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DIRECT DECOMPOSITIONS OF UNIFORM GROUPS

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Abstract. Uniform groups are extensions of rigid completely decomposable groups by a finite direct sum of cyclic primary groups all of the same order. The direct decompositions of uniform groups are completely determined by an algorithm that is realised by a MAPLE procedure.

1. Introduction. Call a decomposition of a group into indecomposable summands an *indecomposable decomposition* for short. In the theory of torsion-free abelian groups, the existence of many radically different indecomposable decompositions has been an unpleasant surprise initially, and a fascinating topic subsequently. Most examples and results dealt with *almost completely decomposable groups* that are, by definition, finite extensions of finite rank completely decomposable groups, and are relatively accessible. Main references are [Jon57], [Jon59], [Cor61], [FL70], [BM80], [Bla83], [Yak89], [BY90], [YK93], [BM94], but, in some form or fashion, the topic is touched upon in almost every publication on almost completely decomposable groups. On the other hand, there are also uniqueness results, the most remarkable and general of these being the theorem of Faticoni and Schultz ([FS96]; for a different proof see [MV97] or [Mad00, Section 10.4]), which says that an indecomposable decomposition of a group that is the extension of a completely decomposable group by a finite primary group is unique up to near-isomorphism. Indecomposable decompositions of extensions of completely decomposable groups by cyclic primary groups, called crq-groups, were shown to be unique up to isomorphism in [MV95] (or see [Mad00, Section 6.5]).

In this paper we consider a class of almost completely decomposable groups whose members trivially have unique indecomposable decompositions and we endeavour to find these. The class is that of rigid p-local groups.

A *type* τ is an isomorphism class of rational groups, and a *rational group* is an additive subgroup of the rationals containing $\mathbb{Z}$. We frequently

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abuse notation and use τ for a representative of the class τ. The groups $G(\tau)$, $G^\sharp(\tau)$ are the usual (pure) *type subgroups* of G. A group G is *rigid* if $\operatorname{rk}(G(\tau)/G^\sharp(\tau)) \leq 1$ for every type τ and its *critical typeset* $\mathrm{T}_{\mathrm{cr}}(G) = \{\tau : G(\tau)/G^\sharp(\tau) \neq 0\}$ is an antichain. An almost completely decomposable X is *p-local* for a prime p if $X/\mathrm{R}(X)$ is a (finite) p-group where $\mathrm{R}(X)$ is the regulator of X. For a rigid almost completely decomposable group X the *regulator* is simply $\mathrm{R}(X) = \sum_{\varrho\in\mathrm{T}_{\mathrm{cr}}(X)} X(\varrho) = \bigoplus_{\varrho\in\mathrm{T}_{\mathrm{cr}}(X)} X(\varrho)$ and $X(\tau) \cong \tau$.

Let X be a rigid p-local almost completely decomposable group and $A = \bigoplus_{\varrho\in\mathrm{T}_{\mathrm{cr}}(X)} X(\varrho)$ its regulator. If $X = Y \oplus Z$, then $\mathrm{T}_{\mathrm{cr}}(X)$ is the disjoint union of $\mathrm{T}_{\mathrm{cr}}(Y)$ and $\mathrm{T}_{\mathrm{cr}}(Z)$ and $Y = (\bigoplus_{\varrho\in\mathrm{T}_{\mathrm{cr}}(Y)} X(\varrho))^X_*$. This shows that Y is a fully invariant subgroup of X. It is an easy consequence (Lemma 3.2) that X has a unique indecomposable decomposition. The problem is to find the partition

$$\mathrm{T}_{\mathrm{cr}}(X) = T_1 \cup \ldots \cup T_n$$

such that

$$X = \bigoplus_{i=1}^{n} \Big(\bigoplus_{\varrho\in T_i} X(\varrho) \Big)^X_*$$

and each summand is indecomposable. The equality of X with the direct sum is equivalent to an index equality, namely,

$$[X : A] = \prod_{i=1}^{n} \Big[\Big(\bigoplus_{\varrho\in T_i} A(\varrho) \Big)^X_* : \Big(\bigoplus_{\varrho\in T_i} A(\varrho) \Big) \Big].$$

In order to compute these indices, it is necessary to assume that the group X is given in some specific form. The natural and usual description (except for form) is the *standard description* developed in [BM98a] (or see [Mad00, Chapter 11]). The *Purification Lemma* (Lemma 2.1; see [BM98a], [Mad00, Lemma 11.4.1]) supplies the practical means for computing the necessary indices. In a sense the problem is now solved: It is a finite task to list the possible partitions of $\mathrm{T}_{\mathrm{cr}}(X)$ and the computations of indices is algorithmic and can be done by computer. Further results must be concerned with making the process more efficient and attractive. We do this by using the fact that the standard description may be assumed to be special without loosing generality ([MMN00]). This leads to a complete success in the case of *uniform groups*, i.e., rigid p-local groups X such that $X/\mathrm{R}(X)$ is a direct sum of mutually isomorphic cyclic groups. These groups were classified by Dugas and Oxford ([DO93], [Mad00, Section 12.5]) and an indecomposability criterion was formulated in geometric language. We establish a method for finding the indecomposable decomposition of such a group and include short MAPLE procedures that implement the algorithm. It is then possible to randomly generate groups (in a certain sense) and determine the experimental frequency of indecomposables among them.

