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HODGE–BOTT–CHERN DECOMPOSITIONS OF MIXED TYPE FORMS ON FOLIATED KÄHLER MANIFOLDS

BY

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Abstract. The Bott–Chern cohomology groups and the Bott–Chern Laplacian on differential forms of mixed type on a compact foliated Kähler manifold are defined and studied. Also, a Hodge decomposition theorem of Bott–Chern type for differential forms of mixed type is proved. Finally, the case of projectivized tangent bundle of a complex Finsler manifold is discussed.

1. Introduction and preliminaries

1.1. Introduction. The importance of Hodge theory both in Riemannian and complex geometry is beyond question. On the other hand the study of geometry and cohomology of hermitian foliations was initiated by I. Vaisman [Va71, Va73, Va77] and the Hodge theory for basic forms, in both transversally riemannian and kählerian cases, was intensively studied by A. El Kacimi Alaoui et al. [E-K, E-K-G, E-K-H]. We also mention that in [Zh06] a Hodge decomposition theorem on Kähler–Finsler manifolds is obtained. Recently, M. Schweitzer [S] developed a Bott–Chern cohomology theory on complex manifolds and obtained a Hodge type decomposition with respect to the associated Bott–Chern Laplacian. Also, L.-S. Tseng and S.-T. Yau [T-Y] presented a systematic study of cohomology and Hodge theory of Bott–Chern type on symplectic manifolds.

The main goal of the present paper is to study a Bott–Chern cohomology theory for differential forms of mixed type and some Hodge type decompositions for such forms on a compact foliated Kähler manifold. Firstly, following [Va71, Va73] and [Va77] we give a short review of decompositions of the classical operators $d, \star, \delta, L, \Lambda$ and C for forms of mixed type on compact foliated Kähler manifolds. Next, we define the Bott–Chern and Aeppli cohomology groups of these forms, discuss the Bott–Chern and Aeppli Laplacian, and prove some Hodge type decomposition theorems for differential forms

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of mixed type. The methods used here are similar to those applied in [S] and [T-Y], and are closely related to those used in [Va77]. Finally, it is shown that such Hodge decompositions of Bott–Chern type hold for differential forms of mixed type on the projectivized tangent bundle of a complex Finsler manifold.

1.2. Preliminaries. Let us begin with a short review of complex analytic foliated manifolds and set up the basic notation and terminology. For more details, see [Va71, Va73, Va77].

Definition 1.1 ([Va77]). A *complex analytic foliated structure* $\mathcal{F}$, briefly c.a.f., of complex codimension n on a complex $(n+m)$-dimensional manifold $\mathcal{M}$ is given by an atlas $\{\mathcal{U},(z^a_\alpha, z^u_\alpha)\}$, with $a, b, \dots$ running in $\{1,\dots,n\}$ and $u, v, \dots$ running in $\{n+1,\dots,n+m\}$, such that for every $U_\alpha, U_\beta \in \mathcal{U}$ with $U_\alpha \cap U_\beta \neq \emptyset$ one has, besides analyticity, $\partial z^a_\beta/\partial z^u_\alpha = 0$.

The leaves of $\mathcal{F}$ are locally given by $z^a_\alpha = \text{const}$. Let $T^{1,0}\mathcal{M}$ be the holomorphic tangent bundle of $\mathcal{M}$, and $T^{0,1}\mathcal{M} = \overline{T^{1,0}\mathcal{M}}$ the antiholomorphic tangent bundle. The tangent vectors of the leaves define the *structural subbundle* $\mathcal{D}^{1,0} = T^{1,0}\mathcal{F}$ of $T^{1,0}\mathcal{M}$ with local bases $\dot{\partial}_u = \{\partial/\partial z^u_\alpha\}$ and the transition functions $(\partial z^u_\beta/\partial z^v_\alpha)$, and $Q^{1,0}\mathcal{F} = T^{1,0}\mathcal{M}/T^{1,0}\mathcal{F}$ is the *transversal bundle* with local bases defined by the equivalence classes $[\partial/\partial z^a_\alpha]$ and the transition functions $(\partial z^a_\beta/\partial z^b_\alpha)$.

Generally, we shall say that objects depending only on leaves are *foliated*; in particular, they may be c.a.f. For instance, $f : \mathcal{M} \to C$ is foliated if $\partial f/\partial z^u_\alpha = \partial f/\partial \overline{z}^u_\alpha = 0$, and it is c.a.f. if moreover $\partial f/\partial \overline{z}^a_\alpha = 0$. A differential form is c.a.f. if it does not contain $dz^u_\alpha, d\overline{z}^u_\alpha$ and has local c.a.f. coefficients. A vector bundle on $\mathcal{M}$ is c.a.f. if it has c.a.f. transition functions (for instance, the transversal bundle is so).

In the following, we suppose that $\mathcal{M}$ is hermitian with metric h. Then the orthogonal bundle $\mathcal{D}^{1,0\perp} = T^{1,0\perp}\mathcal{F}$ of $T^{1,0}\mathcal{F}$, i.e. $T^{1,0}\mathcal{M} = \mathcal{D}^{1,0\perp} \oplus \mathcal{D}^{1,0}$, which is differentially isomorphic to $Q^{1,0}\mathcal{F}$, has local bases of the form

$$\frac{\delta}{\delta z^a} = \frac{\partial}{\partial z^a} - t^u_a \frac{\partial}{\partial z^u} \tag{1.1}$$

(the index α of the coordinate neighborhood will be omitted) and we shall use the bases $\{\delta_a, \dot{\partial}_u\}$, where $\delta_a = \delta/\delta z^a$, to express various vector fields of $\mathcal{X}(\mathcal{M})$. By conjugation, we have $\mathcal{D}^{0,1} = \overline{\mathcal{D}^{1,0}} = \operatorname{span}\{\dot{\partial}_{\overline{u}}\}$ and $\mathcal{D}^{0,1\perp} = \overline{\mathcal{D}^{1,0\perp}} = \operatorname{span}\{\delta_{\overline{a}}\}$, and hence the decomposition of the complexified tangent bundle of $(\mathcal{M},\mathcal{F})$, namely $T_C\mathcal{M} = \mathcal{D}^\perp \oplus \mathcal{D}$, where $\mathcal{D}^\perp = \mathcal{D}^{1,0\perp} \oplus \mathcal{D}^{0,1\perp}$ and $\mathcal{D} = \mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1}$. The corresponding dual cobases are given by

$$\{dz^a, \delta z^u = dz^u + t^u_a dz^a, d\overline{z}^a, \delta\overline{z}^u = d\overline{z}^u + \overline{t^u_a} d\overline{z}^a\}. \tag{1.2}$$

