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THE WEAK PHILLIPS PROPERTY

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Abstract. Let X be a Banach space. If the natural projection $p : X^{***} \to X^*$ is sequentially weak*-weak continuous then the space X is said to have the weak Phillips property. We present several characterizations of the spaces having this property and study its relationships to other Banach space properties, especially the Grothendieck property.

Introduction. Let X be a Banach space. Corresponding to the decomposition $X^{***} = X^* \oplus X^{\perp}$, we have a natural projection $p : X^{***} \to X^*$ that sends each $\mu \in X^{***}$ to its restriction to X, where X is regarded as a subspace of X^{**}. The classical Phillips Lemma [D1, p. 83] says that, for $X = c_0$, the mapping p is sequentially weak*-norm continuous. Motivated by this fact, in [F-Ü], the authors have introduced the so-called Phillips and weak Phillips properties: A Banach space X is said to have the (*weak*) *Phillips property* if the projection $p : X^{***} \to X^*$ is sequentially weak*-(weak) norm continuous.

These two properties and their hereditary versions have been studied, to a certain extent, in the above mentioned paper. Here we present further results on weak Phillips property. It turns out that this property is very closely related to the Grothendieck property (i.e. in X^* weak*-null sequences are weakly null). For instance, X has the weak Phillips property iff, for every separable Banach space Y and every bounded linear operator $T : X^{**} \to Y$, the restriction $T|_X : X \to Y$ of T to X is weakly compact. We recall that X has the *Grothendieck property* iff every bounded operator from X into a separable Banach space is weakly compact. Throughout the paper these kinds of connections will be emphasized.

The class of Banach spaces having the weak Phillips property is much richer than the class of Banach spaces having the Grothendieck property. Every C^*-algebra, more generally every Banach space having the property

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(V) of Pełczyński ([Pe] and [Pf]), and hence a good many of the uniform algebras ([B1], [De] and [S]), have the weak Phillips property. The Banach spaces X such that X^{**} has the Grothendieck property, and the Banach spaces X which are preduals of an L-space also have the weak Phillips property. In particular, every predual of ℓ^1, and more generally, of a von Neumann algebra, has the weak Phillips property.

Below, X and Y are two Banach spaces, $T : X^{**} \to Y$ a (bounded linear) operator and $\widetilde{T} = T|_X : X \to Y$ its restriction to X. The main results of the paper can be summarized as follows:

(a) X has the weak Phillips property iff X^* is weakly sequentially complete and there exists no operator $T : X^{**} \to c_0$ such that $T(X) = c_0$.

(b) If X has the weak Phillips property and we have an operator $T : X^{**} \to Y$ such that $T(X) = Y$ then Y has the Grothendieck property. In particular, X is not complemented in X^{**}, unless it has the Grothendieck property.

(c) If X has the weak Phillips property, Y is complemented in its second dual and the space (Y_1^*, weak^*) is sequentially compact, then every operator $u : X \to Y$ is weakly compact.

(d) If X has the weak Phillips property and the operator $T : X^{**} \to Y$ is such that $\overline{T(X^{**})} = \overline{T(X)}$ then either $\widetilde{T}$ is weakly compact or it fixes a copy of ℓ^1. Moreover, $\widetilde{T}$ is weakly compact iff the space $\overline{T(X)}$ is weakly compactly generated. In the case where Y is complemented in its second dual (this is the case if Y is a dual space, for instance), the same conclusion remains valid for any operator $u : X \to Y$ whose range is dense in Y.

(e) If (i) X has the weak Phillips property and the Dunford–Pettis property (DPP for short), and (ii) Y has the Gelfand–Phillips property and is complemented in its second dual, then every operator $u : X \to Y$ is completely continuous. So, in the case where X also has the reciprocal DPP (e.g. $X = C(K)$), u is also weakly compact. This result generalizes a classical result of Grothendieck stating that a closed subspace of $L^\infty[0,1]$ which is also closed in $L^p[0,1]$ for some p ($1 \le p < \infty$) is finite-dimensional [D-U, p. 178].

(f) If X has the weak Phillips property and we have an operator $T : X^{**} \to C(K)$ such that, for each Borel subset B of K, there exists a bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X whose image under T converges pointwise on K to the characteristic function χ_B of B, then $C(K)$ has the Grothendieck property. This result remains valid if we replace $C(K)$ by a Banach space having the Dieudonné property and χ_B by Baire-1 functionals.