The general case of rigid local groups is more complicated and is left to a subsequent paper.

All "groups" in this paper are abelian, and the torsion-free groups all have finite rank. The expression $\mathbb{M}_{k\times r}(S)$ denotes the set of $k \times r$ matrices with entries in the set S. The set S is usually a ring, in particular the ring of integers $\mathbb{Z}$ will occur and its quotient ring $\mathbb{Z}/e\mathbb{Z}$, but S may also be an abelian group. When $r = 1$ we write $s^{\downharpoonleft} \in \mathbb{M}_{k\times 1}(S)$. Similarly, when $k = 1$ we write $\vec{s} \in \mathbb{M}_{1\times r}(S)$. Frequently we will need to deal with submatrices of a matrix and we will use the following notation. Let $M \in \mathbb{M}_{k\times r}(S)$. Then $M[i\lfloor]$ for $1 \le i \le k$ denotes the submatrix of M consisting of its ith row; $M[\lfloor j]$ for $1 \le j \le r$ denotes the submatrix of M consisting of its jth column; $M[i\lfloor j]$ is the entry of M in the ith row and jth column; $M[\alpha\lfloor]$ for $\alpha \subset \{1, \ldots, k\}$ denotes the submatrix of M formed by the rows with index in α; $M[\lfloor\beta]$ for $\beta \subset \{1, \ldots, r\}$ denotes the submatrix of M formed by the columns with index in β; $M[\alpha\lfloor\beta]$ for $\alpha \subset \{1, \ldots, k\}$ and $\beta \subset \{1, \ldots, r\}$ denotes the submatrix of M formed by deleting all rows with index not listed in α and all columns with index not listed in β.

Greatest common left divisors of integral matrices are only defined up to right invertible factors, hence greatest divisors can and will be assumed to have positive determinant.

For background on almost completely decomposable groups we refer the reader to the survey article [Mad95] or the monograph [Mad00].

2. Background and basic observations. Throughout, X denotes an almost completely decomposable group. The standard description of such a group is

$$X = A + \vec{\mathbb{Z}} N^{-1} a^{\downharpoonleft}$$

where A is a completely decomposable group, $\vec{\mathbb{Z}}$ is the set of all $1 \times k$ integral matrices, N is a non-singular $k \times k$ integral matrix, $a^{\downharpoonleft}$ is a $k \times 1$ matrix of elements in A, and juxtaposition is matrix multiplication in the usual fashion.

It may and will be assumed routinely that $\gcd^A(N, a^{\downharpoonleft}) = I$ since any non-trivial common divisor of N and $a^{\downharpoonleft}$ can be canceled. This implies that $X/A \cong \vec{\mathbb{Z}}/\vec{\mathbb{Z}}N$ and $[X : A] = \det N$. It may be assumed further, without loss of generality, that $N = \operatorname{diag}(d_1, \ldots, d_k)$ with $1 \le d_{i-1} \mid d_i$ for $i = 2, \ldots, k$. In this case there is a clear description of the quotient X/A, namely

$$X/A = \mathbb{Z}(d_1^{-1}a_1 + A) \oplus \ldots \oplus \mathbb{Z}(d_k^{-1}a_k + A), \qquad \operatorname{ord}(d_i^{-1}a_i + A) = d_i.$$

Any almost completely decomposable group can be obtained in this form. The completely decomposable group A can be written as $A = \tau_1 v_1 \oplus \ldots \oplus \tau_r v_r$

(r for rank) where the τ_i are rational groups. We call $\mathcal{V} = \{v_1, \ldots, v_r\}$ a *conditioned basis* of A. Given a conditioned basis $\mathcal{V}$ of A we can write

$$a_i = m_{i1}v_1 + \ldots + m_{ir}v_r,$$

and we obtain a *coordinate matrix* $M = [m_{ij}]$ such that $a^\downharpoonleft = Mv^\downharpoonleft$. The basis $\mathcal{V}$ can be chosen such that M is an integral matrix, $M \in \mathbb{M}_{k\times r}(\mathbb{Z})$ ([FM92], [DO93], [BM98a], [Mad00, Chapter 11]). Since X contains A as a subgroup of finite index, there is a positive integer e such that $eX \subset A$. Given such an e the conditioned basis $\mathcal{V}$ of A may be chosen to be *e-conditioned*, or simply an *e-basis*, which means that for a prime divisor p of e, either τ_i is p-divisible or $\gcd^A(p, v_i) = 1$. This can be done in such a way that the coordinate matrix M of $a^\downharpoonleft$ with respect to $\mathcal{V}$ remains integral.