These cobases allow us to speak of the *type* (p_1, p_2, q_1, q_2) of a differential form by counting in its expression the number of dz^a, $d\overline{z}^a$, δz^u and $\delta\overline{z}^u$, respectively. Thus, we denote by

$$\Omega^{p_1,p_2,q_1,q_2}(\mathcal{M}) = \Lambda^{p_1,p_2}\mathcal{D}^\perp \wedge \Lambda^{q_1,q_2}\mathcal{D}$$

the set of all (p_1,p_2,q_1,q_2)-differential forms, locally given by

$$\varphi = \frac{1}{p_1!p_2!q_1!q_2!}\sum \varphi_{A_{p_1}\overline{B}_{p_2}U_{q_1}\overline{V}_{q_2}} dz^{A_{p_1}} \wedge d\overline{z}^{B_{p_2}} \wedge \delta z^{U_{q_1}} \wedge \delta\overline{z}^{V_{q_2}} \tag{1.3}$$

where we have put $A_{p_1} = (a_1 \dots a_{p_1})$, $B_{p_2} = (b_1 \dots b_{p_2})$, $U_{q_1} = (u_1 \dots u_{q_1})$, $V_{q_2} = (v_1 \dots v_{q_2})$, $dz^{A_{p_1}} = dz^{a_1} \wedge \cdots \wedge dz^{a_{p_1}}$, $d\overline{z}^{B_{p_2}} = d\overline{z}^{b_1} \wedge \cdots \wedge d\overline{z}^{b_{p_2}}$, $\delta z^{U_{q_1}} = \delta z^{u_1} \wedge \cdots \wedge \delta z^{u_{q_1}}$ and $\delta\overline{z}^{V_{q_2}} = \delta\overline{z}^{v_1} \wedge \cdots \wedge \delta\overline{z}^{v_{q_2}}$.

These forms will be referred to as being of *complex type* (p_1+q_1, p_2+q_2), *foliated type* (p_1+p_2, q_1+q_2) and *mixed type* $(p_1, p_2+q_1+q_2)$.

Throughout this paper we consider forms of mixed type and we denote by $\Lambda^{p,q}_{\mathrm{mix}}(\mathcal{M},\mathcal{F})$ the space of all differential forms of mixed type (p,q) on $(\mathcal{M},\mathcal{F})$. According to the above discussion we have (see also [B-C-I])

$$\Lambda^{p,q}_{\mathrm{mix}}(\mathcal{M},\mathcal{F}) = \bigoplus_{r,h} \Omega^{p,r,h,q-r-h}(\mathcal{M},\mathcal{F}) = \bigoplus_{k=0}^{q} \Lambda^{p,k}\mathcal{D}^\perp \wedge \Lambda^{q-k}(\mathcal{D},C).$$

The metric can be locally expressed as

$$h = h_{a\bar{b}} dz^a \otimes d\overline{z}^b + h_{u\overline{v}} \delta z^u \otimes \delta\overline{z}^v \tag{1.4}$$

and its fundamental form is

$$\omega = \omega' + \omega'', \quad \omega' = \frac{i}{2} h_{a\bar{b}} dz^a \wedge d\overline{z}^b, \quad \omega'' = \frac{i}{2} h_{u\overline{v}} \delta z^u \wedge \delta\overline{z}^v. \tag{1.5}$$

We also notice that the metric h is Kähler ($d\omega = 0$) if and only if [Va73]:

$$\begin{aligned} &\text{(a) } \delta_c h_{a\bar{b}} - \delta_a h_{c\bar{b}} = 0, && \text{(b) } \dot{\partial}_u h_{a\bar{b}} - h_{u\overline{v}} \delta_a \overline{t^v_b} = 0,\\ &\text{(c) } \delta_a h_{u\overline{v}} - h_{w\overline{v}} \dot{\partial}_u t^w_a = 0, && \text{(d) } \dot{\partial}_u h_{v\overline{w}} - \dot{\partial}_v h_{u\overline{w}} = 0,\\ &\text{(e) } \delta_c t^u_a - \delta_a t^u_c = 0, && \text{(f) } h_{u\overline{v}} \dot{\partial}_{\overline{w}} t^u_a - h_{u\overline{w}} \dot{\partial}_{\overline{v}} t^u_a = 0. \end{aligned} \tag{1.6}$$

On the manifold $\mathcal{M}$, we can consider the classical scalar product and the operators $d, \star, \delta, L, \Lambda, C$, with respect to forms of complex type [M-K]; following [Va77], we will present the decompositions of these operators with respect to the mixed type.

Let us assume that the manifold $\mathcal{M}$ is compact. The operator d has an obvious decomposition (see [Va71, Va73]),

$$d = \mu + \lambda + \nu \tag{1.7}$$

into three parts of the respective mixed types $(1,0)$, $(0,1)$ and $(2,-1)$.

It is easy to see that in the Kählerian case condition (1.6)(e) implies $\nu = 0$. Hence *the differential forms on a foliated Kähler manifold are organized into a double cochain complex by means of mixed types.*

We denote by $\Omega^{p,q}(\mathcal{M})$ the set of all (p,q)-forms of complex type on $(\mathcal{M},\mathcal{F})$. In order to recall the Hodge star operator $\star$ for differential forms of complex type, according to [Ko87, p. 60], we choose $(1,0)$-forms $\zeta^1,\dots,\zeta^{n+m}\in\Omega^{1,0}(\mathcal{M})$ locally forming a unitary frame field for the holomorphic cotangent bundle $(T^{1,0}\mathcal{M})^*$ so that

$$h=\sum_\alpha \zeta^\alpha\otimes\overline{\zeta}^\alpha \quad\text{and}\quad \omega=i\sum_\alpha\zeta^\alpha\wedge\overline{\zeta}^\alpha. \tag{1.8}$$

For each ordered set of indices $A_p=\{\alpha_1,\dots,\alpha_p\}$, we write $\zeta^{A_p}=\zeta^{\alpha_1}\wedge\dots\wedge\zeta^{\alpha_p},\overline{\zeta}^{A_p}=\overline{\zeta}^{\alpha_1}\wedge\dots\wedge\overline{\zeta}^{\alpha_p}$ and denote by $A_{n+m-p}=\{\alpha_{p+1},\dots,\alpha_{n+m}\}$ a complementary ordered set of indices. We might as well assume that $\alpha_{p+1}<\dots<\alpha_{n+m}$ although this is not essential. The star operator $\star:\Omega^{p,q}(\mathcal{M})\to\Omega^{n+m-q,n+m-p}(\mathcal{M})$ is defined as a linear map (over $\Omega^0(\mathcal{M})=\Omega^{0,0}(\mathcal{M})$) satisfying

$$\star(\zeta^{A_p}\wedge\overline{\zeta}^{B_q})=\varepsilon(A_p,B_q)\zeta^{B_{n+m-q}}\wedge\overline{\zeta}^{A_{n+m-p}}, \tag{1.9}$$

where $\varepsilon(A_p,B_q)=(-1)^{(n+m)p+(n+m)(n+m+2)/2}\sigma(A_pA_{n+m-p})\sigma(B_qB_{n+m-q})$.

