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HARMONIC ANALYSIS FOR SPINORS ON REAL HYPERBOLIC SPACES

BY

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Abstract. We develop the L^2 harmonic analysis for (Dirac) spinors on the real hyperbolic space $H^n(\mathbb{R})$ and give the analogue of the classical notions and results known for functions and differential forms: we investigate the Poisson transform, spherical function theory, spherical Fourier transform and Fourier transform. Very explicit expressions and statements are obtained by reduction to Jacobi analysis on $L^2(\mathbb{R})$. As applications, we describe the exact spectrum of the Dirac operator, study the Abel transform and derive explicit expressions for the heat kernel associated with the spinor Laplacian.

1. Introduction. As is well known since Flensted–Jensen's and Koornwinder's works in the 70's (see [Koo84] for a survey), L^2 spherical harmonic analysis on Riemannian noncompact symmetric spaces G/K of rank one, i.e. on the hyperbolic spaces $H^n(\mathbb{F})$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$) and $H^2(\mathbb{O})$, can be reduced to (noneuclidean) L^2 harmonic analysis on the real line $\mathbb{R}$. Namely, under natural identifications, spherical functions can be realized as Jacobi functions and the spherical Fourier transform of radial functions coincides with a Jacobi transform. This reduction has many applications, for instance in solving classical P.D.E.'s (see e.g. [ADY96] and [AMPS], which both deal with the wider class of Damek–Ricci spaces).

As concerns (homogeneous) vector bundles over hyperbolic spaces, one may wonder if the link between spherical harmonic analysis and Jacobi analysis subsists. The second author ([Ped97], see [Ped98a, Ped98b] for an exposition) showed that this is actually true for the bundle of differential forms over real hyperbolic spaces $H^n(\mathbb{R})$, although the theory is noticeably more complicated.

Because of the strong similarities between Clifford algebras and exterior algebras, one could expect such a phenomenon to be valid also in the case of the spinor bundle. This article proposes a thorough study of the harmonic

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analysis (i.e. of the spherical harmonic analysis with its various applications) of spinors on real hyperbolic spaces, by reduction to Jacobi analysis. Unsurprisingly, the main statements are very close to the ones obtained for the exterior bundle.

Let us come more precisely to the contents of our article. In Section 2 we recall the basic structure of the real hyperbolic spaces $H^n(\mathbb{R}) \simeq G/K = \mathrm{Spin}_e(n,1)/\mathrm{Spin}(n)$ and the definition of the spinor bundle $\Sigma H^n(\mathbb{R})$ over these manifolds. In particular, a spinor field on $H^n(\mathbb{R})$ will be regarded as a function $f : G \to \mathbb{C}^{2^{[n/2]}}$ which has the K-equivariance property

$$f(xk) = \tau_n(k)^{-1} f(x) \qquad (x \in G,\ k \in K),$$

where τ_n is the classical complex spin representation.

In Section 3, we introduce an additional structure for the semisimple Lie group G and use its representation theory to state the abstract Plancherel Theorem for spinors on $H^n(\mathbb{R})$, i.e. the decomposition of the space of L^2 spinors into G-irreducible modules.

In Section 4 we recall first some standard properties of the Dirac operator acting on spinors. Then, by introducing an appropriate Poisson transform, we are able to construct eigenfunctions (eigenspinors) for the generators of the algebra of differential operators acting on smooth sections of $\Sigma H^n(\mathbb{R})$. Our result leads very simply to the determination of the L^2 spectrum of the Dirac operator.

Sections 5 and 6 constitute the central part of our work. First, we construct all spherical functions associated with the spinor bundle $\Sigma H^n(\mathbb{R})$ and express them very explicitly in terms of the so-called hypergeometric Jacobi functions. As a consequence, we show that the spherical Fourier analysis of radial functions on $\Sigma H^n(\mathbb{R})$ can be reduced (in some sense) to the Jacobi analysis of functions defined on $\mathbb{R}$. In particular, this leads to the inversion and Plancherel formulas for the spherical transform, as well as to the Paley–Wiener Theorem (and its analogue in the Schwartz setting).

These results are used first in Section 7 to obtain an analytic version of the Plancherel Theorem stated in Section 3, via the study of the associated Fourier transform. Then we underline in Section 8 the very simple link existing between the spherical Fourier transform and the Abel transform, which allows us to express the latter and its inverse in terms of fractional differential operators. As a simple and immediate application, we solve the heat equation associated with the square of the Dirac operator on spinors (the spinor Laplacian) and find explicit expressions for the heat kernel.

The abstract Plancherel Theorem and the Dirac spectrum were obtained first in [Bun91]. However, as was noticed also by other authors ([Bär98], [GS99]), Bunke's paper contains unfortunately an important error in the case of n even and we think it is worthwhile to correct and restate here these fundamental results. In the same article, Bunke studied Poisson transforms as well, but without realizing them as eigenspinors for the algebra of invariant differential operators, while this observation is the key step of our work. Spherical functions and heat kernels associated with spinors on $H^n(\mathbb{R})$ were determined in [Cam92] and [CH96], but calculations are carried out in the dual compact symmetric space (the sphere $\mathbb{S}^n$) and, as regards spherical functions, the derivation given in [CH96] is incomplete in the odd case. More precise comparisons will be made throughout the text.

2. Basic definitions and notations. Let $n \geq 2$ be an integer. The *n-dimensional real hyperbolic space* is the manifold

$$H^n(\mathbb{R}) := \{x \in \mathbb{R}^{n+1} : L(x,x) = -1,\ x_{n+1} > 0\},$$

where L is the Lorentz form

$$L(x,y) = y_1x_1 + \ldots + y_nx_n - y_{n+1}x_{n+1}.$$

The real hyperbolic space is usually viewed as the rank one symmetric space G'/K' of noncompact type, where $G' = \mathrm{SO}_e(n,1)$ is the identity connected component of the group of orientation preserving isometries of $H^n(\mathbb{R})$ and $K' \simeq \mathrm{SO}(n)$ is the maximal compact subgroup of G which stabilizes the base point $o := (0,\ldots,0,1) \in \mathbb{R}^{n+1}$. However, for our purpose, we shall use instead the equivalent realization $H^n(\mathbb{R}) = G/K$, where $G = \mathrm{Spin}_e(n,1)$ and $K = \mathrm{Spin}(n)$ are the classical two-fold coverings of G' and K' respectively (when $n \geq 3$, both coverings are universal).

Denote by $\mathfrak{g} = \mathfrak{spin}(n,1) \simeq \mathfrak{so}(n,1)$ and $\mathfrak{k} = \mathfrak{spin}(n) \simeq \mathfrak{so}(n)$ the Lie algebras of the groups G and K respectively. As usual, write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the Cartan decomposition of $\mathfrak{g}$. The tangent space $T_o(G/K) \simeq \mathfrak{g}/\mathfrak{k} \simeq \mathfrak{p}$ of $G/K = H^n(\mathbb{R})$ at the origin $o = eK$ will be identified with the vector space $\mathbb{R}^n$ by means of the isomorphism

$$\mathfrak{p} \to \mathbb{R}^n, \qquad X = \begin{pmatrix} 0_n & x \\ {}^t x & 0 \end{pmatrix} \mapsto x, \tag{2.1}$$

so that the Euclidean inner product

$$g_o(X,Y) := {}^t y x \tag{2.2}$$

on $\mathfrak{p}$ induces by translations a G-invariant Riemannian metric g on $H^n(\mathbb{R})$, for which this manifold has constant sectional curvature $\kappa \equiv -1$.

Further results on the structure of $H^n(\mathbb{R})$ and related Lie groups and Lie algebras will be given throughout the paper, when needed.

Let us recall the following well known facts about homogeneous vector bundles on the symmetric spaces G/K (see [Wal73, §5.2]). If τ is a unitary finite-dimensional representation of K on a Hilbert space V_τ, sections of the associated homogeneous vector bundle $G \times_K V_\tau$ over G/K can be regarded as *functions of (right) type* τ *on* G, i.e. as functions $f : G \to V_\tau$ which satisfy the K-equivariance condition

$$f(xk) = \tau(k)^{-1} f(x) \quad (x \in G,\ k \in K).$$

We shall denote by $\Gamma(G,\tau)$ the space of functions of type τ on G, and shall replace the prefix Γ by C, C^∞, C_{c}, C_{c}^∞, L^2 or $\mathcal{S}$ if the sections encountered are respectively continuous, smooth, continuous with compact support, smooth with compact support, square integrable or of Schwartz type. Denoting by $\Gamma(E)$ the sections of any vector bundle E, we then have

$$\Gamma(G \times_K V_\tau) \equiv \Gamma(G,\tau), \tag{2.3}$$

and the induced representation $\mathrm{Ind}_K^G \tau$ of G acts on $\Gamma(G,\tau)$ by left translations:

$$\{(\mathrm{Ind}_K^G \tau)(x) f\}(y) = f(x^{-1}y).$$

Using the standard Hermitian inner product on the complex vector space V_τ, if dx is any Haar measure on G, the inner product on $L^2(G,\tau)$ is given by

$$(f_1, f_2) = \int_G dx\, (f_1(x), f_2(x))_{V_\tau}. \tag{2.4}$$

(An explicit measure on G will not be of use until Section 5.)

Our last reminders concern the definition of the bundle of spinors over the real hyperbolic space. All the material can be found in all generality e.g. in [LM89, Ch. I, §§5–6 and Ch. II, §§1–3].

It is well known that, as any Riemannian symmetric space of noncompact type, the real hyperbolic space $H^n(\mathbb{R})$ is contractible and admits therefore a unique *spin structure*. This means that the principal $\mathrm{SO}(n)$-bundle P' of oriented orthonormal frames on $H^n(\mathbb{R})$ admits a two-fold cover P which is a principal $\mathrm{Spin}(n)$-bundle on $H^n(\mathbb{R})$ such that the respective actions of the structure groups on these two bundles commute with the covering maps. One can then form the *spinor bundle* over $H^n(\mathbb{R})$, whose sections are called *spinors* (or *spinor fields*). Since $P' \simeq G'$ and $P \simeq G$, the spinor bundle is defined as the associated homogeneous vector bundle

$$\Sigma H^n(\mathbb{R}) \equiv G \times_K V_{\tau_n}, \tag{2.5}$$

where τ_n is the classical (unitary) *complex spin representation* of $K \simeq \mathrm{Spin}(n)$ on $V_{\tau_n} = \mathbb{C}^{2^{[n/2]}}$ ($[\cdot]$ stands for the integer part), which is irreducible when n is odd and splits into two irreducible components (the positive and negative *half-spin representations*)

$$\tau_n = \tau_n^+ \oplus \tau_n^-$$

when n is even (see also [Kna96, Ch. V, Problems 24–35], for a comprehensive construction of these representations). Of course, we have in that case

$$\Sigma H^n(\mathbb{R}) = \Sigma^+ H^n(\mathbb{R}) \oplus \Sigma^- H^n(\mathbb{R}) \quad \text{with} \quad \Sigma^\pm H^n(\mathbb{R}) \equiv G \times_K V_{\tau_n^\pm}$$

(n even). We remark that V_{τ_n} is nothing but the space of spinors associated with the complexification of the vector space $\mathfrak{p} \simeq T_o(G/K)$. Recalling (2.3), we shall use the resulting identifications

$$\begin{aligned} \Gamma(\Sigma H^n(\mathbb{R})) &\equiv \Gamma(G, \tau_n), \\ \Gamma(\Sigma^\pm H^n(\mathbb{R})) &\equiv \Gamma(G, \tau_n^\pm) \quad (n \text{ even}). \end{aligned} \tag{2.6}$$

If V is a real vector space, $V_{\mathbb{C}}$ and V^* will denote its complexification and dual space respectively.