From now on we assume that $eX \subset A$, $\mathcal{V} = \{v_1, \ldots, v_r\}$ is an e-basis of A, $A = \tau_1 v_1 \oplus \ldots \oplus \tau_r v_r$, and $a^\downharpoonleft = Mv^\downharpoonleft$ for some coordinate matrix $M \in \mathbb{M}_{k\times r}(\mathbb{Z})$. If it so happens that $\gcd^A(p, v_i) = 1$ for all $i \in \{1, \ldots, r\}$ and all prime divisors of e, then $\gcd^A(N, Mv^\downharpoonleft) = \operatorname{gcld}(N, M)$ ([BM98a, Proposition 5.5], [Mad00, Corollary 11.3.4]). By $\operatorname{gcld}(N, M)$ we denote the greatest common left divisor of the matrices N, M. It is a standard fact, apparently known since the dawn of matrix theory, that the greatest common (left or right) divisor of two integral matrices (of compatible size) exists and can be effectively computed by elementary matrix operations ([Mac46], [BM98b], [BM98a], [Mad00, Section 11.2]). In the following we will restrict ourselves to p-local almost completely decomposable groups, i.e. the case where e is a p-power, say $e = p^d$. In this case a p-divisible critical type τ_i creates a p-divisible direct summand $\tau_i v_i$ of X and these summands are uninteresting for most purposes. We therefore assume that the groups under consideration are *p-reduced*, meaning that there are no non-trivial p-divisible subgroups. In this situation a p^d-basis $\mathcal{V} = \{v_1, \ldots, v_r\}$ is the same as a p-basis and it means that $\gcd^A(p, v_i) = 1$ for each i, or equivalently, $1/p \notin \tau_i$.

The so-called Purification Lemma ([BM98a, Lemma 4.1], [Mad00, Lemma 11.4.1]) will be a convenient and necessary tool. For easy reference we state here the part that will be needed later.

LEMMA 2.1 (Purification Lemma). *Assume that $A = B \oplus C$ is an arbitrary torsion-free abelian group of arbitrary rank, $a^\downharpoonleft = b^\downharpoonleft + c^\downharpoonleft$, where $b^\downharpoonleft \in B^\downharpoonleft$ and $c^\downharpoonleft \in C^\downharpoonleft$. Let*

$$X = A + \overrightarrow{\mathbb{Z}} N^{-1} a^\downharpoonleft$$

be a finite essential extension of A with $\gcd^A(N, a^\downharpoonleft) = I$. Let $N_B = \gcd^A(N, c^\downharpoonleft)$. Then N_B is non-singular and

$$B_*^X = B + \overrightarrow{\mathbb{Z}} N_B^{-1} b^\downharpoonleft \quad \textit{with} \quad \gcd^A(N_B, b^\downharpoonleft) = I.$$

3. Decomposition of rigid local almost completely decomposable groups: basics. The following simple but essential decomposition result applies to rigid almost completely decomposable groups.

LEMMA 3.2. *Let X be any group with an indecomposable decomposition and the property that every direct summand is fully invariant. Then the indecomposable decomposition of X is unique.*

Proof. Let $X = X_1 \oplus \ldots \oplus X_m$ be an indecomposable decomposition of X. Let $X = Y \oplus Z$ with Y indecomposable. Then, Y being fully invariant, $Y = (Y \cap X_1) \oplus \ldots \oplus (Y \cap X_k)$. Since Y is indecomposable we may assume without loss of generality that $Y \subset X_1$. But X_1 is also fully invariant and hence $X_1 = Y \oplus (X_1 \cap Z)$. Since X_1 is indecomposable and $Y \neq 0$ we conclude that $X_1 = Y$. Hence the summands of two indecomposable decompositions must be pairwise equal. ∎

The decompositions of certain p-local crq-groups are a simple and canonical matter and the promising start of an inductive procedure. Recall first that a group is *clipped* if it has no completely decomposable summand, and that every almost completely decomposable group X has a decomposition $X = X_{\mathrm{cd}} \oplus X_{\mathrm{cl}}$ such that X_{cd} is completely decomposable and X_{cl} is clipped, that the summand X_{cd} is unique up to isomorphism, and the summand X_{cl} is unique up to near-isomorphism. This is the so-called *Main Decomposition* of X ([MV95, Theorem 2.3], [Mad00, Theorem 9.2.7]).

LEMMA 3.3. *Let X be a finite essential extension of a completely decomposable group A by a p-primary cyclic group*, *i.e.*, *X is torsion-free and contains A such that X/A is a cyclic p-group. Assume that X has a regulating regulator. If $X = X_{\mathrm{cd}} \oplus X_{\mathrm{cl}}$ is the Main Decomposition of X*, *then X_{cl} is indecomposable.*

Proof. We first show that $X/\mathrm{R}(X)$ is also a cyclic p-group. Since X has a regulating regulator, we have $\mathrm{R}(X) = \sum_{\varrho \in \mathrm{T}_{\mathrm{cr}}(X)} X(\varrho) = X_{\mathrm{cd}} \oplus \mathrm{R}(X_{\mathrm{cl}})$. Because A has finite index in X, we have $A(\tau) = A \cap X(\tau)$ for any type τ and

$$\frac{X(\tau)}{A(\tau)} = \frac{X(\tau)}{A \cap X(\tau)} \cong \frac{A + X(\tau)}{A} \leq \frac{X}{A}.$$