Here $\sigma(A_pA_{n+m-p})$ denotes the sign of the permutation (A_pA_{n+m-p}). Next, we shall consider the dual operator $\overline{\star}$ defined by composition of $\star$ with complex conjugation, that is, $\overline{\star}:\Omega^{p,q}(\mathcal{M})\to\Omega^{n+m-p,n+m-q}(\mathcal{M})$ is given by $\overline{\star}\varphi:=\star\overline{\varphi}=\overline{\star\varphi}$.

The sign in (1.9) is chosen so that

$$(\zeta^{A_p}\wedge\overline{\zeta}^{B_q})\wedge\overline{\star}(\zeta^{A_p}\wedge\overline{\zeta}^{B_q})=i^{n+m}\sum_\alpha\zeta^\alpha\wedge\overline{\zeta}^\alpha=\frac{\omega^{n+m}}{(n+m)!}. \tag{1.10}$$

If $\varphi=\frac{1}{p!q!}\sum\varphi_{A_p\overline{B}_q}\zeta^{A_p}\wedge\overline{\zeta}^{B_q}$ and $\psi=\frac{1}{p!q!}\sum\psi_{A_p\overline{B}_q}\zeta^{A_p}\wedge\overline{\zeta}^{B_q}$ are two (p,q)-forms of complex type on $(\mathcal{M},\mathcal{F})$ then

$$\varphi\wedge\overline{\star}\psi=\left(\frac{1}{p!q!}\sum\varphi_{A_p\overline{B}_q}\overline{\psi_{A_p\overline{B}_q}}\right)\frac{\omega^{n+m}}{(n+m)!}. \tag{1.11}$$

The inner product in the space of (p,q)-forms of complex type on $(\mathcal{M},\mathcal{F})$ is given by setting

$$(\varphi,\psi)=\int_{\mathcal{M}}\varphi\wedge\overline{\star}\psi. \tag{1.12}$$

Now, $\overline{\star}$ sends forms of mixed type (p,q) to forms of mixed type $(n-p,n+2m-q)$. If we write $\varphi,\psi\in\Lambda^{p,q}_{\mathrm{mix}}(\mathcal{M},\mathcal{F})$ as $\varphi=\bigoplus_{h=0}^q\varphi_{p+h,q-h}$ and $\psi=\bigoplus_{h=0}^q\psi_{p+h,q-h}$, where the subscripts indicate the complex type of the respective terms, then their scalar product is given by

$$\langle\varphi,\psi\rangle=\bigoplus_{h=0}^q(\varphi_{p+h,q-h},\psi_{p+h,q-h}). \tag{1.13}$$

As in the classical theory [Ko87, M-K], it follows that

$$(\bar{\star})^{-1}\varphi = (-1)^{\deg\varphi}\bar{\star}\varphi, \qquad \delta = -\bar{\star}d\bar{\star}, \tag{1.14}$$

hence

$$\delta = \mu^* + \lambda^* + \nu^*, \qquad \mu^* = -\bar{\star}\mu\bar{\star}, \qquad \lambda^* = -\bar{\star}\lambda\bar{\star}, \qquad \nu^* = -\bar{\star}\nu\bar{\star}, \tag{1.15}$$

where the summands have mixed types $(-1,0)$, $(0,-1)$ and $(-2,1)$, respectively, and in the kählerian case $\nu^* = 0$. From (1.5) we have $L = L' + L''$, where L' denotes left exterior multiplication by ω' and has mixed type $(1,1)$, and similarly L'' has mixed type $(0,2)$. It also follows that

$$\Lambda = \Lambda' + \Lambda'', \qquad \Lambda' = -\bar{\star}L'\bar{\star}, \qquad \Lambda'' = -\bar{\star}L''\bar{\star}, \tag{1.16}$$

where the summands have mixed types $(-1,-1)$ and $(0,-2)$, respectively.

The operator $C : \Omega^*(\mathcal{M}) \to \Omega^{p,q}(\mathcal{M})$ is defined by $C\varphi = \sum_{p,q} i^{p-q}\varphi_{p,q}$ for any $\varphi = \sum_{p,q}\varphi_{p,q} \in \Omega^*(\mathcal{M})$, where the subscripts indicate the complex type of the respective terms. If φ is of mixed type (p,q) with $\varphi = \bigoplus_{h=0}^{q}\varphi_{p+h,q-h}$, where the subscripts indicate the complex type of the respective terms, then

$$C\varphi = i^{p-q}\bigoplus_{h=0}^{q}(-1)^h\varphi_{p+h,q-h}, \tag{1.17}$$

which shows that C preserves the mixed type and $C^{-1} = (-1)^{p-q}C$. We have

Proposition 1.1 ([Va77]). *On a compact foliated Kähler manifold the following relations hold:*

$$\Lambda'\mu - \mu\Lambda' + \Lambda''\lambda - \lambda\Lambda'' = -C^{-1}\lambda^*C, \tag{1.18}$$

$$\Lambda''\mu - \mu\Lambda'' = 0, \qquad \Lambda'\lambda - \lambda\Lambda' = -C^{-1}\mu^*C. \tag{1.19}$$

If we apply the above formulas to a homogeneous form of mixed type (p,q) and use (1.17) we obtain

$$\Lambda'\mu - \mu\Lambda' + \Lambda''\lambda - \lambda\Lambda'' = i\lambda^*, \qquad \Lambda'\lambda - \lambda\Lambda' = -i\mu^*. \tag{1.20}$$

Proposition 1.2. *If $(\mathcal{M},\mathcal{F},\omega)$ is a compact foliated Kähler manifold, then*

$$\mu^2 = \lambda^2 = \mu\lambda + \lambda\mu = 0, \qquad \mu^{*2} = \lambda^{*2} = \mu^*\lambda^* + \lambda^*\mu^* = 0, \tag{1.21}$$

$$\mu^*\lambda + \lambda\mu^* = 0, \mu\lambda^* + \lambda^*\mu = 0. \tag{1.22}$$

Proof. Relations (1.21) follow since $d^2 = \delta^2 = 0$, and (1.22) follows by applying (1.20) and the first equation in (1.19). ∎

2. Bott–Chern cohomology of forms of mixed type. In this section we define the Bott–Chern cohomology groups of differential forms of mixed type on a compact foliated Kähler manifold $(\mathcal{M},\mathcal{F},\omega)$.