3. The abstract Plancherel Theorem for $L^2(G, \tau_n)$. By "abstract Plancherel Theorem" for $L^2(G, \tau_n)$, we understand the decomposition of this space into G-irreducible modules via an appropriate Fourier transform. Since for any unitary finite-dimensional representation τ of K, one has

$$L^2(G, \tau) \simeq \{L^2(G) \otimes V_\tau\}^K, \tag{3.1}$$

(the upper index K indicates restriction to the subspace of $(R \otimes \tau)(K)$-invariant vectors in $L^2(G) \otimes V_\tau$ where R denotes the restriction of the right regular representation of G on $L^2(G)$), the abstract Plancherel Theorem for $L^2(G, \tau_n)$ comes from the one for $L^2(G)$, which is known (see e.g. [KS71, §11] or [Kna86, Ch. XIII]).

Let us begin with some more structure for the real rank one semisimple Lie group $G = \mathrm{Spin}_e(n, 1)$ and its associated Lie algebra $\mathfrak{g}$. The material can be found for instance in [Kna86]. Consider the element

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathfrak{p}. \tag{3.2}$$

Then $\mathfrak{a} := \mathbb{R}e_1$ is a Cartan subspace in $\mathfrak{p}$, and the corresponding analytic Lie subgroup A of G is parametrized by $a_t := \exp(te_1)$. Let $\alpha \in \mathfrak{a}^*$ be defined by $\alpha(te_1) = t$. Then $R(\mathfrak{g}, \mathfrak{a}) = \{\pm\alpha\}$ is a restricted root system of $(\mathfrak{g}, \mathfrak{a})$ with positive subsystem $R^+(\mathfrak{g}, \mathfrak{a}) = \{\alpha\}$ and corresponding Weyl group $W = W(\mathfrak{g}, \mathfrak{a}) \simeq \{\pm \mathrm{id}\}$. Later on, we shall often use the identification

$$\mathfrak{a}^*_{\mathbb{C}} \xrightarrow{\simeq} \mathbb{C}, \qquad \lambda\alpha \mapsto \lambda. \tag{3.3}$$

Let $\mathfrak{n} = \mathfrak{g}_\alpha$ be the (unique) positive root subspace and N the corresponding (abelian) analytic subgroup of G. Then the classical half-sum of roots in $R^+(\mathfrak{g}, \mathfrak{a})$ (counted with their multiplicities) reduces simply to $\varrho = (n-1)\alpha/2$, and will always be considered as a scalar via (3.3). Recall the classical decompositions

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}, \qquad G = KAN \qquad \text{(Iwasawa)},$$
$$G = K\{a_t : t \geq 0\}K \qquad \text{(Cartan)}.$$

Let $M \simeq \mathrm{Spin}(n-1)$ be the centralizer of A in K, with Lie algebra $\mathfrak{m}$, and $P := MAN$ be the usual (minimal) parabolic subgroup of G associated with A and N. Given $\sigma \in \widehat{M}$ (if H is a Lie group, $\widehat{H}$ stands for its unitary dual) and $\lambda \in \mathfrak{a}^*_{\mathbb{C}} \simeq \mathbb{C}$, the action

$$(\sigma \otimes e^{i\lambda} \otimes \mathbf{1})(ma_t n) = e^{i\lambda t}\sigma(m)$$

defines a representation of P on the space V_σ. Then the *principal series representation* $\pi_{\sigma,\lambda} := \mathrm{Ind}_P^G(\sigma \otimes e^{i\lambda} \otimes \mathbf{1})$ of G acts on the space

$$\begin{aligned} H_{\sigma,\lambda} &:= L^2(G, \sigma \otimes e^{i\lambda} \otimes \mathbf{1}) \\ &= \{f : G \to V_\sigma : f(xma_t n) = e^{-(i\lambda+\varrho)t}\sigma(m)^{-1}f(x),\ f|_K \in L^2(K)\} \end{aligned}$$

by left translations: $\pi_{\sigma,\lambda}(g)f(h) = f(g^{-1}h)$. With this parametrization, $\pi_{\sigma,\lambda}$ is unitary if and only if λ is real. Moreover, unitary principal series representations are always irreducible, except maybe when $\lambda = 0$. Recall also that, as a K-module, $H_{\sigma,\lambda}$ is isomorphic (for any λ) to the space $L^2(K,\sigma)$ of square integrable functions of type σ on K, on which $\pi_{\sigma,\lambda}$ acts by the rule

$$\pi_{\sigma,\lambda}(x)f(k) = e^{-(i\lambda+\varrho)H(x^{-1}k)}f(\underline{k}(x^{-1}k)) \qquad (x \in G,\ k \in K) \tag{3.4}$$

if we write $x = \underline{k}(x)e^{H(x)}\underline{n}(x)$ according to the Iwasawa decomposition $G = KAN$.

Let us introduce the subspaces $\mathfrak{q} \subset \mathfrak{p}$ and $\mathfrak{l} \subset \mathfrak{k}$ by the following decompositions into orthogonal components for the Killing form on $\mathfrak{g}$:

$$\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{q}, \qquad \mathfrak{k} = \mathfrak{m} \oplus \mathfrak{l}. \tag{3.5}$$

Note that $\mathfrak{q}$ (resp. $\mathfrak{l}$) is the orthogonal projection of $\mathfrak{n}$ onto $\mathfrak{p}$ (resp. $\mathfrak{k}$) along $\mathfrak{p}$ (resp. $\mathfrak{k}$) and that $\mathfrak{q} \simeq \mathfrak{l} \simeq \mathbb{R}^{n-1}$. On the other hand, it is well known that $G/P \simeq K/M \simeq \mathbb{S}^{n-1}$ (the $(n-1)$-dimensional sphere in $\mathbb{R}^n$) is the boundary of the hyperbolic space G/K. Thus, similarly to (2.5), the space of spinors over K/M can be realized as the homogeneous vector bundle ([1])

$$\varSigma\mathbb{S}^{n-1} \equiv K \times_M V_{\sigma_{n-1}},$$

where σ_{n-1} is the complex spin representation of $M \simeq \mathrm{Spin}(n-1)$ on $V_{\sigma_{n-1}} \simeq \mathbb{C}^{2^{[(n-1)/2]}}$, i.e. on the spinor space associated with the complexified tangent space $\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}}$ of $K/M = \mathbb{S}^{n-1}$ at the origin, which can be thought of either as $\mathfrak{l}_{\mathbb{C}}$, $\mathfrak{q}_{\mathbb{C}}$ or $\mathbb{C}^{n-1}$. Notice that $\sigma_{n-1} = \tau_{n-1}$, but we shall continue

([1]) Recall that the circle S^1, unlike higher dimensional spheres, admits *two* inequivalent spin structures (see e.g. [DT86]). The identification given here corresponds to the so-called nontrivial one.

in practice to use a different notation to emphasize the group that acts (M or K).

The following result is easy and well known (use e.g. [GW98, Propositions 6.2.3–6.2.4 and Theorems 8.1.3–8.1.4]).

LEMMA 3.1. (I) *Let n be even. Then $\tau_n^{\pm}|_M = \sigma_{n-1} \in \widehat{M}$.*

(II) *Let n be odd. Then $\tau_n|_M = \sigma_{n-1}^{+} \oplus \sigma_{n-1}^{-}$, with $\sigma_{n-1}^{\pm} \in \widehat{M}$.*

In both cases, the decomposing factors occur with multiplicity one.

We now state the abstract Plancherel Theorem for $L^2(G, \tau_n)$.

THEOREM 3.2. *Set $\mathbb{R}_+^* =]0, \infty[$.*

(I) *Let n be even. Then there exists a Plancherel measure $d\nu_{\sigma_{n-1}}$ on $\mathbb{R}_+^*$ such that*

$$L^2(G, \tau_n) = L^2(G, \tau_n^+) \oplus L^2(G, \tau_n^-) \simeq \mathbb{C}^2 \otimes \int_{\mathbb{R}_+^*}^{\oplus} d\nu_{\sigma_{n-1}}(\lambda)\, H_{\sigma_{n-1},\lambda}.$$

(II) *Let n be odd. Then there exist Plancherel measures $d\nu_{\sigma_{n-1}^+}$ and $d\nu_{\sigma_{n-1}^-}$ on $\mathbb{R}_+^*$ such that*

$$L^2(G, \tau_n) \simeq \int_{\mathbb{R}_+^*}^{\oplus} d\nu_{\sigma_{n-1}^+}(\lambda)\, H_{\sigma_{n-1}^+,\lambda} \ \oplus \int_{\mathbb{R}_+^*}^{\oplus} d\nu_{\sigma_{n-1}^-}(\lambda)\, H_{\sigma_{n-1}^-,\lambda}.$$

Proof. As mentioned in §1, the statement of this theorem was given first in [Bun91, Proposition 2.2], with an important error in the even case. However we think that there is no need to repeat entirely Bunke's proof and shall give only its sketch, indicating where the error occurred.

First step. For all real rank one semisimple connected Lie groups G, Harish-Chandra's Plancherel Theorem can be stated as follows (see e.g. [KS71, §11]): for each $\sigma \in \widehat{M}$, there exists a Plancherel measure $d\nu_\sigma(\lambda)$ on $\mathfrak{a}^*$ such that

$$L^2(G) \simeq \int_{W\backslash(\widehat{M}\times\mathfrak{a}^*)}^{\oplus} d\nu_\sigma(\lambda)\, H_{\sigma,\lambda} \,\widehat{\otimes}\, H_{\sigma,\lambda}^* \oplus \bigoplus_{\pi\in\widehat{G}_d} d_\pi\, H_\pi \,\widehat{\otimes}\, H_\pi^*, \tag{3.6}$$

where:

• $\widehat{G}_d$ is the subset of *discrete series representations* (π, H_π) of G, i.e. of (equivalence classes of) irreducible unitary representations of G having L^2 matrix coefficients, and the number d_π is the formal degree of π. Note that a general result by Harish-Chandra says that the group G admits a discrete series if and only if G and K have equal (complex) rank. In our setting, this can occur only in the case of n even.

• The symbol $\int^{\oplus}$ stands for a "direct integral of Hilbert spaces" (see e.g. [Wal92, Ch. 14]).

• Integrating over $W\backslash(\widehat{M}\times\mathfrak{a}^*)$ means that one has to take care of the possible unitary equivalences between principal series representations induced by the action of the Weyl group W on them (see [Bru56, Theorem 7.2]).

• The symbol $\widehat{\otimes}$ means Hilbert completion.