It follows that $A \subset \mathrm{R}(X)$ and $X(\tau) = A(\tau)^X_*$. Hence X/A maps epimorphically onto $X/\mathrm{R}(X) \cong X_{\mathrm{cl}}/\mathrm{R}(X_{\mathrm{cl}})$ and these isomorphic groups are also p-primary and cyclic. Every direct summand Y of X has a regulating regulator as well. Suppose that $X_{\mathrm{cl}} = Y \oplus Z$. Then $\mathrm{R}(X_{\mathrm{cl}}) = \mathrm{R}(Y) \oplus \mathrm{R}(Z)$ and hence

$$\frac{X_{\mathrm{cl}}}{\mathrm{R}(X_{\mathrm{cl}})} = \frac{Y}{\mathrm{R}(Y)} \oplus \frac{Z}{\mathrm{R}(Z)}.$$

Since $X_{\mathrm{cl}}/\mathrm{R}(X_{\mathrm{cl}})$ is p-primary and cyclic, either $Y/\mathrm{R}(Y) = 0$ or $Z/\mathrm{R}(Z) = 0$. Suppose without loss of generality that $Y = \mathrm{R}(Y)$. Then Y is completely decomposable and since X_{cl} is clipped, Y must be zero. Thus X_{cl} is indecomposable. ∎

A torsion-free group is *block-rigid* if its critical typeset is an antichain. If X is a p-local cyclic essential extension of a block-rigid completely decomposable group given in a standard description, then one can read off whether the group is clipped (equivalently, indecomposable). The following result is included for the sake of completeness and easy reference.

LEMMA 3.4. *Let $A = \tau_1 v_1 \oplus \ldots \oplus \tau_r v_r$ be a block-rigid completely decomposable group and $a = m_1 v_1 + \ldots + m_r v_r \in A$ for integers m_i. Furthermore, let p be a prime and d a positive integer.*

(1) *Assume that $X = A + \mathbb{Z}p^{-d}a$ is clipped. Then A is rigid, $p\tau_i \neq \tau_i$, and $\gcd(p^d, m_i) < p^d$.*

(2) *Assume that A is rigid, $\gcd^A(p, v_i) = 1$, and $\gcd(p^d, m_i) < p^d$. Then $X = A + \mathbb{Z}p^{-d}a$ is clipped and indecomposable.*

Proof. (1) It is well known and easy to see that A must be rigid and that no critical type can be p-divisible. It is even simpler to see that $\tau_i v_i$ is a direct summand of X if m_i is divisible by p^d.

(2) The subgroup $\tau_i v_i$ is a direct summand of X if and only if $X = \tau_i v_i \oplus (\bigoplus_{j\neq i} \tau_j v_j)^X_*$. This is the case if and only if $[(\bigoplus_{j\neq i} \tau_j v_j)^X_* : (\bigoplus_{j\neq i} \tau_j v_j)] = p^d$. By the Purification Lemma this is equivalent to $\gcd^A(p^d, m_j v_j) = p^d$. The hypothesis that $\gcd^A(p, v_s) = 1$ implies that $\gcd^A(p^d, m_j v_j) = \gcd(p^d, m_j)$ and by the second hypothesis $\gcd(p^d, m_j) \neq p^d$. Hence X is clipped and therefore indecomposable by Lemma 3.3. ∎

4. Decomposition of rigid local almost completely decomposable groups: algorithms. We wish to study direct decompositions of a rigid p-local almost completely decomposable group X. In order to eliminate an obvious completely decomposable direct summand we assume that X is p-reduced, meaning that X has no non-trivial p-divisible subgroups. The problem of finding the unique indecomposable decomposition of X amounts to studying partitions

$$\mathrm{T}_{\mathrm{cr}}(X) = T_1 \cup \ldots \cup T_n \tag{4.5}$$

with the property that

$$X = \bigoplus_{i=1}^{n} \Big(\bigoplus_{\varrho \in T_i} X(\varrho) \Big)^X_* . \tag{4.6}$$

For convenience we will call such a partition a *decomposition partition* of X. Eventually we need to find the unique *indecomposable decomposition partition* for which each summand $(\bigoplus_{\varrho\in T_i} X(\varrho))^X_*$ is indecomposable. Listing all the partitions of $\mathrm{T}_{\mathrm{cr}}(X)$ is a finite albeit lengthy task. It is then necessary to test a given partition for being a decomposition partition of X. When we do this, the particular structure of X must come into play and some representation of the group must be assumed upon which a procedure can be based. We only assume that X is a finite essential p-primary extension of a rigid completely decomposable group A. Then $A = \bigoplus_{\varrho\in \mathrm{T}_{\mathrm{cr}}(X)} A(\varrho)$, $A(\tau) = A \cap X(\tau)$, and $X(\tau) = A(\tau)^X_*$. We can now say that (4.5) is a decomposition partition of X if and only if

$$\prod_{i=1}^{n}\Big[\Big(\bigoplus_{\varrho\in T_i} A(\varrho)\Big)^X_* : \Big(\bigoplus_{\varrho\in T_i} A(\varrho)\Big)\Big] = [X : A]. \tag{4.7}$$