Let $\Lambda^{p,q}_{\mathrm{mix}}(\mathcal{M},\mathcal{F})$ be the set of all differential forms of mixed type (p,q) on the foliated Kähler manifold $(\mathcal{M},\mathcal{F},\omega)$. According to the previous section we can consider the following differential complexes:

The differential complex

$$\cdots \xrightarrow{\lambda} \Lambda^{p,q-1}_{\mathrm{mix}}(\mathcal{M},\mathcal{F}) \xrightarrow{\lambda} \Lambda^{p,q}_{\mathrm{mix}}(\mathcal{M},\mathcal{F}) \xrightarrow{\lambda} \cdots \tag{2.1}$$

is called the λ-*complex* of $(\mathcal{M},\mathcal{F},\omega)$; its cohomology groups, defined in [Va71], are explicitly given by

$$H^{p,q}_{\mathrm{mix},\lambda}(\mathcal{M},\mathcal{F}) = \frac{\ker\{\lambda : \Lambda^{p,q}_{\mathrm{mix}}(\mathcal{M},\mathcal{F}) \to \Lambda^{p,q+1}_{\mathrm{mix}}(\mathcal{M},\mathcal{F})\}}{\mathrm{im}\{\lambda : \Lambda^{p,q-1}_{\mathrm{mix}}(\mathcal{M},\mathcal{F}) \to \Lambda^{p,q}_{\mathrm{mix}}(\mathcal{M},\mathcal{F})\}}. \tag{2.2}$$

The differential complex

$$\Lambda^{p-1,q-1}_{\mathrm{mix}}(\mathcal{M},\mathcal{F}) \xrightarrow{\mu\lambda} \Lambda^{p,q}_{\mathrm{mix}}(\mathcal{M},\mathcal{F}) \xrightarrow{\mu\oplus\lambda} \Lambda^{p+1,q}_{\mathrm{mix}}(\mathcal{M},\mathcal{F}) \oplus \Lambda^{p,q+1}_{\mathrm{mix}}(\mathcal{M},\mathcal{F}) \tag{2.3}$$

is called the *Bott–Chern complex* of $(\mathcal{M},\mathcal{F},\omega)$; its cohomology groups

$$H^{p,q}_{\mathrm{mix,BC}}(\mathcal{M},\mathcal{F}) = \frac{\ker\{\mu : \Lambda^{p,q}_{\mathrm{mix}} \to \Lambda^{p+1,q}_{\mathrm{mix}}\} \cap \ker\{\lambda : \Lambda^{p,q}_{\mathrm{mix}} \to \Lambda^{p,q+1}_{\mathrm{mix}}\}}{\mathrm{im}\{\mu\lambda : \Lambda^{p-1,q-1}_{\mathrm{mix}} \to \Lambda^{p,q}_{\mathrm{mix}}\}} \tag{2.4}$$

are called the *Bott–Chern cohomology groups of bidegree (p,q) of differential forms of mixed type on $(\mathcal{M},\mathcal{F},\omega)$*.

It is easy to see that the above discussion gives rise to the canonical maps

$$H^{p,q}_{\mathrm{mix,BC}}(\mathcal{M},\mathcal{F}) \to H^{p+q}_{d}(\mathcal{M},\mathcal{F}), \quad H^{p,q}_{\mathrm{mix,BC}}(\mathcal{M},\mathcal{F}) \to H^{p,q}_{\mathrm{mix},\lambda}(\mathcal{M},\mathcal{F}). \tag{2.5}$$

Here $H^{p+q}_{d}(\mathcal{M},\mathcal{F})$ denotes the $(p+q)$th de Rham cohomology group of $(\mathcal{M},\mathcal{F},\omega)$.

In the following, as usual [A, Bi], we consider the dual of the Bott–Chern cohomology groups, given by

$$H^{p,q}_{\mathrm{mix,A}}(\mathcal{M},\mathcal{F}) = \frac{\ker\{\mu\lambda : \Lambda^{p,q}_{\mathrm{mix}} \to \Lambda^{p+1,q+1}_{\mathrm{mix}}\}}{\mathrm{im}\{\mu : \Lambda^{p-1,q}_{\mathrm{mix}} \to \Lambda^{p,q}_{\mathrm{mix}}\} + \mathrm{im}\{\lambda : \Lambda^{p,q-1}_{\mathrm{mix}} \to \Lambda^{p,q}_{\mathrm{mix}}\}},$$

called the *Aeppli cohomology groups of differential forms of mixed type on* $(\mathcal{M},\mathcal{F},\omega)$.

PROPOSITION 2.1. *The exterior product induces a bilinear map*

$$\wedge : H^{p,q}_{\mathrm{mix,BC}}(\mathcal{M},\mathcal{F}) \times H^{r,s}_{\mathrm{mix,A}}(\mathcal{M},\mathcal{F}) \to H^{p+r,q+s}_{\mathrm{mix,A}}(\mathcal{M},\mathcal{F}). \tag{2.6}$$

Proof. Let $\varphi,\psi \in \Lambda^{p,q}_{\mathrm{mix}}(\mathcal{M},\mathcal{F})$. If φ is d-closed and ψ is $\mu\lambda$-closed then $\varphi\wedge\psi$ is $\mu\lambda$-closed. Also, if φ is d-closed and ψ is d-exact then $\varphi\wedge\psi$ is d-exact and if φ is $\mu\lambda$-exact and ψ is $\mu\lambda$-closed then $\varphi\wedge\psi$ is d-exact.

For the last assertion, we have

$$\varphi\wedge\psi = \mu\lambda\theta\wedge\psi = \tfrac{1}{2}d[(\lambda-\mu)\theta\wedge\psi + (-1)^{p+q}\theta\wedge(\mu-\lambda)\psi]. \ \blacksquare$$

In particular, we have

$$H^{p,q}_{\mathrm{mix,BC}}(\mathcal{M},\mathcal{F}) \times H^{n-p,n+2m-q}_{\mathrm{mix,A}}(\mathcal{M},\mathcal{F}) \to H^{n,n+2m}_{\mathrm{mix,A}}(\mathcal{M},\mathcal{F}) = H^{2n+2m}(\mathcal{M},\mathcal{F}).$$

3. Bott–Chern Laplacian and Hodge decompositions of forms of mixed type. In this section we define the Bott–Chern Laplacian for differential forms of mixed type on a compact foliated Kähler manifold and we obtain a Hodge type decomposition theorem with respect to this Laplacian. The notions are introduced using the same technique as in [S, T-Y].

