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PIERI-TYPE INTERSECTION FORMULAS AND PRIMARY OBSTRUCTIONS FOR DECOMPOSING 2-FORMS

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Abstract. We study the homological intersection behaviour for the Chern cells of the universal bundle of $G(d, Q_n)$, the space of $[d]$-planes in the smooth quadric Q_n in $\mathbb{P}^{n+1}$ over the field of complex numbers. For this purpose we define some auxiliary cells in terms of which the intersection properties of the Chern cells can be described. This is then applied to obtain some new necessary conditions for the global decomposability of a 2-form of constant rank.

1. Introduction. In this article we study from a purely projective-geometric point of view the obstructions to globally decomposing a 2-form. It was shown by Dibağ [3] that the vanishing of certain Chern classes is necessary for such a decomposition. We construct new classes whose nonvanishing implies the nonvanishing of the Chern classes. Moreover some vanishing patterns of these new classes imply the vanishing of the Chern class obstructions. This is achieved by studying the intersection structure of the integral homology generated by the Chern cells. Our methods are purely geometric and determine the required products up to a nonzero multiplicative constant. However this suffices for our purposes since we eventually check for vanishing of obstructions. In the case of maximal planes these coefficients can be explicitly calculated. This is done by Hiller and Boe [7] who consider the case of type B maximal isotropic Grassmannians. Type D (which is a consequence of the result in type B) appeared in [9]. The results of [7] are further reproved by Pragacz and Ratajski [11] by using divided differences. Recently similar calculations in type B were done by Sottile [14]. In the nonmaximal case these calculations are due to Pragacz and Ratajski (see [12]). However to adopt these general formulas for our cases would lead to complicated combinatorial formulas. By checking only nonvanishing conditions we are able to present a purely geometrical argument which suffices for our results.

We denote by $G(d, Q_n)$ the space of complex projective $[d]$-planes lying in the smooth quadric hypersurface Q_n of $\mathbb{P}^{n+1}$. Dibağ has shown that $G(d, Q_n)$

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represents $A^{(n+2)}_{d+1}$, the space of normalized 2-forms in $\mathbb{R}^{n+2}$ of rank $2(d+1)$, on which the Stiefel bundle $V_{n+2,2(d+1)}$ of orthonormal $2(d+1)$-frames in $\mathbb{R}^{n+2}$ induces a principal $U(d+1)$-bundle (see [2, 3]).

In general if ω is a 2-form of constant rank $2(d+1)$ on a trivial $\mathbb{R}^{n+2}$-bundle E over some base space B, then it can be represented by a map $\omega_1 : B \to A^{(n+2)}_{d+1}$. Lifting this map to $V_{n+2,2(d+1)}$ is equivalent to decomposing the 2-form ω globally as $\omega = y_1 \wedge y_{d+2} + \ldots + y_{d+1} \wedge y_{2(d+1)}$ for some 1-forms y_i on E. Then the images $\omega_1^*(\mathbf{c}_i) \in H^{2i}(B;\mathbb{Z})$ of the Chern classes $\mathbf{c}_i \in H^{2i}(A^{(n+2)}_{d+1};\mathbb{Z})$, $i = 0,\ldots,d+1$, of the principal $U(d+1)$-bundle $V_{n+2,2(d+1)}$ necessarily vanish. If E is not trivial then the above geometry is analyzed on a certain subbundle S_ω of E, depending on ω, and its triviality is another necessary condition for the decomposability of ω (see [3]).