The paper also contains several corollaries and applications of these results. The main ingredients of the proofs are the below-mentioned geometric properties of Banach spaces.

1. Notation and preliminaries. Our notation and terminology are quite standard. In general, X and Y will denote two Banach spaces over the field of complex numbers. By X^* we denote the dual space of X, and by X_1 its closed unit ball. We identify X with its canonical image in X^{**}. By an *operator* we mean a bounded linear operator. For $x \in X$ and $f \in X^*$, we denote by $\langle x, f\rangle$, or $\langle f, x\rangle$, the evaluation of f on x. The letter p will always denote the natural projection from $X^{***} = X^* \oplus X^\perp$ onto X^*.

We now recall the definitions of some of the properties used in the paper. The reader can find more information about these notions in the indicated references. We shall give more precise references and information at the places where these notions are used.

(a) *The weak Phillips property.* The space X is said to have the weak Phillips property if the projection p sends weak*-convergent sequences in X^{***} to weakly convergent sequences in X^* [F-Ü].

(b) *The Grothendieck property.* The space X is said to have the Grothendieck property if in X^* weak*-convergent sequences are weakly convergent [D-U, p. 179].

(c) *The Dunford–Pettis property.* The Banach space X is said to have the DPP if every weakly compact operator from X into any other Banach space is completely continuous, i.e. it sends weakly Cauchy sequences into norm convergent sequences ([G] and [Di.2]).

(d) *The reciprocal Dunford–Pettis property.* The Banach space X is said to have the reciprocal DPP if every completely continuous operator from X into any other Banach space is weakly compact [G].

(e) *The property* (V) *of Pełczyński.* The Banach space X is said to have the property (V) if every unconditionally converging operator from X into any other Banach space is weakly compact [Pe].

(f) *The Dieudonné property.* Let $B_1(X)$ be the subspace of X^{**} consisting of Baire-1 functionals (i.e. each $m \in B_1(X)$ is the weak*-limit of a sequence in X). The Banach space X is said to have the Dieudonné property if every operator T from X into a Banach space Y such that $T^{**}(B_1(X)) \subseteq Y$ is weakly compact [G].

(g) *The Gelfand–Phillips property.* The Banach space X is said to have the Gelfand–Phillips property if every limited subset of X is relatively compact. We recall that a subset A of X is limited if weak*-null sequences in X^* converge uniformly on A to zero [B-D].

For any undefined notation and terminology, we refer the reader to the books [L-T, Vols. I and II] or [D1].

2. The weak Phillips property. In this section we present several characterizations of the weak Phillips property and a number of results related to this property. As mentioned in the introduction, this property is closely related to the Grothendieck property, hence to property (V). Throughout the paper we shall try to exhibit the connections among these properties.

Throughout the paper, X and Y will be two general Banach spaces. For any operator $T : X^{**} \to Y$, we denote by $\widetilde{T}$ the restriction of T to X. Observe that $\widetilde{T}^* = p \circ T^*$.

We make repeated use of the following theorem due to Josefson–Nissenzweig [D1, p. 219]. The statement we adopt here is taken from [D-S].

Theorem 2.1. *Let $T : X \to Y$ be an operator such that T^* fixes a copy of ℓ^1 and such that whenever, for a sequence (y_n^*) in the unit ball of Y^*, the sequence $(T^*(y_n^*))$ is equivalent to the unit basis of ℓ^1, the sequence $(T^*(y_n^*))$ does not converge weak* to zero. Then T fixes a copy of ℓ^1.* ■

We shall need the following two lemmas.

Lemma 2.2. *Suppose that X^* is weakly sequentially complete. Then either, for every operator $T : X^{**} \to c_0$, $\widetilde{T}$ is weakly compact, or there exists an operator $S : X^{**} \to c_0$ such that $S(X^{**}) = S(X) = c_0$.*

Proof. It is clear that a weakly compact operator from X into c_0 cannot be onto so that the above two cases are exclusive. Suppose $\widetilde{T}$ is not weakly compact. Then, c_0 being separable and X^* weakly sequentially complete, there exists a weak*-null sequence (y_n^*) in the unit ball of $\ell^1 = c_0^*$ such that the sequence $\widetilde{T}^*((y_n^*))$ has no weakly convergent subsequence. By Rosenthal's ℓ^1-theorem, we conclude that $\widetilde{T}^*((y_n^*))$ has a subsequence, denoted again by $\widetilde{T}^*((y_n^*))$, which is equivalent to the unit vector basis of ℓ^1. From this we deduce that the sequence (y_n^*) is also equivalent to the unit vector basis of ℓ^1 and that $\widetilde{T}^*$ is an isomorphism from the closed linear span of the sequence (y_n^*) onto that of $\widetilde{T}^*((y_n^*))$. So $\widetilde{T}^*$ fixes a copy of ℓ^1.