It is easy to see that in the kählerian case the Laplace operator $\Delta = d\delta + \delta d$ admits the decomposition $\Delta = \Delta_\mu + \Delta_\lambda$, where

$$\Delta_\mu = \mu\mu^* + \mu^*\mu, \qquad \Delta_\lambda = \lambda\lambda^* + \lambda^*\lambda. \tag{3.1}$$

Proposition 3.1. *Let $(\mathcal{M},\mathcal{F},\omega)$ be a compact foliated Kähler manifold. Then $\Delta_\mu = \Delta_\lambda$.*

Proof. Using (1.20), we have

$$\begin{aligned}\Delta_\mu &= \mu i(\Lambda'\lambda - \lambda\Lambda') + i(\Lambda'\lambda - \lambda\Lambda')\mu\\ &= i\mu\Lambda'\lambda - i\mu\lambda\Lambda' + i\Lambda'\lambda\mu - i\lambda\Lambda'\mu\end{aligned}$$

and

$$\begin{aligned}\Delta_\lambda &= -\lambda i(\Lambda'\mu - \mu\Lambda' + \Lambda''\lambda - \lambda\Lambda'') - i(\Lambda'\mu - \mu\Lambda' + \Lambda''\lambda - \lambda\Lambda'')\lambda\\ &= -i\lambda\Lambda'\mu + i\lambda\mu\Lambda' - i\lambda\Lambda''\lambda - i\Lambda'\mu\lambda + i\mu\Lambda'\lambda + i\lambda\Lambda''\lambda\\ &= i\mu\Lambda'\lambda - i\mu\lambda\Lambda' + i\Lambda'\lambda\mu - i\lambda\Lambda'\mu. \ \blacksquare\end{aligned}$$

According to [Va71], Δ_λ is an elliptic operator and the following Hodge decomposition holds:

$$\Lambda^{p,q}_{\mathrm{mix}}(\mathcal{M},\mathcal{F}) = \ker\Delta_\lambda \oplus \operatorname{im}\lambda \oplus \operatorname{im}\lambda^*. \tag{3.2}$$

We now define the *Bott–Chern Laplacian* for differential forms of mixed type (p,q) by

$$\Delta^{\mathrm{BC}}_{\mathrm{mix}} = \mu\lambda(\mu\lambda)^* + \mu^*\mu + \lambda^*\lambda. \tag{3.3}$$

This operator is self-adjoint, i.e. $\langle \Delta^{\mathrm{BC}}_{\mathrm{mix}}\varphi,\psi\rangle = \langle\varphi, \Delta^{\mathrm{BC}}_{\mathrm{mix}}\psi\rangle$. For a form $\varphi \in \Lambda^{p,q}_{\mathrm{mix}}(\mathcal{M},\mathcal{F})$ we have

$$\begin{aligned}\langle \Delta^{\mathrm{BC}}_{\mathrm{mix}}\varphi,\varphi\rangle &= \langle \mu\lambda(\mu\lambda)^*\varphi + \mu^*\mu\varphi + \lambda^*\lambda\varphi,\varphi\rangle\\ &= \langle(\mu\lambda)^*\varphi,(\mu\lambda)^*\varphi\rangle + \langle\mu\varphi,\mu\varphi\rangle + \langle\lambda\varphi,\lambda\varphi\rangle\\ &= \|(\mu\lambda)^*\varphi\|^2 + \|\mu\varphi\|^2 + \|\lambda\varphi\|^2,\end{aligned}$$

where $\|\varphi\|^2 = \langle\varphi,\varphi\rangle$. Thus, we obtain

Proposition 3.2. $\Delta^{\mathrm{BC}}_{\mathrm{mix}}\varphi = 0$ *if and only if* $(\mu\lambda)^*\varphi = \mu\varphi = \lambda\varphi = 0$.

We denote the space of $\Delta^{\mathrm{BC}}_{\mathrm{mix}}$-harmonic (p,q)-forms of mixed type by $\mathcal{H}^{p,q}_{\mathrm{mix,BC}}(\mathcal{M},\mathcal{F})$.

Following the ideas of [T-Y], we now show that $H^{*,*}_{\mathrm{mix,BC}}(\mathcal{M},\mathcal{F})$ is finite-dimensional by analyzing the space of its harmonic forms. Firstly, we introduce a related elliptic fourth-order differential operator:

$$\widetilde{\Delta}^{\mathrm{BC}}_{\mathrm{mix}} = \mu\lambda\lambda^*\mu^* + \lambda^*\mu^*\mu\lambda + \lambda^*\mu\mu^*\lambda + \mu^*\lambda\lambda^*\mu + \lambda^*\lambda + \mu^*\mu. \tag{3.4}$$

This operator has the same kernel as $\Delta^{\mathrm{BC}}_{\mathrm{mix}}$. Indeed,

$$0 = \langle\varphi, \widetilde{\Delta}^{\mathrm{BC}}_{\mathrm{mix}}\varphi\rangle = \|\mu\varphi\|^2 + \|\lambda\varphi\|^2 + \|(\mu\lambda)^*\varphi\|^2 + \|\mu\lambda\varphi\|^2 + \|\mu^*\lambda\varphi\|^2 + \|\lambda^*\mu\varphi\|^2$$

and the three additional terms clearly do not give any additional conditions and are automatically zero if one requires that $\mu\varphi = \lambda\varphi = 0$. Essentially, the presence of the second-order differential terms ensures that the spaces $\ker\Delta^{\mathrm{BC}}_{\mathrm{mix}}$ and $\ker\widetilde{\Delta}^{\mathrm{BC}}_{\mathrm{mix}}$ coincide. We notice that such a fourth-order differential operator is considered in [E-K-G] for basic forms on transversally kählerian foliations. We have an analogous result for the operator $\widetilde{\Delta}^{\mathrm{BC}}_{\mathrm{mix}}$:

Proposition 3.3. *If $(\mathcal{M},\mathcal{F},\omega)$ is a compact foliated Kähler manifold, then*

$$\widetilde{\Delta}^{\mathrm{BC}}_{\mathrm{mix}} = \Delta_\lambda\Delta_\lambda + \mu^*\mu + \lambda^*\lambda.$$