Second step. Using (3.1) and (3.6), we get

$$L^2(G,\tau_n) \simeq \int_{W\backslash(\widehat{M}\times\mathfrak{a}^*)}^{\oplus} d\nu_\sigma(\lambda)\, H_{\sigma,\lambda} \mathbin{\widehat{\otimes}} \mathrm{Hom}_K(H_{\sigma,\lambda},V_{\tau_n}) \oplus \bigoplus_{\pi\in\widehat{G}_d} d_\pi\, H_\pi \mathbin{\widehat{\otimes}} \mathrm{Hom}_K(H_\pi,V_{\tau_n}),$$

where $\mathrm{Hom}_K(U_1,U_2)$ denotes the space of K-intertwining operators between the K-modules U_1 and U_2.

Third step. On the one hand, one uses Lemma 3.1 and Frobenius reciprocity to decide whether the spaces $\mathrm{Hom}_K(H_{\sigma,\lambda},V_{\tau_n})$ are nonzero. Note that the nontrivial element $w=m'M$ of the Weyl group $W\simeq M'/M$ (M' denotes the normalizer of A in K) acts on the relevant principal series representations by the rules

$$\begin{aligned} w\cdot\pi_{\sigma_{n-1},\lambda} &= \pi_{\sigma_{n-1},-\lambda} \quad (n \text{ even});\\ w\cdot\pi_{\sigma^{\pm}_{n-1},\lambda} &= \pi_{\sigma^{\mp}_{n-1},-\lambda} \quad (n \text{ odd}). \end{aligned} \tag{3.7}$$

On the other hand, if $\pi\in\widehat{G}_d$, the space $\mathrm{Hom}_K(H_\pi,V_{\tau_n})$ is always trivial: by Theorem 6 of [Thi77], it can be made clear that the two irreducible unitary representations considered by Bunke when n is even do not belong indeed to a discrete series of G, but only to *limits of discrete series* (see e.g. [Kna96, §XII.7] for a definition), whose matrix coefficients are $L^{2+\varepsilon}$, with $\varepsilon>0$. Therefore these representations cannot occur in the Plancherel formula. ∎

REMARK 3.3. The Plancherel measures will be calculated explicitly in Section 5. The definition and study of the Fourier transform realizing the equivalences in the theorem is postponed to Section 7.

4. Eigenfunctions for the algebra $\mathbb{D}(G,\tau_n)$. In this section, we exhibit eigenfunctions for the algebra $\mathbb{D}(G,\tau_n)$ of left-invariant differential operators acting on the space $C^\infty(G,\tau_n)$ of smooth spinors on $H^n(\mathbb{R})$, by applying Poisson transforms to spinors defined on the boundary of the manifold. An immediate corollary will be the determination of the L^2 spectrum of the Dirac operator.

4.1. *The Dirac operator.* For any $xK\in G/K=H^n(\mathbb{R})$, denote by $\gamma_{xK}: T^*_{xK}(G/K)\to \mathrm{End}\,\Sigma_{xK}(G/K)$ the natural extension of the Clifford multiplication to fibres of the spinor bundle. Let also $\nabla: C^\infty(\Sigma(G/K))\to C^\infty(T^*(G/K)\otimes\Sigma(G/K))$ denote the covariant derivative acting on smooth sections of $\Sigma(G/K)$, which can be thought of either as the connection naturally induced by the canonical connection on the K-principal bundle $G \to G/K$ (take $\mathfrak{p}$ as horizontal space), or as the lift of the Levi-Civita connection to the spin bundle.

As usual (see e.g. [LM89, §II.5]), the *Dirac operator* acting on $C^\infty(\Sigma(G/K))$ is defined to be the composition of the covariant derivative ∇ with the Clifford multiplication. Namely, if $(e_j)_{j=1}^n$ is any orthonormal frame on G/K with dual coframe $(e_j^*)_{j=1}^n$, for any $f \in C^\infty(\Sigma(G/K))$ we have the local expression

$$D\, f(xK) = \sum_{j=1}^{n} \gamma_{xK}(e_j^*(xK))\, \nabla_{e_j} f(xK). \tag{4.1}$$

It is well known that D is an elliptic, essentially self-adjoint and G-invariant first-order differential operator. By the Cartan decomposition $G = (\exp \mathfrak{p})K$, it is therefore completely determined by its expression at the origin $xK = eK$. Proceeding e.g. as in [BOS94, §3], we see that D has the following expression in terms of the identification (2.6): if $f \in C^\infty(G, \tau_n)$, then for any $x \in G$,

$$D\, f(x) = \sum_{j=1}^{n} \gamma_j\, f(x : e_j), \tag{4.2}$$

where $(e_j)_{j=1}^n$ is an orthonormal basis of $\mathfrak{p}$ with dual basis $(e_j^*)_{j=1}^n$, and where we have set $\gamma_j := \gamma_{eK}(e_j^*)$ and used the classical Harish-Chandra notation for derivation: if $f \in C^\infty(G)$, then

$$\begin{aligned} &f(X_1 \ldots X_m : x : Y_1 \ldots Y_n) := \\ &\quad \frac{\partial}{\partial r_1}\bigg|_0 \cdots \frac{\partial}{\partial r_l}\bigg|_0 \frac{\partial}{\partial s_1}\bigg|_0 \cdots \frac{\partial}{\partial s_m}\bigg|_0 f(\exp r_1X_1 \ldots \exp r_lX_l \cdot x \cdot \exp s_1Y_1 \ldots \exp s_mY_m) \end{aligned} \tag{4.3}$$

for any $X_1, \ldots, X_l, Y_1, \ldots, Y_m \in \mathfrak{g}$.

Finally, let us say a few words about the matrices γ_j. By the general theory of Clifford algebras, they must be $2^{[n/2]} \times 2^{[n/2]}$ complex matrices that satisfy the usual Clifford relations

$$\gamma_j\gamma_k + \gamma_k\gamma_j = -2\delta_{jk}I \quad (j, k = 1, \ldots, n). \tag{4.4}$$

Of course, these relations do not determine the γ_j's in a unique way. However, it is well known that for n even, all sets $\{\gamma_j\}$, $\{\gamma_j'\}$ satisfying (4.4) are conjugate (i.e., $\gamma_j' = P\gamma_jP^{-1}$ for some invertible matrix P), while for n odd there are two conjugacy classes of gamma matrices, corresponding to the two inequivalent spaces of spinors of the complex Clifford algebra. Thus, in what follows, we can and will take for the γ_j's the following classical (generalized) *Pauli matrices*. Since they are constructed by induction, let us add temporarily a parameter and denote by $\gamma_j^{(n)}$ the matrix acting on the spinor vector space $\mathbb{C}^{2^{[n/2]}}$. Define

$$\gamma_1^{(1)} := i, \quad \gamma_1^{(2)} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_2^{(2)} := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{4.5}$$

as well as, for $n \geq 3$ odd,

$$\gamma_1^{(n)} := \begin{pmatrix} i\,I & 0 \\ 0 & -i\,I \end{pmatrix}, \quad \gamma_j^{(n)} := \gamma_{j-1}^{(n-1)} \quad (j = 2, \ldots, n), \tag{4.6}$$

and, for $n \geq 4$ even,

$$\gamma_1^{(n)} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \gamma_j^{(n)} := \begin{pmatrix} 0 & \gamma_{j-1}^{(n-1)} \\ \gamma_{j-1}^{(n-1)} & 0 \end{pmatrix} \quad (j = 2, \ldots, n). \tag{4.7}$$

Remark 4.1. A general result (or, more simply in our case, (4.5) and (4.7)) shows that D interchanges positive and negative spinors when n is even. Moreover, D^2 is positive, hence has real positive L^2 spectrum. This implies easily that the spectrum of D must be symmetric about zero in the even case. Actually, this property is also true in the odd case, since it is easy to see that there is an element in K which acts as minus identity on $\mathfrak{p}$. Hence it maps D to $-D$ and if λ is in the spectrum of D, so is $-\lambda$.

Remark 4.2. Let E_{jk} denote the $n \times n$ matrix having (j,k) entry 1, and zeros elsewhere. It is well known that the real Lie algebra $\mathfrak{so}(n)$ is generated by the elements $E_{jl} - E_{lj}$ for $j < l$. Then, using the identification $\mathfrak{k} = \mathfrak{spin}(n) \simeq \mathfrak{so}(n)$, it is easily checked that the differential of τ_n is given by

$$\tau_n(E_{jl} - E_{lj}) = -\tfrac{1}{2}\gamma_j^{(n)}\gamma_l^{(n)} = -\tfrac{1}{4}[\gamma_j^{(n)}, \gamma_l^{(n)}]. \tag{4.8}$$

Moreover, it is very well known (see e.g. [GW98, Propositions 6.2.3–6.2.4]) that the highest weight of τ_n equals $\frac{1}{2}\varepsilon_1 + \ldots + \frac{1}{2}\varepsilon_l$ if $n = 2l+1$, whereas the highest weight of $\tau_n^{\pm}$ equals $\frac{1}{2}\varepsilon_1 + \ldots + \frac{1}{2}\varepsilon_{l-1} \pm \frac{1}{2}\varepsilon_l$ if $n = 2l$ (here, the ε_j's denote the standard linear functionals on the Cartan subalgebra of $\mathfrak{so}(n)$).

4.2. *The algebra $\mathbb{D}(G, \tau_n)$.* We recall here Gaillard's explicit description of the algebra $\mathbb{D}(G, \tau_n)$, as stated in [Gai93, Theorem 1].

If n is even, denote by $e^{\pm} : C^\infty(G, \tau_n) \to C^\infty(G, \tau_n^{\pm})$ the canonical projections and set $\mathbb{D}^0 := \mathbb{C}e^+ \oplus \mathbb{C}e^-$.

Theorem 4.3. (I) *Let n be even. Then $\mathbb{D}(G, \tau_n)$ is the algebra generated over $\mathbb{D}^0$ by D subject to the sole relation $D = e^+De^- + e^-De^+$, and $\mathbb{D}(G, \tau_n^{\pm}) \simeq \mathbb{C}[e^{\pm}D^2e^{\pm}] = \mathbb{C}[D^2|_{C^\infty(G, \tau_n^{\pm})}]$.*

(II) *Let n be odd. Then $\mathbb{D}(G, \tau_n) \simeq \mathbb{C}[D]$.*

Remark 4.4. Notice that, when n is even, $\mathbb{D}(G, \tau_n)$ is *not* commutative. On the other hand, one expects the commutativity of $\mathbb{D}(G, \tau_n)$ (n odd) and

$\mathbb{D}(G,\tau_n^\pm)$ (n even) since it is known that $\mathbb{D}(G,\tau)$ has this important property for any $\tau \in \widehat{K}$ when $G/K = H^n(\mathbb{R})$ (see [Dei90]).