Now, since c_0 does not contain a copy of ℓ^1, by the above theorem, there exists a sequence (z_n) in the unit ball of ℓ^1 such that the sequence $\widetilde{T}^*((z_n))$ is equivalent to the unit vector basis of ℓ^1 and converges weak* to zero. Passing to a subsequence and translating it, we can assume that $z_n \to 0$ in the weak* topology of ℓ^1. Then the sequence $(T^*(z_n))$ converges to zero in the weak* topology of X^{***} and is equivalent to the unit vector basis of ℓ^1 since otherwise $(T^*(z_n))$, and hence $\widetilde{T}^*((z_n))$, would have a weakly Cauchy subsequence. Let $\mu_n = T^*(z_n)$. Then both sequences (μ_n) and $(p(\mu_n))$ are weak*-null and equivalent to the unit vector basis of ℓ^1. Hence the operator $S : X^{**} \to c_0$ defined by $S(m) = (\langle \mu_n, m\rangle)$ is such that $S(X^{**}) = S(X) = c_0$. ■

Lemma 2.3. *Suppose that Y does not contain an isomorphic copy of ℓ^∞. Then any operator $u : X^* \to Y$ is unconditionally converging.*

Proof. For a contradiction, suppose that X has a subspace M isomorphic to c_0 on which u acts as an isomorphism. Let $i : M \to X^*$ be the natural injection. Then, since M^{**} is isomorphic to ℓ^∞ and Y does not contain a copy of ℓ^∞, by a result of Rosenthal [R], the operator $u \circ p \circ i^{**} : M^{**} \to X^{***} \to X^* \to Y$ is weakly compact. Hence its restriction to M, which is just $u|_M$, is also weakly compact, which is not possible since this mapping is an isomorphism between two copies of c_0. ∎

From the paper [F-Ü] we recall the following two results: (a) The space X has the weak Phillips property iff, for every operator $T : X^{**} \to c_0$, $\widetilde{T}$ is weakly compact. (b) If X has the weak Phillips property then X^* is weakly sequentially complete.

We now present several characterizations of the weak Phillips property.

THEOREM 2.4. *For any Banach space X, the following assertions are equivalent*:

(a) *X has the weak Phillips property.*

(b) *X^* is weakly sequentially complete and there exists no operator $T : X^{**} \to c_0$ such that $T(X) = c_0$.*

(c) *For any nonreflexive Banach space Y not containing a copy of ℓ^1 there exists no operator $T : X^{**} \to Y$ such that $T(X) = Y$.*

(d) *For any separable Banach space Y and every operator $T : X^{**} \to Y$, $\widetilde{T}$ is weakly compact.*

(e) *For any Banach space Y and every operator $T : X^{**} \to Y$, $\widetilde{T}^*$ is sequentially weak*-weak continuous.*

Proof. By Lemma 2.2 and the characterization of the weak Phillips property recalled above, the equivalence (a)⇔(b), as well as the implication (c)⇒(b), are clear. To prove (b)⇒(c), suppose that (b) holds. Let Y be a nonreflexive Banach space containing no copy of ℓ^1 and $T : X^{**} \to Y$ be an operator such that $T(X) = Y$. Then, since Y is nonreflexive and X^* is weakly sequentially complete, by Rosenthal's ℓ^1-theorem, there exists a sequence (y_n^*) in Y_1^* such that the sequence $\widetilde{T}^*(y_n^*)$ is equivalent to the unit vector basis of ℓ^1. Then (y_n^*) is also equivalent to the unit vector basis of ℓ^1 so that $\widetilde{T}^*$ fixes a copy of ℓ^1. Since Y does not contain a copy of ℓ^1, by Theorem 2.1 above, there exists a sequence (y_n^*) in Y_1^* such that the sequence $\widetilde{T}^*(y_n^*)$ is weak*-null and equivalent to the unit vector basis of ℓ^1. Hence, since $\overline{T(X)} = Y$, the sequence (y_n^*) is also weak*-null and equivalent to the unit vector basis of ℓ^1 so that the operator $\varphi : Y \to c_0$, defined by $\varphi(y) = (\langle y_n^*, y\rangle)$, is surjective. Then the operator $S = \varphi \circ T : X^{**} \to c_0$ is such that $S(X) = c_0$. As this contradicts (b), we conclude that (b)⇒(c) holds.