We define some cohomology classes $\mathrm{PD}\,\Omega_i \in H^{l-i}(A^{(n+2)}_{d+1};\mathbb{Z})$, where $l = \dim A^{(n+2)}_{d+1}$ and $i = 0,\ldots,d+1$, and show that if $\omega_1^*(\mathrm{PD}\,\Omega_s) = 0$ for some $0 \le s \le d+1$, and $\omega_1^*(\mathrm{PD}\,\Omega_{s+i}) \neq 0$ for $i = 1,\ldots,d+1-s$, then $\omega_1^*(\mathbf{c}_i) = 0$ for $i = 1,\ldots,d+1-s$. Moreover if $\omega_1^*(\mathrm{PD}\,\Omega_s) \neq 0$ for some s, then $\omega_1^*(\mathbf{c}_i) \neq 0$ for all $i = 1,\ldots,d+1-s$ (see Theorem 3 and Corollary 4). When $n = 2d$, a trivial line bundle splits off the universal bundle on each irreducible component, V_0 and V_1, of $G(d,Q_n)$ forcing $\mathbf{c}_{d+1}$ to vanish. In this case $s > 0$ if it exists. These results occupy the last section after we establish in Section 3 the intersection properties of Chern cells.

For background on intersection problems we refer to [1, 5, 6]. For recent applications one can refer to [10, 12, 13, 14]. For the existence and the decomposability of 2-forms see [2, 3, 8]. Finally, for 2-forms on spheres see [2, 4].

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2. Preliminaries. In this section we summarize some constructions which help us to understand the geometry of the space of d-planes lying in a smooth quadric Q_n. First we define a set of points which we propose to call the skeleton points and use in describing flags and Schubert cells. We refer to [13, pp. 203–207] for further details and here briefly describe the main lines for completeness.

For two points $p = (p_1,\ldots,p_N)$ and $q = (q_1,\ldots,q_N)$ in $\mathbb{C}^N$ we say that p and q are *f-orthogonal* if $p\cdot q = p_1q_1+\ldots+p_Nq_N = 0$, and *m-orthogonal* if $p\cdot\overline{q} = p_1\overline{q}_1+\ldots+p_N\overline{q}_N = 0$, where the overbar denotes complex conjugation. When p and q are used as homogeneous coordinates of the corresponding points in the projective space the same terminology prevails.

Assume $n = 2m$. The *skeleton points* of Q_{2m} is a set of $2m + 2$ points in $\mathbb{P}^{2m+1}$ chosen as follows:

(i) Choose p_0 in Q_{2m} arbitrarily. This means that p_0 is f-orthogonal to itself.

(ii) Once $p_0, \ldots, p_{k-1}$ are chosen, where $1 \leq k \leq m$, choose p_k as any point of Q_{2m} which is both f-orthogonal and m-orthogonal to the join $p_0 \vee \ldots \vee p_k$ but not in the join itself. The set of points in $\mathbb{P}^{2m+1}$ satisfying these conditions is a $2(m-k)$-dimensional subspace so a choice is always possible.

(iii) After having chosen $p_0, \ldots, p_m$, the remaining $m+1$ points are chosen as the complex conjugates of these, indexed as follows:

$$p_{m+i} = \overline{p}_{m+1-i}, \qquad i = 1, \ldots, m+1.$$

Here again the overbar denotes complex conjugation.

These points $p_0, \ldots, p_{2m+1}$ all lie in Q_{2m} and their join is the whole space $\mathbb{P}^{2m+1}$. The particular way we choose and index them enables us to build a link between geometry and algebra. This can be seen in the following construction.

For any subset L of $I_{2m+1} = \{0, 1, \ldots, 2m+1\}$ define S_L as the intersection of Q_{2m} with the join of those skeleton points whose index is in L:

$$S_L = Q_n \cap \Big(\bigvee_{j \in L} p_j \Big).$$

The link between geometry and algebra comes into play at this stage: the dimension of S_L is determined by the indexing set L. For this define a particular subset of L which affects the dimension of S_L:

$$J(L) = \{ i \in I_m \mid i \in L \text{ and } 2m+1-i \in L \}.$$

In other words $J(L)$ contains the indices of only those skeleton points among $p_0, \ldots, p_m$ whose complex conjugates also lie in S_L.