Proof. Using (1.21) and (1.22), by direct calculation one gets

$$\begin{aligned}
\widetilde{\Delta}^{\mathrm{BC}}_{\mathrm{mix}} &= \mu\lambda\lambda^*\mu^* + \lambda^*\mu^*\mu\lambda + \lambda^*\mu\mu^*\lambda + \mu^*\lambda\lambda^*\mu + \lambda^*\lambda + \mu^*\mu\\
&= \mu\mu^*\lambda\lambda^* + \mu^*\mu\lambda^*\lambda + \mu^*\mu\lambda\lambda^* + \mu\mu^*\lambda^*\lambda + \lambda^*\lambda + \mu^*\mu\\
&= (\mu\mu^* + \mu^*\mu)(\lambda\lambda^* + \lambda^*\lambda) + \lambda^*\lambda + \mu^*\mu\\
&= \Delta_\mu\Delta_\lambda + \lambda^*\lambda + \mu^*\mu = \Delta_\lambda\Delta_\lambda + \lambda^*\lambda + \mu^*\mu. \ \blacksquare
\end{aligned}$$

Theorem 3.1. *Let $(\mathcal{M},\mathcal{F},\omega)$ be a compact foliated Kähler manifold. Then*

(i) $\dim\mathcal{H}^{p,q}_{\mathrm{mix,BC}}(\mathcal{M},\mathcal{F}) < \infty$;

(ii) *There is an orthogonal decomposition*

$$\Lambda^{p,q}_{\mathrm{mix}}(\mathcal{M},\mathcal{F}) = \mathcal{H}^{p,q}_{\mathrm{mix,BC}}(\mathcal{M},\mathcal{F}) \oplus \mathrm{im}(\mu\lambda) \oplus (\mathrm{im}\,\mu^* + \mathrm{im}\,\lambda^*). \tag{3.5}$$

(iii) *There are canonical isomorphisms*

$$\mathcal{H}^{p,q}_{\mathrm{mix,BC}}(\mathcal{M},\mathcal{F}) \cong H^{p,q}_{\mathrm{mix,BC}}(\mathcal{M},\mathcal{F}) \cong H^{p,q}_{\mathrm{mix},\lambda}(\mathcal{M},\mathcal{F}).$$

Proof. (i) Because only the highest order differential contributes to the principal symbol of a Laplace operator, by the calculations of $\widetilde{\Delta}^{\mathrm{BC}}_{\mathrm{mix}}$ from Proposition 3.3, it follows that the principal symbol of $\widetilde{\Delta}^{\mathrm{BC}}_{\mathrm{mix}}$ is equal to that of the square of the operator Δ_λ, so it is positive. $\widetilde{\Delta}^{\mathrm{BC}}_{\mathrm{mix}}$ is thus elliptic and hence its kernel, $\mathcal{H}^{p,q}_{\mathrm{mix,BC}}(\mathcal{M},\mathcal{F})$, is finite-dimensional.

Assertion (ii) then follows directly by applying elliptic theory. For (iii), using the decomposition of (ii), we have

$$\ker(\mu+\lambda) = \mathcal{H}^{p,q}_{\mathrm{mix,BC}}(\mathcal{M},\mathcal{F}) \oplus \mathrm{im}(\mu\lambda). \tag{3.6}$$

Indeed, for a form $\varphi \in \Lambda^{p,q}_{\mathrm{mix}}(\mathcal{M},\mathcal{F})$ given by $\varphi = \psi + \mu\lambda\theta + \mu^*\theta_1 + \lambda^*\theta_2$, where $\psi \in \mathcal{H}^{p,q}_{\mathrm{mix,BC}}(\mathcal{M},\mathcal{F})$, we have $\mu\varphi = \lambda\varphi = 0$ if and only if

$$\begin{aligned} 0 &= \langle \theta_1, \mu(\mu^*\theta_1 + \lambda^*\theta_2)\rangle + \langle \theta_2, \lambda(\mu^*\theta_1 + \lambda^*\theta_2)\rangle \\ &= \langle \mu^*\theta_1 + \lambda^*\theta_2, \mu^*\theta_1 + \lambda^*\theta_2\rangle = \|\mu^*\theta_1 + \lambda^*\theta_2\|^2, \end{aligned}$$

which gives $\mu^*\theta_1 + \lambda^*\theta_2 = 0$, i.e. the desired decomposition (3.6). Thus every cohomology class of $H^{*,*}_{\mathrm{mix,BC}}(\mathcal{M},\mathcal{F})$ contains a unique harmonic representative and $\mathcal{H}^{p,q}_{\mathrm{mix,BC}}(\mathcal{M},\mathcal{F}) \cong H^{p,q}_{\mathrm{mix,BC}}(\mathcal{M},\mathcal{F})$, proving the first isomorphism of (iii). Furthermore, we have

$$\ker \widetilde{\Delta}^{\mathrm{BC}}_{\mathrm{mix}} = \ker \Delta_\lambda. \tag{3.7}$$

Indeed, if $\varphi \in \ker \widetilde{\Delta}^{\mathrm{BC}}_{\mathrm{mix}}$ then by the calculations of $\widetilde{\Delta}^{\mathrm{BC}}_{\mathrm{mix}}$ from Proposition 3.3,

$$0 = \langle \varphi, \widetilde{\Delta}^{\mathrm{BC}}_{\mathrm{mix}}\varphi\rangle = \|\Delta_\lambda\varphi\|^2 + \|\mu\varphi\|^2 + \|\lambda\varphi\|^2,$$

which says that $\varphi \in \ker \Delta_\lambda$. Conversely, if $\varphi \in \ker \Delta_\lambda$, then by Proposition 3.1, $\varphi \in \ker \Delta_\mu$, hence $\lambda\varphi = \mu\varphi = 0$ and so $\varphi \in \ker \widetilde{\Delta}^{\mathrm{BC}}_{\mathrm{mix}}$. Finally, similarly to the above we have $\mathcal{H}^{p,q}_{\mathrm{mix},\lambda}(\mathcal{M},\mathcal{F}) \cong H^{p,q}_{\mathrm{mix},\lambda}(\mathcal{M},\mathcal{F})$, and now the second isomorphism of (iii) follows from (3.7). ∎

Corollary 3.1. *If $(\mathcal{M},\mathcal{F},\omega)$ is a compact foliated Kähler manifold, then $H^{p,q}_{\mathrm{mix,BC}}(\mathcal{M},\mathcal{F})$ is finite-dimensional.*

Now, let us define the *Aeppli Laplacian* for differential forms of mixed type (p,q) by

$$\Delta^{\mathrm{A}}_{\mathrm{mix}} = \mu\mu^* + \lambda\lambda^* + (\mu\lambda)^*\mu\lambda. \tag{3.8}$$

It is not elliptic, but the related operator

$$\widetilde{\Delta}^{\mathrm{A}}_{\mathrm{mix}} = \mu\mu^* + \lambda\lambda^* + \lambda^*\mu^*\mu\lambda + \mu\lambda\lambda^*\mu^* + \mu\lambda^*\lambda\mu^* + \lambda\mu^*\mu\lambda^* \tag{3.9}$$

is elliptic when $(\mathcal{M},\mathcal{F},\omega)$ is a compact foliated Kähler manifold.