When Y is separable, the space (Y_1^*, weak^*) is sequentially compact, so the equivalence (a)$\Leftrightarrow$(d) is clear. To prove that (a)$\Rightarrow$(e), let (y_n^*) be a weak* convergent sequence in Y^*. Then the sequence $(T^*(y_n^*))$ weak* converges in X^{***}. Hence $\widetilde{T}^*(y_n^*) = p \circ T^*(y_n^*)$ converges weakly in $X^{*\cdot}$. For the reverse implication it is enough to take $Y = c_0$. ■

REMARKS 2.5. (a) We recall that the Banach space X has the Grothendieck property iff X^* is weakly sequentially complete and there exists no onto operator $T : X \to c_0$ [Rä, p. 16]. Part (b) of the preceding theorem is an analog of this characterization of the Grothendieck property. From these results and Lemma 2.2 it is clear that if X or X^{**} has the Grothendieck property then X has the weak Phillips property.

(b) By Lemma 2.3, it is clear that if X or X^{**} has property (V), or even the apparently weaker property (V_1) (i.e. every unconditionally converging operator from X into c_0 is weakly compact; this notion has been introduced by Räbiger in his thesis [Rä, p. 18]) then X has the weak Phillips property. However, the class of Banach spaces having the weak Phillips property and the class of Banach spaces having property (V_1) do not coincide. For instance, the space Y constructed by Bourgain–Delbaen [B-De] has the weak Phillips property but not property (V_1) since it is not reflexive and does not contain a copy of c_0.

(c) If X is complemented in its second dual then X has the weak Phillips property iff it has property (V_1). Indeed, in this case every operator $u : X \to c_0$ extends to X^{**}, so it is unconditionally converging by Lemma 2.3.

(d) From the preceding theorem it follows that the quotient spaces and complemented subspaces of a Banach space having the weak Phillips property have the weak Phillips property. Actually, a slightly stronger result is true: If X has the weak Phillips property and we have an operator $u : X \to Y$ such that $\overline{u^{**}(X^{**})} = Y^{**}$ then Y also has the weak Phillips property. This is easy to see.

(e) If $X = (\bigoplus X_n)_0$ and each X_n is an M-ideal in its second dual then X is also an M-ideal in its second dual, so has property (V) [HWW, p. 111], hence the weak Phillips property. Now let $X = (\bigoplus \ell_n^1)_0$. Then X has the weak Phillips property but X^{**} does not since it contains a complemented copy of ℓ^1 (see e.g. [HWW, p. 163]).

(f) Every predual of an L-space has the weak Phillips property since its second dual, being an M-space, has property (V). In particular, every predual of ℓ^1 has the weak Phillips property.

(g) In part (c) of the preceding theorem, if instead of assuming that Y is separable we assume that (Y_1^*, weak^*) is sequentially compact then the conclusion still holds.

From the preceding remarks the first conclusion to draw is that the class of Banach spaces having the weak Phillips property is quite large and rich in content: It contains all the C^*-algebras [Pf]; a good many of the uniform algebras ([B1] and [S]), in particular the disk algebra $A(D)$ [De] and the Hardy class $H^\infty(D)$ [B2]; the spaces which are M-ideals in their second duals (these spaces have property (V)), and spaces whose duals are isomorphic to a predual of a von Neumann algebra.

From part (d) of the preceding theorem we deduce the following corollary.

Corollary 2.6. *Suppose that*

(a) *X has the weak Phillips property*,

(b) *Y is complemented in its second dual and the space* (Y_1^*, weak^*) *is sequentially compact.*

Then every operator $u : X \to Y$ is weakly compact. ∎

Thus, for instance, if X has the weak Phillips property, every operator from X into the Lebesgue space $L^1([0,1])$, or into any weakly compactly generated dual space Y^*, is weakly compact.