In [13, Lemma 1.3] we proved that

$$\dim_{\mathbb{C}} S_L = \begin{cases} \#L - 2 & \text{if } J(L) \neq \emptyset, \\ \#L - 1 & \text{if } J(L) = \emptyset. \end{cases}$$

Now we are ready to construct two dual flags for Q_{2m}. The first one is called the *A-flag* and consists of a nested sequence of subvarieties of Q_{2m},

$$A_0 \subsetneq \ldots \subsetneq A_{m-1} \subsetneq A_{m_0}, A_{m_1} \subsetneq A_{m+1} \subsetneq \ldots \subsetneq A_{2m} = Q_{2m},$$

such that $A_i - A_{i-1}$ is an open cell of dimension i, and each A_i is chosen as follows:

(i) $A_i = S_{\{0,1,\ldots,i\}}$ for $i = 0, \ldots, m-1$,
(ii) $A_{m_0} = S_{\{0,1,\ldots,m\}}$ and $A_{m_1} = S_{\{0,1,\ldots,m-1,m+1\}}$,
(iii) $A_{m+i} = S_{\{0,1,\ldots,m+1+i\}}$ for $i = 1, \ldots, m$.

Note that $\dim A_i = i$ for $i = 0, \dots, 2m$, where the indices m_0 and m_1 are considered to be different as indices but both equal to m as values.

The second flag is the dual flag, called the *B-flag*. It also consists of a nested sequence of subvarieties of Q_{2m},

$$B_0 \subseteq \dots \subseteq B_{m-1} \subseteq B_{m_0}, B_{m_1} \subseteq B_{m+1} \subseteq \dots \subseteq B_{2m} = Q_{2m},$$

where each B_i is chosen as follows:

(i) $B_i = S_{\{2m+1,2m,\dots,2m+1-i\}}$ for $i = 0, \dots, m-1$,

(ii) $B_{m_0} = S_{\{2m+1,2m,\dots,m+2,m\}}$ and $B_{m_1} = S_{\{2m+1,2m,\dots,m+1\}}$ if m is even, $B_{m_1} = S_{\{2m+1,2m,\dots,m+2,m\}}$ and $B_{m_0} = S_{\{2m+1,2m,\dots,m+1\}}$ if m is odd,

(iii) $B_{m+i} = S_{\{2m+1,2m,\dots,m-i\}}$ for $i = 1, \dots, m$.

Note again that $\dim B_i = i$ for $i = 0, \dots, 2m$.

For the corresponding constructions in the $n = 2m + 1$ case we refer the reader to [13, pp. 205–207].

To conclude this section we summarize the construction of Schubert cells on $G(d, Q_{2m})$, the space of d-planes in Q_{2m}. In the notation of [3], $G(d, Q_{2m})$ is $A_{d+1}^{(n+2)}$. A *Schubert symbol* for $G(d, Q_{2m})$ is a finite sequence of integers $a = (a_0, \dots, a_d)$, $d \le m$, satisfying the conditions

(i) $0 \le a_0 < \dots < a_d \le 2m$,

(ii) $a_i + a_j \ne 2m$ for $i < j$. This condition is to avoid assigning different Schubert symbols to the same cell. See [3, p. 506].

Here again m_0 and m_1 are used as two different entities but both having the value m, so if one of them appears in the sequence a the other does not according to (ii).

The *Schubert cell* corresponding to the Schubert symbol a is a subvariety of $G(d, Q_{2m})$ defined as

$$\Omega_{a_0 \dots a_d} = \{P \in G(d, Q_{2m}) \mid \dim_{\mathbb{C}}(P \cap A_{a_i}) \ge i\}.$$

Here A_{a_i} denotes the corresponding member of the A-flag. It turns out that

$$\dim_{\mathbb{C}} \Omega_{a_0 \dots a_d} = a_0 + \dots + a_d - d(d+1) + e$$

where

$$e = \#\{(a_i, a_j) \mid i < j \text{ and } a_i + a_j < n\}.$$

For further details on the intersection properties of these cells we refer to [3, 13].