In order to compute the indices in (4.7) further specifications are necessary. Naturally we assume that the group X is given in standard description:

$$\begin{aligned}
&X = A + \overrightarrow{\mathbb{Z}} N^{-1} M v^{\downharpoonleft},\ A = \tau_1 v_1 \oplus \ldots \oplus \tau_r v_r,\\
&\mathcal{V} = \{v_1, \ldots, v_r\} \text{ is a } p\text{-basis of } A,\ \text{i.e., } \gcd{}^A(p, v_i) = 1,\\
&N = \operatorname{diag}(p^{d_1}, \ldots, p^{d_k}) \text{ with } 1 \le d_1 \le \ldots \le d_k =: d,\\
&M \in \mathbb{M}_{k\times r}(\mathbb{Z}),\ \text{and } \gcd{}^A(N, M v^{\downharpoonleft}) = I.
\end{aligned} \tag{4.8}$$

Under these conditions $\gcd^A(N, Mv^{\downharpoonleft}) = \operatorname{gcld}(N, M)$ and the greatest common divisor can be computed by column reduction of the augmented matrix $[N \mid M]$ ([BM98a, Theorem 3.3], [Mad00, Section 11.2]). The Purification Lemma now provides a criterion for testing partitions of $\mathrm{T}_{\mathrm{cr}}(X)$ for being decomposition partitions. In fact, (4.5) is a decomposition partition of X if and only if

$$(\det N_1)\ldots(\det N_n) = \det N \quad \text{where} \tag{4.9}$$
$$N_i = \operatorname{gcld}(N, M[\downharpoonleft\{1,\ldots,r\}\setminus T_i]).$$

Note that A need not be the regulator of X and that the Purification Lemma works in this less restricted situation. The indecomposable decomposition partition is found when the summands have no proper decomposition partitions.

We will first deal with the case of uniform groups treated by Dugas–Oxford ([DO93]). This is, by definition, the case $d := d_1 = \ldots = d_k$. It was shown in [DO93] and [MMN00] that in case $N = p^d I_k$ is a scalar matrix it may be assumed that M is of the form

(4.10) $M = [E|F]$,

$$E = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}, \qquad F = \begin{bmatrix} m_{1,k+1} & \dots & m_{1r} \\ m_{2,k+1} & \dots & m_{2r} \\ \dots & \dots & \dots \\ m_{k,k+1} & \dots & m_{kr} \end{bmatrix}.$$

In the following we do not need that N is a scalar matrix, but we do need the special form (4.10) of M. So let $N = \operatorname{diag}(p^{d_1}, \dots, p^{d_k})$ as before. To simplify the discourse we note that each row of the coordinate matrix M determines and corresponds to a generator of X, namely the generator

$$x_i = p^{-d_i}(v_i + m_{i,k+1}v_{k+1} + \dots + m_{ir}v_r),$$

and we call $\operatorname{supp}(i) = \{\tau_j : m_{ij} \not\equiv 0 \bmod p^{d_i}\}$ the *support* of x_i or of the ith row $M[i\rfloor]$ of M. We can write

$$x_i = p^{-d_i} \sum\{m_{ij}v_j : \tau_j \in \operatorname{supp}(i)\} + \sum\{p^{-d_i}m_{ij}v_j : \tau_j \notin \operatorname{supp}(i)\},$$

where $\sum\{p^{-d_i}m_{ij}v_j : \tau_j \notin \operatorname{supp}(i)\} \in A$. Thus x_i can be replaced by $x_i - \sum\{p^{-d_i}m_{ij}v_j : \tau_j \notin \operatorname{supp}(i)\}$, so that we may assume without loss of generality that $\tau_j \notin \operatorname{supp}(i)$ if and only if $m_{ij} = 0$.

The following theorem contains the observations that lead to an efficient determination of the decomposition partition of X.

THEOREM 4.11. *Let $A = \tau_1 v_1 \oplus \dots \oplus \tau_r v_r$ be a rigid completely decomposable group and p a prime such that $1/p \notin \tau_i$ for each i. Assume that $X = A + \overrightarrow{\mathbb{Z}}N^{-1}Mv^{\rfloor}$ with*

$$N = \operatorname{diag}(p^{d_1}, \dots, p^{d_k}), \qquad M = \begin{bmatrix} 1 & 0 & \dots & 0 & m_{1,k+1} & \dots & m_{1r} \\ 0 & 1 & \dots & 0 & m_{2,k+1} & \dots & m_{2r} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & m_{k,k+1} & \dots & m_{kr} \end{bmatrix}.$$

Then the following hold:

(1) *For $i \in \{1, \dots, k\}$, the group $L_i := (\bigoplus\{\tau_j v_j : \tau_j \in \operatorname{supp}(i)\})_*^X$ is pure, fully invariant in X, and a cyclic extension of $A_i = \bigoplus\{\tau_j v_j : \tau_j \in \operatorname{supp}(i)\}$ as follows:*

$$L_i = A_i + \mathbb{Z}p^{-d_i}\Big(v_i + \sum\{m_{ij}v_j : \tau_j \in \operatorname{supp}(i),\ j > k\}\Big).$$

(2) *The subgroups L_i are indecomposable, and the indecomposable partition of X is the finest partition $\mathrm{T}_{\mathrm{cr}}(X) = T_1 \cup \dots \cup T_n$ with the property that for each $i \in \{1, \dots, k\}$ there is $j \in \{1, \dots, n\}$ such that $\operatorname{supp}(i) \subset T_j$.*

Proof. (1) We use the Purification Lemma to compute the purification L_i. Accordingly, the greatest common divisor $\operatorname{gcld}(\operatorname{diag}(p^{d_1}, \dots, p^{d_k}), M[\rfloor \mathrm{T}_{\mathrm{cr}}(X) \setminus \operatorname{supp}(i)])$ must be computed. This is done by reducing the augmented matrix

$$[\operatorname{diag}(p^{d_1}, \ldots, p^{d_k}) \mid M[\downharpoonright \mathrm{T}_{\mathrm{cr}}(X) \setminus \operatorname{supp}(i)]$$

to column reduced form. The augmented matrix contains the columns of the identity matrix except for its ith column. In row i all entries are divisible by p^{d_i}. It is clear that the augmented matrix reduces by column transformations to (zero columns omitted)

$$\begin{array}{cl}\begin{bmatrix} 1 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & p^{d_i} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & \ldots & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow \text{row } i \\ \\ \\ \end{array}\end{array}$$

The claim follows from the Purification Lemma by omitting redundant generators.

(2) The subgroup L_i is indecomposable by Lemma 3.3 and fully invariant in X. Note that

$$X = \bigoplus \Big\{\tau_j v_j : \tau_j \notin \bigcup_i \operatorname{supp}(i)\Big\} \oplus \sum_{j=1}^{k} L_j. \tag{4.12}$$

Let $X = X_1 \oplus \ldots \oplus X_n$ be the indecomposable decomposition of X. Then $L_i = L_i \cap X_1 \oplus \ldots \oplus L_i \cap X_n$ and since L_i is indecomposable, there is $j \in \{1, \ldots, n\}$ such that $L_i \subset X_j$. Hence the groups L_i are distributed over the indecomposable summands X_j in some way. If X_j contains no group L_i, then the critical type(s) of X_j is outside $\bigcup_i \operatorname{supp}(i)$ and therefore X_j is one of the summands of $\bigoplus\{\tau_j v_j : \tau_j \notin \bigcup_i \operatorname{supp}(i)\}$. If X_j does contain certain groups L_i, then $X_j = \sum\{L_i : L_i \subset X_j\}$ by (4.12). The claim is now clear. ■

The following MAPLE V program implements an algorithm for finding the decomposition partition, hence the indecomposable decomposition of a rigid uniform group. We first observe that we only need to distinguish between entries of $M[i\downharpoonright]$ divisible by p^{d_i} and entries not divisible by p^{d_i}. Replacing the entries of row i divisible by p^{d_i} by 0 and those not divisible by p^{d_i} by 1, we obtain a $(0,1)$-matrix, again called M. In the procedure `findec` row i of the $(0,1)$-matrix M is multiplied by i to obtain a matrix L. Initially the value of `findec` is set to $[0, \ldots, 0]$ meaning that initially each $\tau_i v_i$ is allowed to be in a different summand of X. By successively adding rows of the matrix L to `findec` one can tell whether there is an overlap of the supports of the current row i with previous rows. If so the value i replaces previous values indicating that the ith row placed certain critical types into one part of the decomposition partition. The first procedure `diag1tok` is called by the main program `findec`. We will explain the procedure below with an example.

```
################################################################
# Create a k by k diagonal matrix with diagonal entries 1,...,k
################################################################
 diag1tok := proc(k)
  local diag1tok, i, j;
  diag1tok := matrix(k,k):
    for i from 1 to k
      do
        for j from 1 to k
          do
            if i<>j then diag1tok[i,j] := 0
            else diag1tok[i,j] := i
            fi;
          od;
        od;
    diag1tok;
    end;
################################################################
# Procedure to determine the direct decomposition of a group
# with a (0,1)-representing matrix M = [I|F] of size k x r.
# The output is a list of non-negative integers 0,1,...,k of size r.
# 0 in position j means that tau_j v_j is a direct summand.
# i > 0 means all tau_j v_j marked by the entry i belong to the same
# indecomposable summand.
################################################################
findec := proc(M)
  local findec, k, r, L, i, entries, j, s;
  k := rowdim(M);                       # row dimension
  r := coldim(M);                       # column dimension
  L := multiply(diag1tok(k),M);
  findec := matrix(1,r,0);              # list of zeros
  for i from 1 to k
    do
      findec := matadd(findec,submatrix(L,i..i,1..r));
      findec;                           # see result
      entries := convert(findec,set);
      for j from i+1 to 2*i-1
        do
          if member(j,entries) then
            for s from 1 to r
              do
                if findec[1,s] = j-i or findec[1,s] = j
```

```
                then findec[1,s] := i fi;
              od;
          fi;
        od;
    od;
  findec;
            end;
```

