{id})\circ\Delta_{\mathcal{X}}(h) = ((US((\mathcal{X}^{(1)}h_1)\mathcal{X}^{-(1)}))U^{-1})((\mathcal{X}^{(2)}h_2)\mathcal{X}^{-(2)})$$
$$= \alpha(US((\mathcal{X}^{(1)}h_1)\mathcal{X}^{-(1)}))(U^{-1}\alpha((\mathcal{X}^{(2)}h_2)\mathcal{X}^{-(2)}))$$
$$= (\alpha(U)S(\mathcal{X}^{(1)}(\alpha(h_1)\mathcal{X}^{-(1)})))((S(\mathcal{X}'^{-(1)})\mathcal{X}'^{-(2)})(\mathcal{X}^{(2)}(\alpha(h_2)\mathcal{X}^{-(2)})))$$
$$= (U(S(\alpha(h_1)\mathcal{X}^{-(1)})S(\mathcal{X}^{(1)})))(\alpha\circ S(\mathcal{X}'^{-(1)})(\mathcal{X}'^{-(2)}(\alpha(\mathcal{X}^{(2)})(h_2\alpha(\mathcal{X}^{-(2)})))$$
$$= ((US(\alpha(h_1)\mathcal{X}^{-(1)}))\alpha\circ S(\mathcal{X}^{(1)}))(\alpha\circ S(\mathcal{X}'^{-(1)})(\alpha(\mathcal{X}'^{-(2)}\mathcal{X}^{(2)})(\alpha(h_2)\mathcal{X}^{-(2)})))$$
$$= ((\alpha(US(\alpha(h_1)\mathcal{X}^{-(1)}))S(\mathcal{X}^{(1)}))\alpha\circ S(\mathcal{X}'^{-(1)}))((\mathcal{X}'^{-(2)}\mathcal{X}^{(2)})(h_2\alpha(\mathcal{X}^{-(2)})))$$
$$= ((US(\alpha(h_1)\mathcal{X}^{-(1)}))S(\mathcal{X}'^{-(1)}\mathcal{X}^{(1)}))((\mathcal{X}'^{-(2)}\mathcal{X}^{(2)})(h_2\alpha(\mathcal{X}^{-(2)})))$$

$= \alpha(US(\alpha(h_1)\mathcal{X}^{-(1)}))(\alpha(h_2)\mathcal{X}^{-(2)})$
$= \alpha(U(S(\mathcal{X}^{-(1)})S(\alpha(h_1))))(\alpha(h_2)\mathcal{X}^{-(2)})$
$= U(\alpha\circ S(\mathcal{X}^{-(1)})S(h_1))(\alpha(h_2)\mathcal{X}^{-(2)})$
$= ((US\circ\alpha(\mathcal{X}^{-(1)}))\alpha\circ S(h_1))(\alpha(h_2)\mathcal{X}^{-(2)})$
$= (US(\mathcal{X}^{-(1)}))(\alpha\circ S(h_1)(h_2\alpha(\mathcal{X}^{-(2)}))) = (US(\mathcal{X}^{-(1)}))((S(h_1)h_2)\mathcal{X}^{-(2)})$
$= \varepsilon(h)(\alpha(U)S(\mathcal{X}^{-(1)}))\alpha(\mathcal{X}^{-(2)}) = \varepsilon(h)U(S(\mathcal{X}^{-(1)})\mathcal{X}^{-(2)})$
$= \varepsilon(h)UU^{-1} = \varepsilon(h)1_H.$

On the other hand, in the same way

$$m\circ(\mathrm{id}\otimes S_{\mathcal{X}})\circ\Delta_{\mathcal{X}}(h) = \varepsilon(h)1_H.$$

So $S_{\mathcal{X}}$ is an antipode for $H_{\mathcal{X}}$. Thus, we finally obtain a new quasitriangular Hom-Hopf algebra $(H_{\mathcal{X}}, \alpha, \mathcal{R}_{\mathcal{X}})$. ∎

Remark 5.4. Under the assumption of Theorem 5.3, if H is commutative, then the quasitriangular Hom-Hopf algebra $H_{\mathcal{R}}$, twisted by $\mathcal{R}$, is just H with the same multiplication, comultiplication, unit, counit and antipode, but only with the different Hom-quasitriangular structure $\mathcal{R}^{21}$, namely, the opposite Hom-quasitriangular structure.

From Theorem 5.3, we know that

$$\Delta_{\mathcal{R}}(h) = (\mathcal{R}\Delta(h))\mathcal{R}^{-1}, \quad \mathcal{R}_{\mathcal{R}} = (\mathcal{R}^{21}\mathcal{R})\mathcal{R}^{-1}, \quad S_{\mathcal{X}}(h) = (US(h))U^{-1},$$

for all $h \in H_{\mathcal{X}}$. Here $\mathcal{R}^{21} = \mathcal{R}^{(2)}\otimes\mathcal{R}^{(1)}$ and $U = \mathcal{R}^{(1)}S(\mathcal{R}^{(2)})$. Since H is commutative and $\mathcal{R}$ is α-invariant, we have

$$\begin{aligned}\Delta_{\mathcal{R}}(h) &= (\mathcal{R}\Delta(h))\mathcal{R}^{-1} = (\Delta(h)\mathcal{R})\mathcal{R}^{-1} = \alpha^{\otimes 2}(\Delta(h))(\mathcal{R}\mathcal{R}^{-1})\\ &= \alpha^{\otimes 2}(\Delta(h))(1_H\otimes 1_H) = \Delta(h),\\ \mathcal{R}_{\mathcal{R}} &= (\mathcal{R}^{21}\mathcal{R})\mathcal{R}^{-1} = \mathcal{R}^{21}(\mathcal{R}\mathcal{R}^{-1}) = \mathcal{R}^{21},\\ S_{\mathcal{X}}(h) &= (US(h))U^{-1} = \alpha(S(h))(UU^{-1}) = \alpha(S(h))1_H = S(h).\end{aligned}$$

That is, the twisted quasitriangular Hom-Hopf algebra $H_{\mathcal{R}}$ is just H, except for the opposite Hom-quasitriangular structure $\mathcal{R}^{21}$.

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