3. Intersecting Chern cells. Following Dibağ, the ith *Chern cell* Ω_i of the principal $U(d+1)$-bundle $V_{n+2,d+1}(A_{d+1}^{(2n)}; U(d+1))$ is defined in terms of Schubert cells on $G(d, Q_n)$ as

$$\Omega_i = \Omega_{0 \dots \widehat{(d-i+1)} \dots d+1}, \quad 0 \le i \le d+1,$$

when $n \geq 2d+3$. (Here $(\widehat{d-i+1})$ means "omit $d-i+1$"). The restriction on n ensures that the condition $a_i + a_j \neq n$ for $i < j$ holds in the Schubert symbols corresponding to the Chern cells, which is important to avoid redundant representations.

It turns out that the homology duals of the Chern cells play a crucial part in the intersection behaviour of the Chern classes. These are defined as follows:

$$\Delta_j = \Omega_{n-d-1\ldots(\widehat{n-d-1+j})\ldots n}, \quad 0 \leq j \leq d+1.$$

Note that Δ_j is the "dual" of Ω_j, i.e. $\Omega_j^t = \Delta_j$ in the notation of [3]. A direct calculation shows that $\dim_{\mathbb{C}} \Omega_i = \operatorname{codim}_{\mathbb{C}} \Delta_i = i$ for $0 \leq i \leq d+1$.

The intersection properties of the Chern cells can now be described fully in terms of the dual cells: the ith Chern cell nontrivially intersects a cell if and only if this cell is the jth dual Chern cell with a j not greater than i, and in that case the intersection is precisely a multiple of the $(i-j)$th Chern cell. We can now formulate this in the following theorem;

THEOREM 1. *Let Ω_i and Δ_j be as defined above and let Ω be any Schubert cell of $G(d, Q_n)$ with $n \geq 2d+3$. Then*

$$\Omega_i \cdot \Omega \neq 0 \ \text{ if and only if } \ \Omega = \Delta_j \text{ for some } j \text{ with } 0 \leq j \leq i \leq d+1.$$

Moreover in that case we have

$$\Omega_i \cdot \Delta_j = \alpha \Omega_{i-j}, \quad 0 \leq j \leq i \leq d+1,$$

where α is a nonzero integer.

REMARK. P. Pragacz has communicated these coefficients as powers of 2. In fact Hiller and Boe [7] have shown that for the maximal plane case, i.e. the $n = 2d$ case, these coefficients are indeed powers of 2 (see also [11, 14]). We will deal with the $2d \leq n < 2d+3$ cases in the next section. However we are only interested in the obstruction-theoretical properties of these intersections so it only matters for us if the coefficients are zero or not. By appealing to some general facts about Schubert cycles and Bruhat order it is possible to give a shorter proof of this theorem but we prefer this approach which is elementary and exhibits the inner workings of geometry.

Proof of Theorem 1. We will give the proof for the $n = 2m$ case which reflects the main geometric ideas involved. The $n = 2m+1$ case is similar and is omitted. First we note that Ω_0 is a point and hence nontrivially intersects only Δ_0, with $\alpha = 1$. Next let $1 \leq i \leq d$: Suppose Ω_i intersects nontrivially a Schubert cell Ω whose Schubert symbol is $a = (a_0, \ldots, a_d)$. Let P be a d-plane in $G(d, Q_n)$ which lies in the intersection $\Omega_i \cdot \Omega$. Then P must satisfy simultaneously the Schubert conditions dictated by the two symbols $(0, \ldots, (\widehat{d-i+1}), \ldots, d+1)$ and $(a_0, \ldots, a_d)$ of Ω_i and Ω respectively. If we use the A-flag of $G(d, Q_n)$, the first symbol $(0, \ldots, \widehat{(d-i+1)}, \ldots, d+1)$ implies that the d-plane P contains the join $p_0 \vee \ldots \vee p_{d-i}$ and itself lies in the join $p_0 \vee \ldots \vee p_{d+1}$. Here we used the description of the spaces A_i of the A-flag. Next we use the dual B-flag to interpret the second symbol. The a_0 of a now requires that P intersects the space B_{a_0}, but $\dim_{\mathbb{C}} B_{a_0} = a_0$. The d-dimensional plane q lies in the $(d+1)$-dimensional join $p_0 \vee \ldots \vee p_{d+1}$. Then this join must have at least a point in common with B_{a_0}, which forces $(d+1) + a_0 \geq n$ or equivalently $a_0 \geq n-d-1$. Combining this with the general properties of Schubert symbols we have