Remark 4.5. A maybe more concrete description of the algebra when n is even was kindly communicated to us by P.-Y. Gaillard: $\mathbb{D}(G,\tau_n)$ is isomorphic to the subalgebra of the algebra of 2×2 matrices with coefficients in $\mathbb{C}[z]$ which is generated by the elements

$$E^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad d^+ = \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}, \quad d^- = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}.$$

4.3. *Poisson transforms.* Let $\tau \in \widehat{K}$ denote alternately $\tau_n^\pm$ if n is even and τ_n is n is odd. Write $\widehat{M}(\tau)$ for the set of representations $\sigma \in \widehat{M}$ that occur in the restriction of τ to M with multiplicity one (according to Lemma 3.1), so that, by Frobenius reciprocity,

$$\dim \mathrm{Hom}_K(C^\infty(K,\sigma), V_\tau) = \dim \mathrm{Hom}_M(V_\tau, V_\sigma) = 1.$$

Fix $\sigma \in \widehat{M}(\tau)$ and define, for any $\lambda \in \mathbb{C}$,

$$\begin{aligned} &P_\sigma^\tau : C^\infty(G, \sigma\otimes e^{i\lambda}\otimes \mathbf{1}) \to V_\tau \\ &f \mapsto \sqrt{\frac{\dim\tau}{\dim\sigma}} \int_K dk\, \tau(k) f(k), \end{aligned} \tag{4.9}$$

where dk is the Haar measure on K normalized by $\int_K dk = 1$. Then P_σ^τ is a generator of the one-dimensional space

$$\mathrm{Hom}_K(C^\infty(G,\sigma\otimes e^{i\lambda}\otimes\mathbf{1}), V_\tau) \simeq \mathrm{Hom}_K(C^\infty(K,\sigma), V_\tau).$$

This last identity shows that P_σ^τ does not depend on the parameter λ and is therefore well defined. We shall also handle its adjoint operator:

$$J_\sigma^\tau : V_\tau \to C^\infty(K,\sigma), \qquad \xi \mapsto \sqrt{\frac{\dim\tau}{\dim\sigma}}\, P_\sigma(\tau(\cdot)^{-1}\xi), \tag{4.10}$$

where P_σ denotes the orthogonal projection of V_τ onto its σ-isotypical component $V_\tau(\sigma) \simeq V_\sigma$. With our normalization, J_σ^τ is an isometry (use Schur's orthogonality relations) and $P_\sigma^\tau \circ J_\sigma^\tau = \mathrm{Id}_\tau$ (use Schur's lemma).

Now, for $f \in C^\infty(G,\sigma\otimes e^{i\lambda}\otimes\mathbf{1})$, set

$$\phi_{\sigma,\lambda}^\tau(x) f := P_\sigma^\tau \circ \pi_{\sigma,\lambda}(x^{-1}) f \qquad (\forall x \in G). \tag{4.11}$$

Then the map

$$C^\infty(G,\sigma\otimes e^{i\lambda}\otimes\mathbf{1}) \to C^\infty(G,\tau), \qquad f \mapsto \phi_{\sigma,\lambda}^\tau(\cdot) f,$$

is continuous, linear and G-equivariant, and we shall call it the *Poisson transform on* $C^\infty(G,\sigma\otimes e^{i\lambda}\otimes\mathbf{1})$.

Remark 4.6. The definition of the Poisson transform given above is consistent in different or more general settings and is inspired by various previous works: [Bun91], [Olb94], [Ven94], [Ped98a], etc.

Remark 4.7. A simple calculation (using (3.4)) yields the following integral formula:

$$\phi^{\tau}_{\sigma,\lambda}(x)f = \sqrt{\frac{\dim\tau}{\dim\sigma}} \int_K dk\, e^{-(i\lambda+\varrho)H(xk)}\tau(k)\, f(\underline{k}(xk)).$$

When n is even and $\tau = \tau_n^{\pm}$, then $\sigma = \sigma_{n-1}$ and we introduce the shorter notations

$$P^{\pm} = P^{\tau_n^{\pm}}_{\sigma_{n-1}}, \qquad J^{\pm} = J^{\tau_n^{\pm}}_{\sigma_{n-1}}, \qquad \phi^{\pm}(\lambda,\cdot) = \phi^{\tau_n^{\pm}}_{\sigma_{n-1},\lambda}. \tag{4.12}$$

(Note the simplification $\sqrt{\dim\tau/\dim\sigma}\, P_\sigma = \mathrm{Id}$.) Here, the Poisson transform maps spinors defined on the boundary of $H^n(\mathbb{R})$ into half-spinors defined on the interior (see the discussion preceding Lemma 3.1). Similarly, when n is odd and $\tau = \tau_n$, then $\sigma = \sigma^{\pm}_{n-1}$ and we put

$$P_{\pm} = P^{\tau_n}_{\sigma^{\pm}_{n-1}}, \qquad J_{\pm} = J^{\tau_n}_{\sigma^{\pm}_{n-1}}, \qquad \phi_{\pm}(\lambda,\cdot) = \phi^{\tau_n}_{\sigma^{\pm}_{n-1},\lambda}. \tag{4.13}$$

In this case, the Poisson transform maps half-spinors defined on the boundary of $H^n(\mathbb{R})$ into spinors defined on the interior.

We can now give the main result of this section.

Theorem 4.8. *The Poisson transforms $\phi^{\pm}(\lambda,\cdot)$ and $\phi_{\pm}(\lambda,\cdot)$ are respectively eigenfunctions for (the generators of) the algebras $\mathbb{D}(G,\tau_n^{\pm})$ (n even) and $\mathbb{D}(G,\tau_n)$ (n odd). More precisely,*

(i) *if n is even and $f \in C^\infty(G,\sigma_{n-1}\otimes e^{i\lambda}\otimes \mathbf{1})$, then for all $x \in G$,*

$$D\,\phi^{\pm}(\lambda,x)f = \pm i\lambda\,\phi^{\mp}(\lambda,x)f, \tag{4.14}$$

$$D^2\phi^{\pm}(\lambda,x)f = \lambda^2\,\phi^{\pm}(\lambda,x)f; \tag{4.15}$$

(ii) *if n is odd and $f_\pm \in C^\infty(G,\sigma^{\pm}_{n-1}\otimes e^{i\lambda}\otimes \mathbf{1})$, then for all $x \in G$,*

$$D\,\phi_{\pm}(\lambda,x)f_\pm = \pm\lambda\,\phi_{\pm}(\lambda,x)f_\pm, \tag{4.16}$$

$$D^2\phi_{\pm}(\lambda,x)f_\pm = \lambda^2\,\phi_{\pm}(\lambda,x)f_\pm. \tag{4.17}$$

Proof. Let us start with case (ii). Choose the basis elements e_i of $\mathfrak{p}$ to be the ones corresponding to the canonical orthonormal basis of $\mathbb{R}^n$ via the identification (2.1). (Note that this is consistent with (3.2).) For brevity, set $\tau = \tau_n$ and $\sigma^{\pm} = \sigma^{\pm}_{n-1}$. Then, by (4.2), for any $f_\pm \in C^\infty(G,\sigma^{\pm}\otimes e^{i\lambda}\otimes \mathbf{1})$,

$$\begin{aligned} D\,\phi_\pm(\lambda,x)f_\pm &= \sum_{j=1}^{n} \gamma_j\,\phi_\pm(\lambda,x:e_j)f_\pm \\ &= \sum_{j=1}^{n} \gamma_j\,P_\pm \circ \pi_{\sigma^\pm,\lambda}(-e_j)\circ \pi_{\sigma^\pm,\lambda}(x^{-1})f_\pm. \end{aligned}$$

Now, fix $v \in V_\tau$. Using the following properties of adjoint operators:

$$\gamma_j^* = -\gamma_j, \qquad \pi_{\sigma,\lambda}(-e_j)^* = \pi_{\sigma,\overline{\lambda}}(e_j),$$

we get

$$\begin{aligned} (4.18)\qquad & (D\,\phi_\pm(\lambda,x)f_\pm, v)_{V_\tau} \\ &= -\sum_{j=1}^{n}(\pi_{\sigma^\pm,\lambda}(x^{-1})f_\pm, \pi_{\sigma^\pm,\overline{\lambda}}(e_j)J_\pm\gamma_j v)_{L^2(K,\sigma^\pm)} \\ &= (\pi_{\sigma^\pm,\lambda}(x^{-1})f_\pm, T_\pm v)_{L^2(K,\sigma^\pm)}, \end{aligned}$$

where $T_\pm : V_\tau \to H_{\sigma^\pm,\overline{\lambda}}|_K \simeq L^2(K,\sigma^\pm)$ is the operator defined by the last equality.

LEMMA 4.9. *$T_\pm \in \mathrm{Hom}_K(V_\tau, H_{\sigma^\pm,\overline{\lambda}})$, i.e.*

$$\pi_{\sigma^\pm,\overline{\lambda}}(k)\circ T_\pm \circ \tau(k^{-1}) = T_\pm \qquad (\forall k \in K).$$

Proof. Recall the fundamental identity satisfied by the Clifford map

$$(4.19)\qquad \tau(k)\circ\gamma(X^*)\circ\tau(k^{-1}) = \gamma(\mathrm{Ad}^\vee(k)X^*) \qquad (X^* \in \mathfrak{p}^*),$$

where $k \mapsto \mathrm{Ad}^\vee(k)$ is the contragredient representation of the (restricted) adjoint representation $k \mapsto \mathrm{Ad}(k)|_{\mathfrak{p}}$. Writing

$$\mathrm{Ad}^\vee(k)e_j^* = \sum_{l=1}^{n}\mathrm{Ad}(k^{-1})_{jl}\,e_l^*$$

and using (4.19) as well as the intertwining property for $J_\pm$, we have

$$\begin{aligned} &\pi_{\sigma^\pm,\overline{\lambda}}(k)\circ T_\pm\circ\tau(k^{-1}) \\ &\quad= -\sum_j \pi_{\sigma^\pm,\overline{\lambda}}(k)\circ\pi_{\sigma^\pm,\overline{\lambda}}(e_j)\circ J_\pm\circ\gamma_j\circ\tau(k^{-1}) \\ &\quad= -\sum_j \pi_{\sigma^\pm,\overline{\lambda}}(k)\circ\pi_{\sigma^\pm,\overline{\lambda}}(e_j)\circ\pi_{\sigma^\pm,\overline{\lambda}}(k^{-1})\circ\pi_{\sigma^\pm,\overline{\lambda}}(k)\circ J_\pm\circ\gamma_j\circ\tau(k^{-1}) \\ &\quad= -\sum_j \pi_{\sigma^\pm,\overline{\lambda}}(\mathrm{Ad}(k)e_j)\circ J_\pm\circ\tau(k)\circ\gamma_j\circ\tau(k^{-1}) \\ &\quad= -\sum_j \pi_{\sigma^\pm,\overline{\lambda}}(\mathrm{Ad}(k)e_j)\circ J_\pm\circ\gamma(\mathrm{Ad}^\vee(k)e_j^*) \end{aligned}$$

$$= -\sum_{j,l,m} \mathrm{Ad}(k)_{lj}\, \pi_{\sigma^\pm,\overline{\lambda}}(e_l) \circ J_\pm \circ \mathrm{Ad}(k^{-1})_{jm}\, \gamma_m$$

$$= -\sum_l \pi_{\sigma^\pm,\overline{\lambda}}(e_l) \circ J_\pm \circ \gamma_l = T_\pm,$$

which gives the statement. ∎

Since $\mathrm{Hom}_K(V_\tau, H_{\sigma^\pm,\overline{\lambda}})$ is one-dimensional, it follows from the lemma that $T_\pm = c_\pm J_\pm$ for some complex number $c_\pm$ (depending on λ) to be determined. To find $c_\pm$ we calculate $T_\pm v$ at $k = e$. On the one hand,

$$T_\pm v(e) = c_\pm J_\pm v(e) = \sqrt{2}\, c_\pm P_{\sigma^\pm} v. \tag{4.20}$$

On the other hand, if we recall (3.4), the definition of $T_\pm$ implies

$$\begin{aligned} T_\pm v(e) &= -\sum_j \frac{d}{ds}\bigg|_0 \pi_{\sigma^\pm,\overline{\lambda}}(\exp se_j) \circ J_\pm\, \gamma_j\, v(e) \\ &= -\sqrt{2} \sum_j \frac{d}{ds}\bigg|_0 e^{-(i\overline{\lambda}+\varrho)H(\exp -se_j)} P_{\sigma^\pm} \circ \tau(\underline{k}(\exp -se_j)^{-1})\gamma_j\, v. \end{aligned}$$

We now need a technical result. Denote by E_{jk} the $(n+1)\times(n+1)$ matrix having (j,k) entry 1, and zeros elsewhere.