From part (b) of the preceding theorem and part (a) of the above remark, the next result is immediate as well.

Corollary 2.7. *Suppose that we have an operator $T : X^{**} \to X$ such that $T(X) = X$. Then X has the weak Phillips property iff it has the Grothendieck property.* ∎

Thus a dual space, and more generally any Banach space which is complemented in its second dual, has the weak Phillips property iff it has the Grothendieck property. So, a Banach space having property (V) or (V_1) is not complemented in its second dual unless it has the Grothendieck property. Next we present a slightly more general result.

Proposition 2.8. *Suppose that X has the weak Phillips property and $T : X^{**} \to Y$ is an operator such that $\widetilde{T}^{**}X^{**}$ is dense in Y^{**} (this is the case if $\widetilde{T}$ is onto). Then Y has the Grothendieck property.*

Proof. Let (y_n^*) be a weak*-null sequence in Y^*. Then, by part (d) of the above theorem, the sequence $(\widetilde{T}(y_n^*))$ is weakly null. Hence, for $m \in X^{**}$, $\langle \widetilde{T}(y_n^*), m\rangle = \langle y_n^*, \widetilde{T}^{**}(m)\rangle \to 0$. As $\widetilde{T}^{**}X^{**}$ is dense in Y^{**} and the sequence (y_n^*) is bounded, we conclude that $y_n^* \to 0$ weakly in Y^*. So Y^* has the Grothendieck property. ∎

It is clear that in this proposition, the hypothesis that $\overline{\widetilde{T}^{**}X^{**}} = Y^{**}$ cannot be replaced by $\overline{\widetilde{T}(X)} = Y$. Indeed, the natural injection $i : C([0,1]) \to L^1[0,1]$ is weakly compact, so i^{**} maps $C([0,1])^{**}$ into $L^1[0,1]$ and i has dense range but $L^1[0,1]$ does not have the Grothendieck property.

As is well known, every operator from a space having the Grothendieck property into any other Banach space is either weakly compact or fixes a copy of ℓ^1 [D-S]. Next we present an analog of this result.

THEOREM 2.9. *Suppose that* X *has the weak Phillips property and* $T : X^{**} \to Y$ *is an operator such that* $\overline{T(X^{**})} = \overline{T(X)}$. *Then either* $\widetilde{T}$ *is weakly compact or it fixes a copy of* ℓ^1. *Moreover*, $\widetilde{T}$ *is weakly compact iff* $\overline{T(X)}$ *is weakly compactly generated.*

Proof. Since $\overline{T(X^{**})} = \overline{T(X)}$, if $\widetilde{T}$ is weakly compact then $\overline{T(X)}$ is weakly compactly generated. Conversely, if $Z = \overline{T(X)}$ is weakly compactly generated then (Z^*, weak^*) is sequentially compact [D1, p. 228]. Hence, by part (d) of Theorem 2.4, $\widetilde{T}$ is weakly compact.

Now suppose that $\widetilde{T}$ is not weakly compact. Then, taking into account the fact that X^* is weakly sequentially complete and using Rosenthal's ℓ^1-theorem, we find a sequence (y_n^*) in the unit ball of Y^* such that the sequence $(\widetilde{T}^*(y_n^*))$ is equivalent to the unit vector basis of ℓ^1. As in the proof of Lemma 2.2, we conclude that $\widetilde{T}^*$ fixes a copy of ℓ^1. Now let, if there is any, (y_n^*) be a sequence in the unit ball of Y^* such that the sequence $(\widetilde{T}^*(y_n^*))$ is equivalent to the unit basis of ℓ^1 and $\widetilde{T}^*(y_n^*) \to 0$ in (X^*, weak^*). Since $\overline{T(X^{**})} = \overline{T(X)}$, $T^*(y_n^*) \to 0$ in (X^{***}, weak^*). Hence, since X has the weak Phillips property, $\widetilde{T}^*(y_n^*) \to 0$ weakly in X^*, which contradicts the fact that the sequence $(\widetilde{T}^*(y_n^*))$ is equivalent to the unit basis of ℓ^1. From this, by Theorem 2.1 above, we conclude that $\widetilde{T}$ fixes a copy of ℓ^1. ■

As a useful corollary of this theorem we give the next result.