$$n-d-1 \leq a_0 < \ldots < a_d \leq n.$$

This means that we have to choose $d+1$ integers from the interval $[n-d-1, n]$. But there are only $d+2$ integers in this interval so we take all the integers from this interval except one

$$(a_0, \ldots, a_d) = (n-d-1, \ldots, \widehat{(n-d-1+j)}, \ldots, n), \quad 0 \leq j \leq d+1.$$

This is precisely the definition of Δ_j and thus the first part of the theorem is proved. That j cannot exceed i will follow from the proof of the second part of the theorem.

To prove that part, we assume that the intersection of Ω_i with Δ_j is nonempty. We may again assume that $i > 0$. Assume that P is a d-plane lying in the nonempty intersection $\Omega_i \cdot \Delta_j$. We know from the above analysis that P must contain the join $p_0 \vee \ldots \vee p_{d-i}$ and must lie in the $(d+1)$-dimensional space defined by the join $p_0 \vee \ldots \vee p_{d+1}$. These conditions are imposed on P because it belongs to Ω_i. Now we inspect what further conditions will be imposed on P by forcing it to belong to Δ_j as well.

The Schubert symbol of Δ_j is

$$(a_0, \ldots, a_d) = (n-d-1, \ldots, \widehat{(n-d-1+j)}, \ldots, n), \quad 1 \leq j \leq d+1.$$

If we use the B-flag, the number a_0 imposes that

$$\dim_{\mathbb{C}} P \cap (p_{2m+2} \vee \ldots \vee p_{d+1}) \geq 0.$$

But since P lies only in $p_0 \vee \ldots \vee p_{d+1}$ the condition imposed by a_0 holds if and only if P contains the point p_{d+1}.

In the same vein we argue that since the integers $a_0, \ldots, a_{j-1}$ are consecutive the condition

$$\dim_{\mathbb{C}} P \cap (p_{2m+2} \vee \ldots \vee p_{d+2-j}) \geq j-1$$

can hold if and only if P contains the join $p_{d+1} \vee \ldots \vee p_{d+2-j}$. The other integers in the Schubert symbol a do not impose any further conditions on P.

In view of these arguments we find that if the d-plane P lies in the nonempty intersection $\Omega_i \cdot \Delta_j$ then it must contain the following list of skeleton points:

$$p_0, \ldots, p_{d+1} \quad \text{and} \quad p_{d+2-j}, \ldots, p_{d+1}.$$

The first part of the list is derived from the fact that $P \in \Omega_i$ and the second part from the fact that $P \in Q_{2m}$. But there are altogether $d+1-(i-j)$ skeleton points in this list and hence their join, which necessarily belongs to P, has dimension $d-(i-j)$. Since P is a d-plane we must have $i-j \geq 0$ or $i \geq j$.

We thus find the description for all $P \in \Omega_i \cdot \Delta_j$: each such P must live in the join $p_0 \vee \ldots \vee p_{d+1}$ and must contain a $(d+1-(i-j))$-dimensional subspace of this join. This is the description of Ω_{i-j}.

This completes the proof of the theorem. ∎

4. The unstable cases. The cases when $2d \leq n \leq 2d+2$ are called the *unstable cases* (the terminology belongs to Dibağ, see [3]). The theorem of the previous section holds verbatim in the unstable cases if we provide the correct definitions of the Δ_j's. In the following subsections we describe the necessary modifications in the definitions to make the theorem hold.

4.1. *The $n = 2d+2$ case.* In this case the Chern cycles are defined as

$$\Omega_i = \Omega_{0\ldots\widehat{(m-i)}\ldots m_0} + \Omega_{0\ldots\widehat{(m-i)}\ldots m_1}, \qquad 0 \leq i \leq m.$$

Here note that $m = d+1$. For these Chern cycles we define the following dual Chern cycles:

$$\Delta_j = \Omega_{m_0\ldots\widehat{(m+j)}\ldots n} + \Omega_{m_1\ldots\widehat{(m+j)}\ldots n}, \qquad 1 \leq j \leq m.$$

Our theorem of the previous section now holds verbatim with these definitions.