Lemma 4.10. *We have*

$$\frac{d}{ds}\bigg|_0 H(\exp -se_j) = \begin{cases} -e_1 & \text{if } j = 1, \\ 0 & \text{if } j = 2,\dots,n; \end{cases} \tag{4.21}$$

$$\frac{d}{ds}\bigg|_0 \underline{k}(\exp -se_j)^{-1} = \begin{cases} 0 & \text{if } j = 1, \\ E_{j1} - E_{1j} & \text{if } j = 2,\dots,n. \end{cases} \tag{4.22}$$

Proof. The left-hand side in (4.21) is the projection of $-e_j$ onto $\mathfrak{a}$ along $\mathfrak{k}\oplus\mathfrak{n}$, hence the result. The left-hand side in (4.22) is the projection of e_j onto $\mathfrak{k}$ along $\mathfrak{a}\oplus\mathfrak{n}$. Recall the (isomorphic) subspaces $\mathfrak{q}\subset\mathfrak{p}$ and $\mathfrak{l}\subset\mathfrak{k}$ defined by (3.5). Let $\overline{\mathfrak{n}} = \theta(\mathfrak{n})$, where θ is the Cartan involution on $\mathfrak{g}$ associated with $\mathfrak{k}$. Then a classical result says that $\mathfrak{l}\oplus\mathfrak{q} = \mathfrak{n}\oplus\overline{\mathfrak{n}}$. More precisely, let us introduce the basis elements

$$\begin{aligned} X_j &:= E_{1j} - E_{j1} + E_{n+1,j} + E_{j,n+1}, \\ \theta X_j &= E_{1j} - E_{j1} - E_{n+1,j} - E_{j,n+1}, \\ Y_j &:= \frac{X_j - \theta X_j}{2} = E_{n+1,j} + E_{j,n+1} = e_j, \\ Z_j &:= \frac{X_j + \theta X_j}{2} = E_{1j} - E_{j1} \end{aligned} \tag{4.23}$$

of $\mathfrak{n}$, $\overline{\mathfrak{n}}$, $\mathfrak{q}$ and $\mathfrak{l}$ respectively ($j = 2,\dots,n$). Therefore, if $j\neq 1$, then e_j decomposes uniquely as $e_j = X_j - Z_j$ and the result follows. ∎

Applying the lemma and using successively equations (4.8) and (4.4), we see that

$$\begin{aligned} T_{\pm}v(e) &= -\sqrt{2}\,P_{\sigma^{\pm}}\Big\{(i\overline{\lambda}+\varrho)\gamma_1+\sum_{j=2}^{n}\tau(E_{j1}-E_{1j})\gamma_j\Big\}v \\ &= -\sqrt{2}\,i\overline{\lambda}\,P_{\sigma^{\pm}}\,\gamma_1\,v. \end{aligned}$$

If we write $v=v_{+}+v_{-}$ according to the decomposition $V_\tau=V_{\sigma^+}\oplus V_{\sigma^-}$ and use the explicit expression of γ_1 given in (4.6), we obtain finally

$$T_{\pm}v(e)=\pm\sqrt{2}\,\overline{\lambda}\,v_{\pm}.$$

Comparison with (4.20) yields $c_{\pm}=\pm\overline{\lambda}$. Eventually, returning to (4.18) we have

$$\begin{aligned} (D\,\phi_{\pm}(\lambda,x)f_{\pm},v)_{V_\tau} &= \pm\lambda(P_{\pm}\circ\pi_{\sigma^{\pm},\lambda}(x^{-1})f_{\pm},v)_{V_\tau} \\ &= \pm\lambda(\phi_{\pm}(\lambda,x)f_{\pm},v)_{V_\tau}, \end{aligned}$$

and this proves (4.16) and (4.17).

Consider now case (i). Again, set $\tau^{\pm}=\tau_n^{\pm}$, $\tau=\tau^{+}\oplus\tau^{-}$ and $\sigma=\sigma_{n-1}$ for short. Define also

$$P=P^{+}+P^{-},\qquad J=J^{+}+J^{-},\qquad \phi=\phi^{+}+\phi^{-}.$$

Let $f\in C^\infty(G,\sigma\otimes e^{i\lambda}\otimes\mathbf{1})$. Then $\phi(\lambda,\cdot)f\in C^\infty(G,\tau)$ and proceeding as in the previous case, we obtain, for any $v\in V_\tau$,

$$(D\,\phi(\lambda,x)f,v)_{V_\tau}=(\pi_{\sigma,\lambda}(x^{-1})f,Tv)_{L^2(K,\sigma)}, \tag{4.24}$$

where $T:=-\sum_j\pi_{\sigma,\overline{\lambda}}(e_j)\circ J\circ\gamma_j$ is an element of $\mathrm{Hom}_K(V_\tau,L^2(K,\sigma))$. However, this space is now two-dimensional, with generators J^{+} and J^{-}. It follows that there exist two complex numbers c^{+},c^{-} (depending on λ) such that $T=c^{+}\,J^{+}+c^{-}\,J^{-}$. As before, let us evaluate Tv at $k=e$. On the one hand,

$$\begin{aligned} Tv(e) &= (c^{+}\,J^{+}+c^{-}\,J^{-})v(e)=c^{+}\,J^{+}v^{+}(e)+c^{-}\,J^{-}v^{-}(e) \\ &= c^{+}v^{+}+c^{-}v^{-}, \end{aligned} \tag{4.25}$$

where we have written $v=v^{+}+v^{-}$ according to the decomposition $V_\tau=V_{\tau^+}\oplus V_{\tau^-}$. On the other hand, a similar calculation to the one in the odd case shows that

$$Tv(e)=i\overline{\lambda}(v^{+}-v^{-}).$$

Comparing this with (4.25) we get $c^{\pm}=\pm i\overline{\lambda}$. Then (4.24) gives

$$(D\phi(\lambda,x)f,v)_{V_\tau}=(-i\lambda\phi^{+}(\lambda,x)f+i\lambda\phi^{-}(\lambda,x)f,v)_{V_\tau}.$$

But we know that $D:C^\infty(G,\tau^{\pm})\to C^\infty(G,\tau^{\mp})$ and we immediately obtain (4.14) and (4.15). ∎

The theorem has an easy consequence: we remark that, together with Theorem 3.2, it implies that the spectrum of D^2 (resp. of D) is purely continuous and constituted by the positive half-line $\mathbb{R}_+$ (resp. by the line $\mathbb{R}$) when n is even (resp. n is odd). By Remark 4.1, we immediately get the following result, which corrects Corollary 4.6 of [Bun91]. See also [Bär98, §3], for a purely geometrical calculation of the Dirac spectrum in our setting.

COROLLARY 4.11. *The L^2 discrete spectrum of the Dirac operator on $H^n(\mathbb{R})$ is empty, while its continuous L^2 spectrum is $\mathbb{R}$.*

REMARK 4.12. Actually, the corollary can be obtained without using Theorem 4.8, by representation-theoretical arguments combined with the Lichnerowicz formula. Let us give a broad outline of this alternative method, since we shall use farther some of the results.

Let ∇^* denote the formal L^2 adjoint of the covariant derivative. If s stands for the scalar curvature of any spin manifold, the famous *Lichnerowicz formula* says that

$$D^2 = \nabla^*\nabla + \frac{s}{4}\,\mathrm{Id} \tag{4.26}$$

on smooth spinors (the operator $\nabla^*\nabla$ is *Bochner's Laplacian*). On the other hand, one can see (e.g. [BOS94, §3]) that, for sections of any homogeneous vector bundle $G \times_H V_\tau$ defined over a reductive symmetric space G/H,

$$\nabla^*\nabla = -\Omega_{\mathfrak{g}} - \Omega_{\mathfrak{h}},$$

where $\Omega_{\mathfrak{g}}$ and $\Omega_{\mathfrak{h}}$ stand for the *Casimir operators* (see e.g. [Kna86, §VIII.3]) in the enveloping algebras of $\mathfrak{g}$ and $\mathfrak{h}$ respectively (viewed here as differential operators acting on sections of the bundle). If τ is supposed to be irreducible, $\Omega_{\mathfrak{h}}$ acts as multiplication by a scalar $\tau(\Omega_{\mathfrak{h}})$ on $C^\infty(G,\tau)$ which can be expressed in terms of the highest weight of τ. In our case and with our normalization of curvature, it is easily seen that

$$\tau(\Omega_{\mathfrak{k}}) = -n(n-1)/8, \qquad s = -n(n-1) \tag{4.27}$$

if $\tau = \tau_n$ (n odd) as well as if $\tau = \tau_n^\pm$ (n even). Thus we get

$$D^2 = -\Omega_{\mathfrak{g}} - \frac{n(n-1)}{8}\,\mathrm{Id}\,.$$

This identity is often referred to as (a special case of) the *Parthasarathy formula*. Finally, our Plancherel Theorem (Theorem 3.2) shows that it suffices to apply this identity to vectors in the relevant principal series representations spaces. Again, it is not hard to check that for any such $\pi_{\sigma,\lambda}$,

$$\pi_{\sigma,\lambda}(-\Omega_{\mathfrak{g}}) = \left\{\lambda^2 + \varrho^2 - \frac{(n-1)(n-2)}{8}\right\}\mathrm{Id} = \left\{\lambda^2 + \frac{n(n-1)}{8}\right\}\mathrm{Id},$$

and, eventually,

$$\pi_{\sigma,\lambda}(D^2) = \lambda^2 \,\mathrm{Id}.$$

Then we conclude by Remark 4.1.

REMARK 4.13. As far as the L^2 spectrum of the Dirac operator on symmetric spaces is concerned, progress has been made recently: the discrete spectrum of D has been determined in [GS99] for all noncompact Riemannian symmetric spaces, and the continuous spectrum of D has been calculated in the rank one case in [CP00].