We will now execute the procedure on an example. Let

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix M has $k = 5$ rows and $r = 12$ columns. Let $X(M)$ denote the rigid group belonging to M. The matrix L is obtained by multiplying row i of M by i:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 3 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

The initial value of `findec` is

$$\texttt{findec} = [0,0,0,0,0,0,0,0,0,0,0,0].$$

The interpretation is that the critical typeset is partitioned into singletons. The i-loop starts with $\boxed{i=1}$. The first row of L is added to `findec` to produce

$$\texttt{findec} = [1,0,0,0,0,0,0,1,1,1,1,0] \quad \text{with} \quad \texttt{entries} = \{0,1\}.$$

The j-loop runs from $j = i+1 = 2$ to $j = 2i-1 = 1$, hence nothing is executed. So far the decomposition partition consists of seven singletons and the set $\{\tau_1, \tau_8, \tau_9, \tau_{10}, \tau_{11}\}$.

Next $\boxed{i=2}$. The value of `findec` becomes

$$\texttt{findec} = [1,2,0,0,0,2,2,1,1,1,1,0] \quad \text{with} \quad \texttt{entries} = \{0,1,2\}.$$

The only value of the j-loop is $j = 3$, which is not one of the entries and the value of `findec` is not changed.

$\boxed{i=3}$. Then

$$\texttt{findec} = [1,2,3,0,0,2,2,4,1,4,4,0] \quad \text{with} \quad \texttt{entries} = \{0,1,2,3,4\}.$$

The j-loop starts with $j = i+1 = 4$, which is one of the entries. It signals that there is an overlap of supports of the first and third row of M. The values $j-i=1$ and $j=4$ in `findec` are replaced by $i=3$ to produce

$$\texttt{findec} = [3,2,3,0,0,2,2,3,3,3,3,0].$$

The second and last value $j=5$ is not an entry, so nothing more is done. At this point the decomposition partition consists of three singletons, $\{\tau_1,\tau_3,\tau_8,\tau_9,\tau_{10},\tau_{11}\}$, and $\{\tau_2,\tau_6,\tau_7\}$.

$\boxed{i=4}$. Then

$$\texttt{findec} = [3,2,3,4,0,6,2,3,3,3,3,0] \quad \text{with} \quad \texttt{entries} = \{0,2,3,4,6\}.$$

The j-loop starts with $j=i+1=5$, which is not one of the entries. The next value $j=6$ is an entry and the values $j-i=2$ and $j=6$ in `findec` are replaced by $i=4$ to produce

$$\texttt{findec} = [3,4,3,4,0,4,4,3,3,3,3,0].$$

The third and last value $j=6$ is not an entry, so nothing more is done. Now the decomposition partition consists of two singletons, $\{\tau_1,\tau_3,\tau_8,\tau_9,\tau_{10},\tau_{11}\}$, and $\{\tau_2,\tau_4,\tau_6,\tau_7\}$.

$\boxed{i=5}$. Then

$$\texttt{findec} = [3,4,3,4,5,4,4,3,3,3,3,5] \quad \text{with} \quad \texttt{entries} = \{3,4,5\}.$$

The j-loop starts with $j=i+1=6$ and ends with $j=2i-1=9$ and none of these is an entry, so the final value and output of the procedure is

$$\texttt{findec} = [3,4,3,4,5,4,4,3,3,3,3,5],$$

which says that the group $X(M)$ has the indecomposable decomposition

$$X(M) = (\tau_1 v_1 \oplus \tau_3 v_3 \oplus \tau_8 v_8 \oplus \tau_9 v_9 \oplus \tau_{10} v_{10} \oplus \tau_{11} v_{11})_*^X \oplus (\tau_2 v_2 \oplus \tau_4 v_4 \oplus \tau_6 v_6 \oplus \tau_7 v_7)_*^X \oplus (\tau_5 v_5 \oplus \tau_{12} v_{12})_*^X.$$

Using the random matrix generator of MAPLE V we produced representing matrices of the form $M=[I\,|\,F] \in \mathbb{M}_{k\times r}(\{0,1\})$ where I is the identity matrix of size k, and computed the proportion of indecomposable groups among these. The following table contains the results.

k	5	5	5	5	5	5	5	5
r	6	7	8	9	10	15	20	20
No. of groups	100	100	100	100	100	100	100	200
No. of indecomposables	2	25	36	56	77	79	59	119