4.2. *The $n = 2d+1$ case.* The Chern cycles for $i = 0, \ldots, d+1$ are defined as

$$\Omega_i = 2\Omega_{0\ldots\widehat{(d+1-i)}\ldots d,d+i}.$$

Define the required "duals" as

$$\Delta_j = \Omega_{d+1-j,d+1\ldots\widehat{(d+j)}\ldots 2d+1}, \qquad j = 0, \ldots, d+1.$$

4.3. *The $n = 2d$ case.* This is the maximal plane case. There are two disjoint, irreducible families of d-planes in Q_n. Call these families V_0 and V_1. The Schubert cells of $G(d, Q_n)$ are evenly divided among these families. It suffices to consider V_0 only. The V_1 case is obtained simply by reversing the marking of d in the following definitions.

First assume that d is even. Ω_0 is defined as

$$\Omega_0 = 2\Omega_{0\ldots d_0}.$$

The nontrivial Chern cycles are defined as

$$\Omega_i = 2\Omega_{0\ldots\widehat{(d-i)}\ldots d_1,d+i}, \qquad i = 1,\ldots,d.$$

Finally, define $\Omega_{d+1} = 0$. The "duals" are then defined as

$$\begin{aligned}\Delta_0 &= \Omega_{d_0\ldots 2d},\\ \Delta_j &= \Omega_{d-j,d_1\ldots\widehat{(d+j)}\ldots 2d}, \qquad j = 1,\ldots,d,\\ \Delta_{d+1} &= 0.\end{aligned}$$

When d is odd, to obtain the symbolism in V_0 reverse the indexing of d in the definition of Chern cycles but leave the indexing of the "duals" the same.

With these definitions Theorem 1 holds. Because of the significance of the maximal plane case we quote this result separately as a corollary to Theorem 1.

COROLLARY 2. *In the $n = 2d$ case we also have*

$$\Omega_i \cdot \Delta_j = \alpha\Omega_{i-j}$$

for $0 \le j \le i \le d$ where α is a nonzero integer.

Proof. We give the proof when d is even. The odd case is similar. Let Λ be a d-plane in the intersection of $\Omega_i \cdot \Delta_j$. Assume that a set of skeleton points $p_0,\ldots,p_{2d+1}$ is fixed. We interpret the Schubert conditions of Ω_i with respect to the A-flag and those of Δ_j with respect to the B-flag. Then Λ lives in $p_0 \vee\ldots\vee p_{d+i+1}$, and must have a point p in $p_{d+j+1} \vee\ldots\vee p_{d+i+1}$. Therefore the complex conjugate of this join, which is $p_{d-i} \vee\ldots\vee p_{d-j}$, can contribute only $i-j-1$ to the dimension to Λ, i.e. $\dim(\Lambda \cap (p_0 \vee\ldots\vee p_{d-j})) = d-j-1$. Since by the Schubert conditions of Ω_i the join $p_0 \vee\ldots\vee p_{d-i-1}$ belongs to Λ, it follows that Λ also contains $p_{d-j+1} \vee\ldots\vee p_{d-1}$ and p_{d+1}. These conditions completely describe any Λ in the intersection. To prove the corollary we translate these descriptions to Schubert conditions. For this purpose define a new set of skeleton points $q_0,\ldots,q_{2d+1}$ as follows:

- $q_t = p_t$ for $t = 0,\ldots,d-i-1$.
- $q_{d-i+t} = p_{d-j+1+t}$ for $t = 0,\ldots,j-2$.
- $q_{d-(i-j)-1} = p_{d+1}$.
- $q_{d-(i-j)+t} = p_{d-i+t}$ for $t = 0,\ldots,(i-j)-1$.
- $q_d = p_{d+j}$. (This is to respect the V_0, V_1 formalism of maximal planes in a quadric.)
- $q_{d+t} = \overline{q}_{d-t+1}$ for $t = 1,\ldots,d+1$.