5. Radial and spherical functions associated with $\Sigma H^n(\mathbb{R})$

5.1. *Definitions and basic facts.* The classical notion of radial functions on a symmetric space extends to the bundle case as follows. If τ is a finite-dimensional representation of K, a function $F : G \to \operatorname{End} V_\tau$ is a *τ-radial function on G* if it satisfies the double K-equivariance condition

$$F(k_1 x k_2) = \tau(k_2)^{-1} F(x)\, \tau(k_1)^{-1} \qquad (x \in G,\ k_1, k_2 \in K).$$

The set of τ-radial functions on G will be denoted by $\Gamma(G,\tau,\tau)$, with the same possible changes for the prefix Γ as for the space $\Gamma(G,\tau)$. Let us point out the following facts:

(i) if F is τ-radial then for all $v \in V_\tau$, the function $x \mapsto F(x)v$ is of type τ;

(ii) conversely, a τ-radial function F is uniquely determined by any of the functions of type τ: $x \mapsto F(x)v$, as long as $v \neq 0$;

(iii) a τ-radial function is completely determined by its restriction to the subgroup $A = \{a_t\}$ of G, and even by its restriction to $\{a_t : t \geq 0\}$.

The L^2 inner product on $C^\infty_{\mathrm{c}}(G,\tau,\tau)$ is given by

$$(F,H) := \int_G dx\, (F(x), H(x))_{\mathrm{HS}} = \int_G dx\ \operatorname{tr}\{F(x)H(x)^*\} \tag{5.1}$$

where the subscript "HS" stands for "Hilbert–Schmidt". The space $C^\infty_{\mathrm{c}}(G,\tau,\tau)$ is also equipped with an algebra structure for the convolution product

$$(F * H)(x) = \int_G dy\, F(y^{-1}x)H(y) = \int_G dy\, F(y)H(xy^{-1}). \tag{5.2}$$

Suppose now that $\tau|_M$ is multiplicity free. Since $F|_A \in \operatorname{End}_M V_\tau$, Schur's lemma implies that $F|_A$ must be scalar on each M-irreducible submodule V_σ of V_τ. Thus for each $\sigma \in \widehat{M}(\tau)$, there exists a function $f_\sigma : \mathbb{R} \to \mathbb{C}$ such that

$$F(a_t)|_{V_\sigma} = f_\sigma(t)\, \mathrm{Id}_{V_\sigma} \qquad (t \in \mathbb{R}).$$

These functions f_σ will be called the *scalar components* of F.

From now on, the Haar measure dx on G is defined such that the following integral formula for the Cartan decomposition $G = K\{a_t : t \geq 0\}K$ holds: for any integrable function f on G,

$$\int_G dx\, f(x) = \int_K dk_1 \int_0^\infty dt\, (2\operatorname{sh} t)^{n-1} \int_K dk_2\, f(k_1 a_t k_2), \tag{5.3}$$

where dt is the Lebesgue measure on $\mathbb{R}$ and dk is normalized as in (4.9). This choice is adopted in order to match the normalization made in Koornwinder's article [Koo84], which will be extensively used in what follows. With the above notations, the inner product (5.1) for τ-radial functions reduces to

$$(F, H) = \sum_{\sigma \in \widehat{M}(\tau)} \dim \sigma \int_0^\infty dt\, (2\operatorname{sh} t)^{n-1} f_\sigma(t)\, \overline{h_\sigma(t)}. \tag{5.4}$$

We now apply the considerations above to our particular setting. Suppose first n is even. Then $\tau = \tau_n^\pm$ restricts to $\sigma = \sigma_{n-1}$ and, if $F^\pm \in \Gamma(G, \tau_n^\pm, \tau_n^\pm)$, we shall denote by $f^\pm$ its unique scalar component. Notice that $f^\pm$ must be an even function, since the nontrivial Weyl group element $w = m'M$ satisfies $m'^{-1} a_t m' = a_{-t}$ and stabilizes $V_{\sigma_{n-1}}$. Moreover, (5.4) reads

$$(F^\pm, H^\pm) = 2^{n/2-1} \int_0^\infty dt\, (2\operatorname{sh} t)^{n-1} f^\pm(t)\, \overline{h^\pm(t)}. \tag{5.5}$$

Suppose now n is odd. Then $\tau = \tau_n$ restricts to $\sigma_{n-1}^+ \oplus \sigma_{n-1}^-$ and if $F \in \Gamma(G, \tau_n, \tau_n)$, we shall denote the two corresponding scalar components by $f^\pm$. This time the Weyl group action yields the relation $f^\pm(-t) = f^\mp(t)$ since it interchanges σ_{n-1}^+ and σ_{n-1}^-. The inner product (5.4) can be rewritten as

$$(F, H) = 2^{(n-3)/2} \int_0^\infty dt\, (2\operatorname{sh} t)^{n-1} \big[f^+(t)\, \overline{h^+(t)} + f^-(t)\, \overline{h^-(t)}\big]. \tag{5.6}$$

In [Cam97b] and [Ped98a] it has been shown that the spherical harmonic analysis on (general) homogeneous vector bundles $G \times_K V_\tau$ over noncompact Riemannian symmetric spaces G/K can be carried out similarly to the classical scalar case (i.e. when $\tau = \mathbf{1}$ is trivial) provided that the convolution algebra $C_c^\infty(G, \tau, \tau)$ is *commutative*. If this condition is satisfied, we say that (G, K, τ) is a *Gelfand triple* (when $\tau = \mathbf{1}$, we recover the classical definition of a Gelfand pair). By a criterion due to Deitmar [Dei90], (G, K, τ) is a Gelfand triple if and only if $\tau|_M$ is multiplicity free or, equivalently, if and only if the algebra $\mathbb{D}(G, \tau)$ of left-invariant differential operators acting on $C^\infty(G, \tau)$ is commutative. Moreover, these conditions are satisfied for any $\tau \in \widehat{K}$ when $G/K = H^n(\mathbb{R})$ (see also [Koo82] for this result).

Assume (G,K,τ) is a Gelfand triple, and let $\Phi \in C^\infty(G,\tau,\tau)$ with $\Phi(e)=\mathrm{Id}$. Then Φ is a *τ-spherical function* on G if Φ is an eigenfunction for the algebra $\mathbb{D}(G,\tau)$, in the sense that there exists a character χ_Φ of $\mathbb{D}(G,\tau)$ such that $D\Phi(\cdot)v = \chi_\Phi(D)\Phi(\cdot)v$ for any $D \in \mathbb{D}(G,\tau)$ and for one nonzero $v \in V_\tau$ (hence for all $v \in V_\tau$). Actually, τ-spherical functions on G have three other equivalent characterizations: as characters of the convolution algebra $C_{\mathrm{c}}^\infty(G,\tau,\tau)$, as solutions of functional equations and as eigenfunctions with respect to convolution with $C_{\mathrm{c}}(G,\tau,\tau)$ (see [Cam97b] or [Ped98a]).

5.2. *Determination of the τ-spherical functions.* With the notations of §4.3, for $\sigma \in \widehat{M}(\tau)$ and $\lambda \in \mathbb{C}$, set

$$\Phi_\sigma^\tau(\lambda,x) := P_\sigma^\tau \circ \pi_{\sigma,\lambda}(x^{-1}) \circ J_\sigma^\tau \qquad (x \in G). \tag{5.7}$$

Clearly, the map $x \mapsto \Phi_\sigma^\tau(\lambda,x)$ is a τ-radial function. As usual, we set for short

$$\Phi^\pm = \Phi_{\sigma_{n-1}}^{\tau_n^\pm} \qquad (n \text{ even}), \tag{5.8}$$

$$\Phi_\pm = \Phi_{\sigma_{n-1}^\pm}^{\tau_n} \qquad (n \text{ odd}). \tag{5.9}$$

Notice that the action of W on the principal series representations induces the relations

$$\Phi^\pm(-\lambda,\cdot) = \Phi^\pm(\lambda,\cdot), \tag{5.10}$$

$$\Phi_\pm(-\lambda,\cdot) = \Phi_\mp(\lambda,\cdot). \tag{5.11}$$

THEOREM 5.1. *Let D be the Dirac operator*

(I) *Suppose n is even. Then*:

(i) *For any $\lambda \in \mathbb{C}$, $\Phi^\pm(\lambda,\cdot)$ is a $\tau_n^\pm$-spherical function. Moreover, for any $v^\pm \in V_{\tau_n^\pm}$,*

$$D^2\Phi^\pm(\lambda,\cdot)v^\pm = \lambda^2\,\Phi^\pm(\lambda,\cdot)v^\pm. \tag{5.12}$$

(ii) *$\Phi^\pm(\lambda,\cdot)$ admits the following representation as an Eisenstein integral*:

$$\Phi^\pm(\lambda,x) = \int_K dk\, e^{-(i\lambda+\varrho)H(xk)}\tau_n^\pm(k\underline{k}(xk)^{-1}).$$

In particular, $\Phi^\pm$ is holomorphic in the variable λ.

(iii) *The $\Phi^\pm(\lambda,\cdot)$'s, when $\lambda \in \mathbb{C}/\{\pm 1\}$, are the only $\tau_n^\pm$-spherical functions on G.*

(II) *Suppose n is odd. Then*:

(i) *For any $\lambda \in \mathbb{C}$, $\Phi_\pm(\lambda,\cdot)$ is a τ_n-spherical function. Moreover, for any $v \in V_{\tau_n}$,*

$$D\,\Phi_\pm(\lambda,\cdot)v = \pm\lambda\,\Phi_\pm(\lambda,\cdot)v. \tag{5.13}$$

(ii) $\Phi_\pm(\lambda,\cdot)$ *admits the following representation as an Eisenstein integral*:

$$\Phi_\pm(\lambda,x) = 2\int_K dk\, e^{-(i\lambda+\varrho)H(xk)}\tau_n(k)\circ P_{\sigma^\pm_{n-1}}\circ\tau_n(\underline{k}(xk)^{-1}).$$

In particular, $\Phi_\pm$ *is holomorphic in the variable* λ.

(iii) *The* $\Phi_\pm(\lambda,\cdot)$*'s, when* $\lambda\in\mathbb{C}/\{\pm1\}$, *are the only* τ_n*-spherical functions on* G.

Proof. Equations (5.12) and (5.13) follow from Theorem 4.8, once we remark the relation $\Phi^\tau_\sigma(\lambda,\cdot)v = \phi^\tau_\sigma(\lambda,\cdot)(J^\tau_\sigma v)$. All other statements are easy consequences of [Ped98a, §3]. ∎

5.3. *τ-spherical functions as Jacobi functions.* In order to express our spherical functions in terms of Jacobi functions, the first step consists in obtaining hypergeometric differential equations for their scalar components. Specifically, we have the following result.

PROPOSITION 5.2. (I) *Let n be even. Then the scalar component* $\varphi^\pm(\lambda,\cdot)$ *of* $\Phi^\pm(\lambda,\cdot)$ *satisfies the differential equation*

$$\left\{\frac{d^2}{dt^2}+2\varrho(\coth t)\frac{d}{dt}+\varrho(\operatorname{ch} t-1)(\operatorname{sh} t)^{-2}+\lambda^2+\varrho^2\right\}\varphi^\pm(\lambda,t)=0. \tag{5.14}$$

(II) *Let n be odd. Then the scalar components* $\varphi^+_\pm(\lambda,\cdot)$ *and* $\varphi^-_\pm(\lambda,\cdot)$ *of* $\Phi_\pm(\lambda,\cdot)$ *satisfy the differential equations*

$$\left\{\frac{d}{dt}+\varrho(\coth t)\pm i\lambda\right\}\varphi^+_\pm(\lambda,t)=\varrho(\operatorname{sh} t)^{-1}\varphi^-_\pm(\lambda,t), \tag{5.15}$$

$$\left\{\frac{d}{dt}+\varrho(\coth t)\mp i\lambda\right\}\varphi^-_\pm(\lambda,t)=\varrho(\operatorname{sh} t)^{-1}\varphi^+_\pm(\lambda,t). \tag{5.16}$$

Proof. Let us state first a technical result. As in the proof of Theorem 4.8, let $(e_j)_{j=1}^n$ be the basis of $\mathfrak{p}$ corresponding to the canonical orthonormal basis of $\mathbb{R}^n$ via the identification (2.1) and define consistently bases for the spaces $\mathfrak{n},\overline{\mathfrak{n}},\mathfrak{q},\mathfrak{l}$ as in (4.23). Denote by Ad the adjoint representation of G on $\mathfrak{g}$. Recall also that $a_t=\exp(te_1)$.