Now, if we denote $\mathcal{H}^{p,q}_{\mathrm{mix,A}}(\mathcal{M},\mathcal{F}) = \ker \widetilde{\Delta}^{\mathrm{A}}_{\mathrm{mix}}$, then by applying ellliptic theory arguments, similar to those in Theorem 3.1, we obtain

Theorem 3.2. *Let $(\mathcal{M},\mathcal{F},\omega)$ be a compact foliated Kähler manifold. Then*

(i) $\dim \mathcal{H}^{p,q}_{\mathrm{mix,A}}(\mathcal{M},\mathcal{F}) < \infty$.

(ii) *There is an orthogonal decomposition*

$$\Lambda^{p,q}_{\mathrm{mix}}(\mathcal{M},\mathcal{F}) = \mathcal{H}^{p,q}_{\mathrm{mix,A}}(\mathcal{M},\mathcal{F}) \oplus (\mathrm{im}\,\mu + \mathrm{im}\,\lambda) \oplus \mathrm{im}(\lambda^*\mu^*). \tag{3.10}$$

(iii) *There is a canonical isomorphism*

$$\mathcal{H}^{p,q}_{\mathrm{mix,A}}(\mathcal{M},\mathcal{F}) \cong H^{p,q}_{\mathrm{mix,A}}(\mathcal{M},\mathcal{F}).$$

COROLLARY 3.2. *If* $(\mathcal{M},\mathcal{F},\omega)$ *is a compact foliated Kähler manifold, then* $H^{p,q}_{\mathrm{mix,A}}(\mathcal{M},\mathcal{F})$ *is finite-dimensional.*

Finally, if $\varphi \in \mathcal{H}^{p,q}_{\mathrm{mix,BC}}(\mathcal{M},\mathcal{F})$ then $\mu\varphi = \lambda\varphi = \mu^*\lambda^*\varphi = 0$ if and only if $\lambda^*(\overline{\star}\varphi) = \mu^*(\overline{\star}\varphi) = \mu\lambda(\overline{\star}\varphi) = 0$, which holds if and only if $\overline{\star}\varphi \in \mathcal{H}^{n-p,n+2m-q}_{\mathrm{mix,A}}(\mathcal{M},\mathcal{F})$, where we have used the relations $\lambda^* = -\overline{\star}\lambda\overline{\star}$ and $\mu^* = -\overline{\star}\mu\overline{\star}$. Thus, $\overline{\star}$ gives an isomorphism

$$H^{p,q}_{\mathrm{mix,BC}}(\mathcal{M},\mathcal{F}) \cong H^{n-p,n+2m-q}_{\mathrm{mix,A}}(\mathcal{M},\mathcal{F}).$$

4. The case of the projectivized tangent bundle of a complex Finsler manifold. In this section it is shown that similar Hodge decompositions of Bott–Chern type are valid for differential forms of mixed type on the total space of the projectivized tangent bundle of a complex Finsler manifold.

Let M be an n-dimensional complex manifold and (z^k), $k = 1,\dots,n$, the complex coordinates in a local chart U. The complexification of the real tangent bundle $T_{\mathbb{R}}M$, denoted by $T_{\mathbb{C}}M$, splits into the direct sum $T^{1,0}M \oplus T^{0,1}M$ of the holomorphic tangent bundle $T^{1,0}M$ and antiholomorphic tangent bundle $T^{0,1}M$. The total space of the holomorphic tangent bundle $\pi : T^{1,0}M \to M$ is in turn a $2n$-dimensional complex manifold with (z^k,η^k), $k = 1,\dots,n$, the induced complex coordinates in the local chart $\pi^{-1}(U)$, where $\eta = \eta^k\partial/\partial z^k \in T^{1,0}_zM$.

A *complex Finsler space* is a pair (M,F), where $F : T^{1,0}M \to \mathbb{R}_+ \cup \{0\}$ is a continuous function satisfying the following conditions:

(i) $L := F^2$ is smooth on $\widetilde{M} := T^{1,0}M - \{\text{zero section}\}$;
(ii) $F(z,\eta) \geq 0$, and equality holds if and only if $\eta = 0$;
(iii) $F(z,\lambda\eta) = |\lambda|F(z,\eta)$ for any $\lambda \in \mathbb{C}$ (homogeneity);
(iv) the hermitian matrix $(g_{i\bar{j}})$ is positive definite, where $g_{i\bar{j}} = \partial^2L/\partial\eta^i\partial\overline{\eta}^j$ is the fundamental metric tensor, or equivalently, the indicatrix $I_z = \{\eta \in T^{1,0}_zM \mid g_{i\bar{j}}(z,\eta)\eta^i\overline{\eta}^j = 1\}$ is strongly pseudoconvex for any $z \in M$.

Consequently, from (iii) we have

$$\frac{\partial L}{\partial\eta^k}\eta^k = \frac{\partial L}{\partial\overline{\eta}^k}\overline{\eta}^k = L, \qquad \frac{\partial g_{i\bar{j}}}{\partial\eta^k}\eta^k = \frac{\partial g_{i\bar{j}}}{\partial\overline{\eta}^k}\overline{\eta}^k = 0, \qquad g_{i\bar{j}}\eta^i\overline{\eta}^j = L. \tag{4.1}$$

Roughly speaking, the geometry of a complex Finsler space involves the geometric objects of the complex manifold $T^{1,0}M$ endowed with a hermitian metric structure defined by $g_{i\bar{j}}$.