If we define an A-flag with respect to this set of skeleton points, the above description of Λ becomes equivalent to the description of Ω_{i-j} as claimed. ■

5. Obstruction classes. We first define the Chern classes as

$$\mathbf{c}_i = \Omega_i^* = \mathrm{PD}\,\Omega_i^t = \mathrm{PD}\,\Delta_i \in H^i(A_{d+1}^{(n+2)};\mathbb{Z})$$

where * denotes the cell dual, PD denotes Poincaré duality and ^{t} denotes homology duality (see [3]). Since boundary operations are zero, the cycles and cocycles constitute the homology and cohomology respectively.

A necessary condition for the decomposability of a 2-form ω of constant rank $2(d+1)$ on a trivial $\mathbb{R}^{n+2}$-bundle E, with $n > 2d$, on some base space B is the vanishing in $H^{2(d+1)}(B;\mathbb{Z})$ of $\omega_1^*(\mathbf{c}_i)$, $0 \le i \le d+1$, where $\mathbf{c}_i \in H^{2(d+1)}(A_{d+1}^{(n+2)};\mathbb{Z})$ is the Chern class of the principal $U(d+1)$-bundle of the Stiefel manifold of orthonormal $2(d+1)$-frames in $\mathbb{R}^{n+2}$ with the projection onto $A_{d+1}^{(n+2)}$ given by $(y_1,\dots,y_{2(d+1)}) \mapsto y_1 \wedge y_{d+1} + \dots + y_{d+1} \wedge y_{2(d+1)}$ and where $\omega_1 : B \to A_{d+1}^{(n+2)}$ represents ω (see [3]).

In this section as an application of our intersection theorem we describe the vanishing of $\omega_1^*(\mathbf{c}_i)$ in terms of the vanishing of $\omega_1^*(\mathrm{PD}\,\Omega_i)$. Intersection of homology cells being Poincaré dual to cup product in cohomology, we have the following relations which follow from Theorem 1:

$$\begin{aligned}\mathrm{PD}\,\Omega_{i-j} &= \mathrm{PD}(\Omega_i \cdot \Delta_j) = (\mathrm{PD}\,\Omega_i) \cup (\mathrm{PD}\,\Delta_j)\\ &= (\mathrm{PD}\,\Omega_i) \cup \mathbf{c}_j, \qquad 0 \le j \le i \le d+1.\end{aligned}$$

We now get our application to obstruction of decomposability:

THEOREM 3. *If $\omega_1^*(\mathrm{PD}\,\Omega_s) \neq 0$ for some fixed s with $0 \le s \le d+1$, then $\omega_1^*(\mathbf{c}_i) \neq 0$ for all $i = 0,\dots,d+1-s$.*

Proof. This follows from the equation

$$\omega_1^*(\mathrm{PD}\,\Omega_s) = \omega_1^*(\mathrm{PD}\,\Omega_{s+i}) \cup \omega_1^*(\mathbf{c}_i).$$

The left hand side being nonzero, each term on the right hand side has to be nonzero. ∎

In particular $\omega_1^*(\mathrm{PD}\,\Omega_0)$ is an obstruction to the vanishing of every $\omega_1^*(\mathbf{c}_i)$. We conclude with the following remark which we record as a corollary.

COROLLARY 4. *If $\omega_1^*(\mathrm{PD}\,\Omega_s) = 0$ and $\omega_1^*(\mathrm{PD}\,\Omega_{s+i}) \neq 0$ for $i = 1,\dots, d+1-s$, then $\omega_1^*(\mathbf{c}_i) = 0$ for $i = 1,\dots,d+1-s$. In particular if $\omega_1^*(\mathrm{PD}\,\Omega_i)$ vanishes for $i = 0$ only, then $\omega_1^*(\mathbf{c}_i) = 0$ for all $i = 1,\dots,d+1$.* ∎

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