LEMMA 5.3. *For* $j=2,\dots,n$ *and* $t>0$, *we have*:

(a) $Y_j = -(\operatorname{sh} t)^{-1}\operatorname{Ad}(a_{-t})Z_j+(\coth t)Z_j$,

(b) $Y_j^2 = (\operatorname{sh} t)^{-2}[\operatorname{Ad}(a_{-t})Z_j]^2+(\coth t)^2 Z_j^2$

$$-2(\operatorname{sh} t)^{-1}(\coth t)\operatorname{Ad}(a_{-t})Z_j\cdot Z_j+(\coth t)e_1.$$

The standard proof is omitted.

Consider Case (I) of the proposition. For simplicity, write $\tau=\tau_n^\pm$ and $\Phi=\Phi^\pm(\lambda,\cdot)$ if there is no ambiguity. By (5.12), we know that $D^2\Phi=$

$\lambda^2\Phi$ and we want to expand the left-hand side of this equality. Recall from Remark 4.12 equations (4.26) and (4.27), i.e. the Lichnerowicz formula in our setting:

$$D^2 = \nabla^*\nabla - \frac{n(n-1)}{4}\,\mathrm{Id}$$

on $C^\infty(G,\tau)$ and, by abuse of notation, on $C^\infty(G,\tau,\tau)$. On the other hand, Bochner's Laplacian $\nabla^*\nabla$ coincides with $-\Omega_{\mathfrak{p}} := -\Omega_{\mathfrak{g}} - \Omega_{\mathfrak{k}}$, and we can always suppose that

$$\Omega_{\mathfrak{p}} = \sum_{j=1}^{n} e_j^2$$

for a suitable choice of orthonormal bases for $\mathfrak{g}$ and $\mathfrak{k}$ (on which the Casimir operators do not depend). Thus

$$D^2\Phi(x) = -\Phi(x:\Omega_{\mathfrak{p}}) - \frac{n(n-1)}{4}\Phi(x) \tag{5.17}$$

and since we can restrict to $x = a_t \in A$, we are left with the calculation of

$$\Phi(a_t:\Omega_{\mathfrak{p}}) = \Phi(a_t:e_1^2) + \sum_{j=2}^{n}\Phi(a_t:e_j^2).$$

We have $\Phi(a_t:e_1) = \frac{d}{dt}\Phi(a_t)$ and, by the previous lemma, for $j=2,\dots,n$,

$$\begin{aligned}\Phi(a_t:e_j^2) &= \Phi(a_t:Y_j^2)\\ &= (\operatorname{sh} t)^{-2}\Phi(a_t:[\mathrm{Ad}(a_{-t})Z_j]^2) + (\coth t)^2\Phi(a_t:Z_j^2)\\ &\quad - 2(\operatorname{sh} t)^{-1}(\coth t)\Phi(a_t:\mathrm{Ad}(a_{-t})Z_j\cdot Z_j) + (\coth t)\Phi(a_t:e_1),\end{aligned}$$

with

$$\begin{aligned}\Phi(a_t:[\mathrm{Ad}(a_{-t})Z_j]^2) &= \Phi(Z_j^2:a_t) = \Phi(a_t)\tau(Z_j^2),\\ \Phi(a_t:Z_j^2) &= \tau(Z_j^2)\Phi(a_t),\\ \Phi(a_t:\mathrm{Ad}(a_{-t})Z_j\cdot Z_j) &= \Phi(Z_j:a_t:Z_j) = \tau(Z_j)\Phi(a_t)\tau(Z_j).\end{aligned}$$

Hence

$$\begin{aligned}\Phi(a_t:\Omega_{\mathfrak{p}}) = &\left\{\frac{d^2}{dt^2} + (n-1)(\coth t)\frac{d}{dt}\right\}\Phi(a_t) + (\operatorname{sh} t)^{-2}\Phi(a_t)\sum_{j=2}^{n}\tau(Z_j^2)\\ &+ (\coth t)^2\sum_{j=2}^{n}\tau(Z_j^2)\Phi(a_t)\\ &- 2(\operatorname{sh} t)^{-1}(\coth t)\sum_{j=2}^{n}\tau(Z_j)\Phi(a_t)\tau(Z_j).\end{aligned}$$

But $\Phi(a_t)$ is a scalar matrix and therefore

$$\tau(Z_j)\Phi(a_t)\tau(Z_j) = \tau(Z_j^2)\Phi(a_t) = \Phi(a_t)\tau(Z_j^2).$$

On the other hand, by (4.27),

$$\sum_{j=2}^{n} \tau(Z_j^2) = \tau(\Omega_{\mathfrak{l}}) = \tau(\Omega_{\mathfrak{k}}) - \tau(\Omega_{\mathfrak{m}}) = -\frac{n-1}{4}\,\mathrm{Id}\,.$$

Hence

$$\begin{aligned}\Phi(a_t : \Omega_{\mathfrak{p}}) = &\left\{\frac{d^2}{dt^2} + (n-1)(\coth t)\frac{d}{dt}\right\}\Phi(a_t)\\ &- \frac{n-1}{4}[2(\operatorname{sh} t)^{-2} + 1 - 2(\operatorname{sh} t)^{-1}(\coth t)]\Phi(a_t).\end{aligned}$$

Finally, substituting (5.17) in (5.12), we get (5.14).

Consider now Case (II) of the proposition. Set again $\tau = \tau_n$ and $\Phi = \Phi_\pm(\lambda, \cdot)$ for short. Restricting to $x = a_t$, we have, by (4.2),

$$D\,\Phi(a_t) = \gamma_1\,\Phi(a_t : e_1) + \sum_{j=2}^{n} \gamma_j\,\Phi(a_t : Y_j).$$

By the previous lemma,

$$\begin{aligned}\Phi(a_t : Y_j) &= -(\operatorname{sh} t)^{-1}\Phi(a_t : \mathrm{Ad}(a_{-t})Z_j) + (\coth t)\Phi(a_t : Z_j)\\ &= (\operatorname{sh} t)^{-1}\Phi(a_t)\tau(Z_j) - (\coth t)\tau(Z_j)\Phi(a_t).\end{aligned}$$

Let φ^+ and φ^- denote the scalar components of Φ and fix $v = v_+ + v_- \in V_\tau = V_{\sigma^+} \oplus V_{\sigma^-}$. Using (4.4), (4.6), (4.7) and (4.8), we get

$$\begin{aligned}D\Phi(a_t)v = &\gamma_1\frac{d}{dt}\Phi(a_t)v + \frac{1}{2}(\operatorname{sh} t)^{-1}\sum_{j=2}^{n}\gamma_j\Phi(a_t)\gamma_j\gamma_1 v\\ &- \frac{1}{2}(\coth t)\sum_{j=2}^{n}\gamma_j^2\gamma_1\Phi(a_t)v\\ = &\,i\begin{pmatrix} \frac{d}{dt}\varphi^+(t)v_+ + \varrho(\coth t)\varphi^+(t)v_+ - \varrho(\operatorname{sh} t)^{-1}\varphi^-(t)v_+ \\ -\frac{d}{dt}\varphi^-(t)v_- - \varrho(\coth t)\varphi^-(t)v_- + \varrho(\operatorname{sh} t)^{-1}\varphi^+(t)v_- \end{pmatrix}.\end{aligned}$$

With this expression, (5.13) implies (5.15) and (5.16). ∎

Before stating the main result of this section, let us collect some definitions and facts about Jacobi functions, taken from [Koo84, §2.1].

Let $\alpha \in \mathbb{C} \smallsetminus \{-1, -2, \ldots\}$, $\beta \in \mathbb{C}$ and $\lambda \in \mathbb{C}$. The *Jacobi function* $\phi_\lambda^{(\alpha,\beta)}$ is, by definition, the hypergeometric function

$$(5.18)\qquad \begin{aligned}&\phi_\lambda^{(\alpha,\beta)}(t)\\ &= {}_2F_1\left(\frac{\alpha+\beta+1+i\lambda}{2}, \frac{\alpha+\beta+1-i\lambda}{2}; \alpha+1; -(\operatorname{sh} t)^2\right) \qquad (t \in \mathbb{R}).\end{aligned}$$

Consider the *Jacobi Laplacian*

$$L_{\alpha,\beta} = \frac{d^2}{dt^2} + \{(2\alpha+1)\coth t + (2\beta+1)\operatorname{th} t\}\frac{d}{dt}. \tag{5.19}$$

Then $\phi = \phi_{\pm\lambda}^{(\alpha,\beta)}$ is the unique analytic solution on $\mathbb{R}$ of the equation

$$(L_{\alpha,\beta} + \lambda^2 + (\alpha+\beta+1)^2)\phi = 0 \tag{5.20}$$

which is even and satisfies $\phi(0) = 1$.