Let $V^{1,0}(\widetilde{M}) \subset T^{1,0}(\widetilde{M})$ be the holomorphic vertical bundle, locally spanned by $\{\partial/\partial\eta^k\}$, and $V^{0,1}(\widetilde{M})$ be its conjugate, locally spanned by $\{\partial/\partial\overline{\eta}^k\}$. A *complex nonlinear connection*, briefly c.n.c., on $\widetilde{M}$ is defined by a complementary complex subbundle to $V^{1,0}(\widetilde{M})$ in $T^{1,0}(\widetilde{M})$, so that $T^{1,0}(\widetilde{M}) = H^{1,0}(\widetilde{M}) \oplus V^{1,0}(\widetilde{M})$. The horizontal subbundle $H^{1,0}(\widetilde{M})$ is locally spanned by

$$\left\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\right\},$$

where $N_k^j(z,\eta)$ are the coefficients of the c.n.c., which obey a specific transformation rule under changes of local charts such that

$$\frac{\delta}{\delta z^k} = \frac{\partial z'^j}{\partial z^k}\frac{\delta}{\delta z'^j}.$$

Obviously, we also have

$$\frac{\partial}{\partial \eta^k} = \frac{\partial z'^j}{\partial z^k}\frac{\partial}{\partial \eta'^j}.$$

The pair $\{\delta/\delta z^k, \partial/\partial\eta^k\}$, $k = 1,\dots,n$, will be called the *adapted frame* of the c.n.c. By conjugation an adapted frame $\{\delta/\delta\overline{z}^k, \partial/\partial\overline{\eta}^k\}$ is obtained on $T^{0,1}(\widetilde{M})$. The dual adapted bases are given by $\{dz^k\}$, $\{\delta\eta^k = d\eta^k + N_j^k dz^j\}$, $\{d\overline{z}^k\}$ and $\{\delta\overline{\eta}^k = d\overline{\eta}^k + N_{\overline{j}}^{\overline{k}} d\overline{z}^j\}$, respectively.

According to [A-P, Ai, Mu], a c.n.c. on (M, F) depending only on the complex Finsler metric F is the Chern–Finsler c.n.c. locally given by

$$\overset{\mathrm{CF}}{N}{}_k^j = g^{\overline{m}j}\partial_k\dot{\partial}_{\overline{m}}(L), \tag{4.2}$$

where $(g^{\overline{m}j})$ is the inverse of $(g_{j\overline{m}})$, and it has an important property:

$$\overset{\mathrm{CF}}{R}{}_{kj}^i = \delta_k \overset{\mathrm{CF}}{N}{}_j^i - \delta_j \overset{\mathrm{CF}}{N}{}_k^i = 0. \tag{4.3}$$

We will consider the adapted frames and coframes with respect to the Chern–Finsler c.n.c. Similarly to (1.3), with respect to the adapted coframes $\{dz^k, d\overline{z}^k, \delta\eta^k, \overline{\delta\eta^k}\}$ of $T^*_{\mathbb{C}}\widetilde{M}$, a (p_1,p_2,q_1,q_2)-form φ on $\widetilde{M}$ is locally given by

$$\varphi = \frac{1}{p_1!p_2!q_1!q_2!}\varphi_{I_{p_1}\overline{J_{p_2}}K_{q_1}\overline{H_{q_2}}}(z,\eta)dz^{I_{p_1}} \wedge d\overline{z}^{J_{p_2}} \wedge \delta\eta^{K_{q_1}} \wedge \delta\overline{\eta}^{H_{q_2}}. \tag{4.4}$$

Such (p_1,p_2,q_1,q_2)-forms on complex Finsler manifolds are defined in [P-M]. See also [Zh09a, Zh09b].

As in the preliminary subsection these forms may be viewed as being of mixed type $(p_1, p_2+q_1+q_2)$ (see also [I]), and we denote by $\Lambda^{p,q}_{\mathrm{mix}}(\widetilde{M})$ the set of all differential forms of mixed type (p,q) on $\widetilde{M}$.

Now we suppose that the manifold M is compact. Note that there is a natural $\mathbb{C}^* = \mathbb{C} - \{0\}$ action on $\widetilde{M}$, and the associated projectivized bundle is defined by $P(\widetilde{M}) = \widetilde{M}/\mathbb{C}^*$ with the projection $p_M : P(\widetilde{M}) \to M$. Each fiber $P_z(T^{1,0}M) = P(T^{1,0}_z M)$ is isomorphic to the $(n-1)$-dimensional complex projective space $P^{n-1}(\mathbb{C})$. The pull-back bundle $p_M^{-1}(T^{1,0}M)$ is a holomorphic vector bundle of rank n over $P(\widetilde{M})$. Thus, the local complex coordinates (z, η) on $T^{1,0}M$ may also be considered as a local complex coordinate system for $P(\widetilde{M})$ as long as $\eta^1, \ldots, \eta^n$ is treated as a homogeneous coordinate system for fibers. All the geometric objects on $T^{1,0}M$ which are invariant under replacing η by $\lambda\eta$, $\lambda \in \mathbb{C}^*$, descend to $P(\widetilde{M})$. The reason for working with $P(\widetilde{M})$ rather than $\widetilde{M}$ is that $P(\widetilde{M})$ is a compact complex manifold with $\dim_{\mathbb{C}} P(\widetilde{M}) = 2n - 1$ whenever M is a compact complex manifold of dimension n (see [Ko75, Ko96]). Also, due to [B-K], the natural hermitian metric on $\widetilde{M}$ given by the Sasaki type lift of the fundamental metric tensor $g_{i\bar{j}}$,

$$G_{\widetilde{M}} = g_{i\bar{j}} dz^i \otimes d\bar{z}^j + g_{i\bar{j}} \delta\eta^i \otimes \delta\overline{\eta}^j,$$

descends to a nondegenerate metric

$$G_{P(\widetilde{M})} = g_{i\bar{j}} dz^i \otimes d\bar{z}^j + (\ln L)_{i\bar{j}} \delta\eta^i \otimes \delta\overline{\eta}^j$$

on the total space $P(\widetilde{M})$, where $(\ln L)_{i\bar{j}} = \partial^2 \ln L / \partial\eta^i \partial\overline{\eta}^j$. See also [Ai, C-W]. The corresponding fundamental form is $\omega_{P(\widetilde{M})} = \omega^h_{P(\widetilde{M})} + \omega^v_{P(\widetilde{M})}$, where

$$\omega^h_{P(\widetilde{M})} = \sqrt{-1}\, g_{i\bar{j}} dz^i \wedge d\bar{z}^j, \qquad \omega^v_{P(\widetilde{M})} = \sqrt{-1}\, (\ln L)_{i\bar{j}} \delta\eta^i \wedge \delta\overline{\eta}^j. \tag{4.5}$$

Let $\Lambda^{p,q}_{\mathrm{mix}}(P(\widetilde{M}))$ be the set of all differential forms of mixed type (p, q) locally given by (4.4) which are invariant under replacing η by $\lambda\eta$, $\lambda \in \mathbb{C}^*$.

Now by (4.3), the exterior derivative d on such forms admits a decomposition $d = \lambda + \mu$ into parts of the respective mixed types $(1, 0)$ and $(0, 1)$. Thus, by the same technique as in the previous sections, we can define the Bott–Chern and Aeppli Laplacian on $\Lambda^{p,q}_{\mathrm{mix}}(P(\widetilde{M}))$, and we can prove that similar Hodge decompositions of Bott–Chern type are valid for the differential forms of mixed type (p, q) on the projectivized tangent bundle of a complex Finsler manifold.

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