COROLLARY 2.10. *Suppose that* X *has the weak Phillips property and that* Y *is complemented in its second dual. Then every operator* $u : X \to Y$ *whose range is dense in* Y *is either weakly compact or fixes a copy of* ℓ^1.

Proof. Let $q : Y^{**} \to Y$ be a projection. Then $T = q \circ u^{**}$ maps X^{**} into Y and $\widetilde{T} = u$. So, by the preceding theorem, the conclusion follows. ■

Since every dual space is complemented in its second dual, from this corollary it follows, for instance, that every operator from a C^*-algebra into a dual space with dense range is either weakly compact or fixes a copy of ℓ^1. This also shows that in this corollary it is not possible to drop the condition that $\overline{u(X)} = Y$. Indeed, the identity mapping $i : c_0 \to \ell^\infty$ is neither weakly compact nor fixes a copy of ℓ^1.

Let (Ω, Σ, μ) be a finite measure space and M be a closed subspace of $L^\infty(\mu)$. Let $i : L^\infty(\mu) \to L^p(\mu)$ $(1 \le p < \infty)$ be the natural injection. A result due to A. Grothendieck [D-U, p. 178] says that if $i(M)$ is closed in $L^p(\mu)$ then M is finite-dimensional. The space $L^\infty(\mu)$ has the weak Phillips property and the DPP; and the space $L^p(\mu)$ has the Gelfand–Phillips property and is complemented in its second dual. Moreover, if $i(M)$ is closed in $L^p(\mu)$, the unit ball of M is a weakly precompact subset of $L^\infty(\mu)$ (i.e. every sequence in M_1 has a weakly Cauchy subsequence); see the proof of [D-U, p. 178, Theorem]. Since, when $i(M)$ is closed in $L^p(\mu)$, by the Open Mapping Theorem, $i(M_1)$ contains a multiple of the unit ball of $i(M)$, the next theorem is a generalization of this result of Grothendieck.

THEOREM 2.11. *Suppose that*

(a) *X has the weak Phillips property and the DPP, and*

(b) *Y is complemented in its second dual and has the Gelfand–Phillips property.*

Then every operator $u : X \to Y$ is completely continuous. Hence, if X also has the reciprocal DPP then u is weakly compact as well.

Proof. Let A be a weakly precompact subset of X. As X has the DPP, every weakly null sequence in X^* converges to zero uniformly on A. Now let $q : Y^{**} \to Y$ be a projection and let $T = q \circ u^{**} : X^{**} \to Y$. Then $\widetilde{T} = u$ and, for any weak*-null sequence (y_n^*) in Y^*, since X has the weak Phillips property, $u^*(y_n^*) \to 0$ weakly in X^*. It follows that $y_n^* \to 0$ uniformly on $u(A)$. Hence, since Y has the Gelfand–Phillips property, $u(A)$ is relatively compact. From this we conclude that u is completely continuous. ∎

The above theorem applies, for instance, to the natural injections $i : H^\infty(D) \to H^p(D)$ and $i : A(D) \to H^p(D)$ $(1 \le p < \infty)$. As an immediate corollary we give the next result.

COROLLARY 2.12. *Let Y be as in the preceding theorem. Then every operator u from a commutative C^*-algebra $C(K)$ into Y is weakly compact (and completely continuous).* ∎

The next result may be used to prove that certain spaces have the Grothendieck property.

PROPOSITION 2.13. *Suppose that X has the weak Phillips property and Y has the Dieudonné property. Let $T : X^{**} \to Y$ be an operator such that $\overline{\widetilde{T}^{**}(X^{**})} \supseteq B_1(Y)$. Then Y has the Grothendieck property.*

Proof. Let (y_n^*) be a weak*-null sequence in Y^*. Then, by Theorem 2.4 above, $\widetilde{T}^*(y_n^*) \to 0$ weakly in X^*. Hence, for $m \in X^{**}$,

$$\langle \widetilde{T}^*(y_n^*), m\rangle = \langle y_n^*, \widetilde{T}^{**}(m)\rangle \to 0.$$

So, since $\overline{\widetilde{T}^{**}(X^{**})} \supseteq B_1(Y)$, $y_n^* \to 0$ for the topology $\sigma(Y^*, B_1(Y))$. As Y has the Dieudonné property, we conclude that $y_n^* \to 0$ weakly in Y^* [G, Prop. 11] so that Y has the Grothendieck property. ∎

Thus, in the preceding proposition, if $Y = C(K)$ and if, for each Borel subset B of K, there exists a bounded sequence (x_n) in X such that $T(x_n) \to \chi_B$ pointwise on K then $C(K)$ has the Grothendieck property.