It appears that the highest probability of indecomposable groups (around 80%) is obtained for rank 15 when X/A has 5 generators. The generators of X/A may be viewed as clamps that hold blocks (summands) of A together. One would therefore expect that the probability of generating indecomposable groups increases with increasing numbers of generators of X/A. This is confirmed empirically by the next table.

k	10	10	10	10	10	10	10	10	10	10
r	12	14	16	20	30	40	50	100	150	200
No. of groups	50	50	50	50	50	50	50	50	50	50
No. of indecomposables	4	26	44	49	49	49	50	48	42	42

QUESTION 4.13. *What is the probability that a randomly chosen* $(0,1)$-*matrix* F *results in an indecomposable group* $X = A + \overrightarrow{\mathbb{Z}}N^{-1}[I_k|F]v^{\downharpoonleft}$?

The following theorem displays another method for decomposing a rigid group that is based on concepts and results from [MM00]. We will omit proofs but some notation is needed. By a "line" of a matrix we mean either a row or a column.

A $(0,1)$-matrix M is said to be in *block diagonal form* or a *block diagonal matrix* if $M = \operatorname{diag}(M_1, \dots, M_k)$ where the blocks M_i are rectangular and all entries outside of the blocks are 0. A $(0,1)$-matrix M, not of size 1 by 1, without 0-lines, is said to be *indecomposed* if it has only the trivial block diagonal form, i.e., $M = \operatorname{diag}(M)$. A $(0,1)$-matrix of size 1 by 1 is indecomposed by convention. $(0,1)$-matrices with 0-lines are considered decomposed.

A matrix A is *permutation equivalent* to a matrix B if there are permutation matrices P, Q, such that $A = PBQ$. A $(0,1)$-matrix is called *indecomposable* if it is not permutation equivalent to a decomposed matrix. The *support* M^* of the $(0,1)$-matrix M is the submatrix obtained by deleting all 0-lines of M. A $(0,1)$-matrix M is called *totally decomposed* if its support $M^* = \operatorname{diag}(M_1, \dots, M_k)$ is a block diagonal matrix with all blocks M_i indecomposable.

There are *total decompositions* of $(0,1)$-matrices relative to permutation equivalence, i.e., block diagonal forms of the support such that the diagonal blocks are indecomposable. It is shown in [MM00] that the size of the indecomposable diagonal blocks of those total decompositions are unique up to rearrangements of the diagonal blocks.

With each matrix $M = [m_{ij}]$ over some ring we associate a $(0,1)$-matrix $M' = [m'_{ij}]$ by agreeing that $m'_{ij} = 1$ if $m_{ij} \neq 0$ and $m'_{ij} = 0$ otherwise. The original matrix M is called *decomposed*, *decomposable*, etc., if the corresponding matrix M' has the respective properties. Since block decompositions are only rearrangements of rows and columns we are allowed to take tacitly the corresponding $(0,1)$-matrix M' instead of the matrix M.

THEOREM 4.14. *Let* $d_1, \dots, d_k$ *be natural numbers. Let* $A = \tau_1 v_1 \oplus \dots$ $\dots \oplus \tau_r v_r$ *be a rigid completely decomposable group and* p *a prime such* *that $1/p \notin \tau_i$ for each i. Assume that $X = A + \overrightarrow{\mathbb{Z}} N^{-1}[E \mid M]v^{\downarrow}$ with $N = \operatorname{diag}(p^{d_1}, \ldots, p^{d_k})$ and unit matrix E. Then the group X is directly decomposable if and only if the matrix M is decomposable. Moreover, the block diagonal structure of the permutation equivalence class of M displays the finest decomposition of X. More precisely, let $\operatorname{diag}(B_1, \ldots, B_h)$ represent the finest diagonal block structure of the permutation equivalence class of M, where B_i is of size m_i by n_i. Then a finest direct decomposition of X has the form $X = \bigoplus_{i=1}^{h} X_i$, where $\operatorname{rk}(X_i) = m_i + n_i$. The regulator quotient of X_i has rank m_i.*

A matrix with entries in $\{0, 1\}$ is called *doubly ordered* if both its rows and its columns are in the lexicographic order determined by $1 > 0$. In [MM00] it is shown that it is enough to double order the matrix M' in order to obtain the matrix with the finest diagonal block structure in the permutation equivalence class of M. There are fast algorithms that double order a matrix.

The matrix M above can be doubly ordered manually by first arranging the columns according to order, then the rows, and then one more time the columns. This produces the doubly ordered matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

The block structure of this matrix shows that the group X corresponding to M has an indecomposable decomposition into indecomposable summands of rank 6, 4, and 2 as before. Without keeping track of the column exchanges, the precise decomposition cannot be told.

We have not tried to analyse the two algorithms (`findec`, double ordering) by way of quantitative efficiency. Double ordering does more than just finding decompositions and therefore is likely to require more computer resources. On the other hand, `findec` uses MAPLE which obscures its actual workings. Our interest was in finding quick and reliable ways to obtain decompositions and indecomposable groups for testing and finding theoretical results.

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