Our final result is a noncomplementation scheme. Suppose that Z is a weakly compactly generated closed subspace of Y and we have an isomorphism $i : c_0 \to Z$ that extends to an operator $\widetilde{i} : \ell^\infty \to Y$. Then Z is not complemented in Y. The next result shows that in this scheme one can use instead of c_0 any nonreflexive Banach space X having the weak Phillips property.

PROPOSITION 2.14. *Suppose that*

(a) X *is nonreflexive and has the weak Phillips property*,

(b) (Z_1, weak^*) *is sequentially compact*, *and*

(c) *we have an* (*into*) *isomorphism* $u : X \to Z$ *that extends to an operator* $S : X^{**} \to Y$.

Then Z *is not complemented in* Y.

Proof. For a contradiction, suppose that we have a projection $q : Y \to Z$. Let $T = q \circ S$. Then $\widetilde{T} = u$, and by Theorem 2.4(d) or (e), u is weakly compact. This is not possible since X is not reflexive and u is an isomorphism. Hence Z is not complemented in Y. ∎

Thus a Banach space Y with weak* sequentially compact dual ball that contains a subspace with the weak Phillips property is not complemented in its second dual unless it is reflexive. At this point we recall that there exist nonreflexive Banach spaces with the weak Phillips property that do not contain a copy of c_0; for instance, the space Y constructed by Bourgain and Delbaen in [B-D] is such a space.

3. Remarks and questions. 1. The characterizations of the weak Phillips property we have given involve the second dual of the space. We do not know whether it is possible to give a characterization of the weak Phillips property involving just the space and its dual.

2. The question whether X having the Grothendieck property does or does not imply that X^{**} has the same property seems to be open. Concerning this question, we have the following connection: X^{**} has the Grothendieck property iff X has the weak Phillips property and X^{**}/X has the Grothendieck property.

3. Suppose X has the weak Phillips property. We know that every complemented subspace of X has the same property. Actually, a somewhat stronger result holds: Suppose that M is a (closed) subspace of X and that we have an operator $q : X^{**} \to M^{**}$ such that $q(M) = M$. We can assume, by the Open Mapping Theorem, that $q(M_1) \supseteq M_1$. Then, for any operator

$T : M^{**} \to c_0$, $T(M_1) \subseteq T(q(M_1)) \subseteq T \circ q(X_1)$. This latter set being relatively weakly compact, we conclude that $\widetilde{T}$ is weakly compact so that M has the weak Phillips property. Thus, in particular, if M^{**} is complemented in X^{**} then M has the weak Phillips property. If M itself is complemented in X^{**} then, by Proposition 2.8, M has the Grothendieck property.

4. We do not know whether X having the weak Phillips property implies, or is implied by, X^* having the property (V*) of Pełczyński. A related question is this: Suppose that X has the weak Phillips property and Y^{**} does not contain a copy of ℓ^∞. Is then every operator $u : X \to Y$ weakly compact? We remark that by Lemma 2.3, u is unconditionally converging.

5. We do not know if the space $K(X)$ of compact operators on a reflexive space X (with the approximation property, for instance) has the weak Phillips property.

6. Suppose that X has the weak Phillips property and let K be a compact (Hausdorff) space. We do not know if the space $C(K, X)$ of continuous functions $\varphi : K \to X$, endowed with the supremum norm, has the weak Phillips property.

7. Suppose that X has the weak Phillips property. We do not know whether every operator $u : X \to X^*$ is weakly compact. In connection with this and with Question 4 above, it is not difficult to see that every operator from X into a Banach space Y having property (V*) is weakly compact. Note that, by Corollary 2.6, in the case when X is separable and does not contain a copy of ℓ^1, every operator $u : X \to X^*$ is weakly compact since then the space (X^{**}, weak^*) is sequentially compact [D1, p. 236, Theorem 10].

8. Suppose that X has the weak Phillips property and Y does not contain a copy of ℓ^1. Let $u : X \to Y$ be an operator with dense range. By Theorem 2.9, u is weakly compact iff it has a (bounded linear) extension $S : X^{**} \to Y$. We do not know if it is possible to drop the density condition in this result.

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