Now, in the definition of a Jacobi function $\phi_\lambda^{(\alpha,\beta)}$, we add two more parameters $l, m \in \mathbb{Z}$, and put (see [Koo84, formula (4.15)])

$$\phi_{\lambda,l,m}^{(\alpha,\beta)}(t) = (\operatorname{sh} t)^l (\operatorname{ch} t)^m \phi_\lambda^{(\alpha+l,\beta+m)}(t). \tag{5.21}$$

One easily checks that $\phi = \phi_{\lambda,l,m}^{(\alpha,\beta)}$ is (the unique, even and satisfying $\phi(0) = 1$ when $l = 0$) solution of the equation

$$\begin{aligned} &\frac{d^2\phi}{dt^2} + \{(2\alpha+1)\coth t + (2\beta+1)\operatorname{th} t\}\frac{d\phi}{dt} \\ &\quad + \{-(2\alpha+l)l(\operatorname{sh} t)^{-2} + (2\beta+m)m(\operatorname{ch} t)^{-2} + \lambda^2 + (\alpha+\beta+1)^2\}\phi = 0. \end{aligned} \tag{5.22}$$

The functions $\phi_{\lambda,l,m}^{(\alpha,\beta)}$ are called *modified* (or *associated*) *Jacobi functions.*

THEOREM 5.4. (I) *Let n be even. Then the scalar component $\varphi^\pm(\lambda,\cdot)$ of $\Phi^\pm(\lambda,\cdot)$ is given by*

$$\varphi^\pm(\lambda,t) = \left(\operatorname{ch}\frac{t}{2}\right)\phi_{2\lambda}^{(n/2-1,n/2)}\left(\frac{t}{2}\right). \tag{5.23}$$

(II) *Let n be odd. Then the scalar components $\varphi_\pm^+(\lambda,\cdot)$ and $\varphi_\pm^-(\lambda,\cdot)$ of $\Phi_\pm(\lambda,\cdot)$ are given by*

$$\varphi_\pm^+(\lambda,t) = \left(\operatorname{ch}\frac{t}{2}\right)\phi_{2\lambda}^{(n/2-1,n/2)}\left(\frac{t}{2}\right) \mp i\frac{2\lambda}{n}\left(\operatorname{sh}\frac{t}{2}\right)\phi_{2\lambda}^{(n/2,n/2-1)}\left(\frac{t}{2}\right), \tag{5.24}$$

$$\varphi_\pm^-(\lambda,t) = \left(\operatorname{ch}\frac{t}{2}\right)\phi_{2\lambda}^{(n/2-1,n/2)}\left(\frac{t}{2}\right) \pm i\frac{2\lambda}{n}\left(\operatorname{sh}\frac{t}{2}\right)\phi_{2\lambda}^{(n/2,n/2-1)}\left(\frac{t}{2}\right). \tag{5.25}$$

Proof. Suppose first n is even. Then, letting $t = 2u$ in (5.14), we easily get the differential equation

$$\left\{\frac{d^2}{du^2} + (n-1)(\coth u + \operatorname{th} u)\frac{d}{du} \right. \\ \left. + (n-1)(\operatorname{ch} u)^{-2} + (2\lambda)^2 + (n-1)^2\right\}\varphi^\pm(\lambda,2u) = 0.$$

Since $t \mapsto \varphi^\pm(\lambda,t)$ is even, analytic and satisfies $\varphi^\pm(\lambda,0) = 1$, we conclude using (5.21) and (5.22).

The case of n odd is slightly more complicated. Put

$$\varphi_\pm := \tfrac{1}{2}(\varphi_\pm^+ + \varphi_\pm^-), \qquad \widetilde{\varphi}_\pm := \tfrac{1}{2}(\varphi_\pm^+ - \varphi_\pm^-). \tag{5.26}$$

The differential system formed by (5.15) and (5.16) is then equivalent to the system formed by the couple of equations

$$\left\{\frac{d}{dt} + \varrho\left(\operatorname{th}\frac{t}{2}\right)\right\}\varphi_\pm(\lambda,t) = \mp i\lambda\,\widetilde{\varphi}_\pm(\lambda,t), \tag{5.27}$$

$$\left\{\frac{d}{dt} + \varrho\left(\coth\frac{t}{2}\right)\right\}\widetilde{\varphi}_\pm(\lambda,t) = \mp i\lambda\,\varphi_\pm(\lambda,t). \tag{5.28}$$

Multiplying (5.28) by $\mp i\lambda$ and substituting (5.27) in (5.28), one gets precisely equation (5.14). Hence, as in Case (I) ($\varphi_\pm(\lambda,\cdot)$ is an even function),

$$\varphi_\pm(\lambda,t) = \left(\operatorname{ch}\frac{t}{2}\right)\phi_{2\lambda}^{(n/2-1,n/2)}\left(\frac{t}{2}\right). \tag{5.29}$$

It remains to determine $\widetilde{\varphi}_\pm$. Recall that

$$\varphi_\pm(\lambda,t) = \sqrt{1-z}\,{}_2F_1(a,b;c;z),$$

with $a = n/2+i\lambda$, $b = n/2-i\lambda$, $c = n/2$, $z = -(\operatorname{sh}(t/2))^2$. A straightforward calculation shows that (5.27) can be written as

$$\mp i\lambda\widetilde{\varphi}_\pm(\lambda,t) = \operatorname{sgn}(t)\sqrt{-z}\left(\frac{n}{2} - (1-z)\frac{d}{dz}\right){}_2F_1(a,b;c;z).$$

By applying formula (24), p. 102, of [EMOT53], we obtain

$$\mp i\lambda\widetilde{\varphi}_\pm(\lambda,t) = -\operatorname{sgn}(t)\sqrt{-z}\,\frac{2\lambda^2}{n}\,{}_2F_1(a,b;c+1;z),$$

or, in terms of Jacobi functions,

$$\widetilde{\varphi}_\pm(\lambda,t) = \mp i\frac{2\lambda}{n}\left(\operatorname{sh}\frac{t}{2}\right)\phi_{2\lambda}^{(n/2,n/2-1)}\left(\frac{t}{2}\right). \tag{5.30}$$

Combining (5.26), (5.29) and (5.30), we finish the proof of the theorem. ∎

Remark 5.5. The expressions obtained in the last theorem for the spherical functions coincide with the ones given in [CH96, §4.2], once one converts Jacobi functions into hypergeometric functions. However, as mentioned in the introduction, the proof of the result is incomplete in the odd case, since the authors have used for the spherical functions the differential equation involving only the square of D, which determines the solution up to sign.

6. Spherical Fourier transform. As was made clear in [Cam97b] and [Ped98b], when (G,K,τ) is a Gelfand triple, the spherical Fourier transform on $C_c^\infty(G,\tau,\tau)$ arises naturally as the Gelfand transform on this commutative algebra. Namely, the *spherical transform* $\mathcal{H}(F)$ of a τ-radial function F

is the collection of functions

$$\lambda \mapsto \{\mathcal{H}^\tau_\sigma(F)(\lambda)\}_{\sigma\in\widehat{M}(\tau)},$$

where the *partial spherical Fourier transforms* $\mathcal{H}^\tau_\sigma(F)$ of F are defined by

$$\mathcal{H}^\tau_\sigma(F)(\lambda) := \frac{1}{\dim\tau}\int_G dx\ \mathrm{tr}\{F(x)\Phi^\tau_\sigma(\lambda, x^{-1})\} \quad (\lambda\in\mathbb{C}). \tag{6.1}$$

As usual, we abbreviate notations in our setting as follows:

$$\mathcal{H}^\pm = \mathcal{H}^{\tau_n^\pm}_{\sigma_{n-1}}, \qquad \mathcal{H}_\pm = \mathcal{H}^{\tau_n}_{\sigma^\pm_{n-1}}.$$

We want to clarify the link between the spherical transform associated with spinors on $H^n(\mathbb{R})$ and certain Jacobi transforms. Recall first from [Koo84, §2] that the *Jacobi transform* $\mathcal{J}^{(\alpha,\beta)}(f)$ of an even (smooth, compactly supported) function $f:\mathbb{R}\to\mathbb{C}$ is defined by

$$\mathcal{J}^{(\alpha,\beta)}(f)(\lambda) := \int_0^\infty dt\,\zeta^{(\alpha,\beta)}(t)\,\phi^{(\alpha,\beta)}_\lambda(t)\,f(t) \quad (\lambda\in\mathbb{C}), \tag{6.2}$$

where $\zeta^{(\alpha,\beta)}(t) := (2\,\mathrm{sh}\,t)^{2\alpha+1}(2\,\mathrm{ch}\,t)^{2\beta+1}$.

Proposition 6.1. (I) *Let n be even. Let $F^\pm \in C^\infty_{\mathrm{c}}(G,\tau_n^\pm,\tau_n^\pm)$, with scalar component $f^\pm$. Then*

$$\mathcal{H}^\pm(F^\pm)(\lambda) = \frac{1}{2}\,\mathcal{J}^{(n/2-1,n/2)}\left(\frac{f^\pm(2\cdot)}{\mathrm{ch}}\right)(2\lambda) \quad (\lambda\in\mathbb{C}). \tag{6.3}$$

(II) *Let n be odd. Let $F\in C^\infty_{\mathrm{c}}(G,\tau_n,\tau_n)$ with scalar components f^+, f^- and put $f := \frac{1}{2}(f^+ + f^-)$ and $\widetilde{f} := \frac{1}{2}(f^+ - f^-)$. Then*

$$\begin{aligned} \mathcal{H}_\pm(F)(\lambda) = {} & \frac{1}{2}\mathcal{J}^{(n/2-1,n/2)}\left(\frac{f(2\cdot)}{\mathrm{ch}}\right)(2\lambda) \\ & \pm \frac{i\lambda}{n}\mathcal{J}^{(n/2,n/2-1)}\left(\frac{\widetilde{f}(2\cdot)}{\mathrm{sh}}\right)(2\lambda) \quad (\lambda\in\mathbb{C}). \end{aligned} \tag{6.4}$$

Proof. In Case (I), (5.5) shows that

$$\mathcal{H}^\pm(F^\pm)(\lambda) = \int_0^\infty dt\,(2\,\mathrm{sh}\,t)^{n-1} f^\pm(t)\varphi^\pm(\lambda,t)$$

and a straightforward calculation using (5.23) yields (6.3).

In Case (II), (5.6) gives easily

$$\mathcal{H}_\pm(F)(\lambda) = \frac{1}{2}\int_0^\infty dt\,(2\,\mathrm{sh}\,t)^{n-1}[f^+(t)\varphi^-_\pm(\lambda,t) + f^-(t)\varphi^+_\pm(\lambda,t)].$$

Using notation (5.26) we can write $\mathcal{H}_\pm(F)(\lambda)$ as the sum of the following two integrals:

$$I_\pm(\lambda) = \int_0^\infty dt\,(2\operatorname{sh} t)^{n-1}\varphi_\pm(\lambda,t)f(t),$$

$$\widetilde{I}_\pm(\lambda) = -\int_0^\infty dt\,(2\operatorname{sh} t)^{n-1}\widetilde{\varphi}_\pm(\lambda,t)\widetilde{f}(t).$$

Now (5.29) shows that $I_\pm(\lambda)$ has the same expression as the right-hand side of (6.3). A similar calculation gives

$$\widetilde{I}_\pm(\lambda) = \pm\frac{i\lambda}{n}\mathcal{J}^{(n/2,n/2-1)}\left(\frac{\widetilde{f}(2\cdot)}{\operatorname{sh}}\right)(2\lambda),$$

which yields the required results. ∎

We now introduce some more function spaces. In the following definitions, R is a (strictly) positive number.

(a) $C_R^\infty(\mathbb{R})$ is the space of C^∞ functions on $\mathbb{R}$ with compact support included in the interval $[-R,R]$.

(b) $\mathrm{PW}_R(\mathbb{C})$ is the space of entire functions h on $\mathbb{C}$ satisfying the condition

$$\forall N\in\mathbb{N},\ \exists C_N>0,\ \forall\lambda\in\mathbb{C},\qquad |h(\lambda)|\le C_N(1+|\lambda|)^{-N}e^{R|\operatorname{Im}\lambda|}.$$

(c) $\mathcal{S}(\mathbb{R})$ is the classical Schwartz space on $\mathbb{R}$.

(d) $L^2(\mathbb{R};d\mu(t))$ is the Hilbert space of square integrable functions on $\mathbb{R}$ with respect to the measure $d\mu(t)$.

In definitions (a)–(d), we may add the subscript "even" to indicate restriction to the subspace of even functions.

(e) $C_R^\infty(G,\tau,\tau)$ is the space of τ-radial functions on G whose support is compact and included in $B_R := \{k_1a_tk_2 \,:\, k_1,k_2\in K, -R\le t\le R\}$.

(f) $\mathcal{S}(G,\tau,\tau)$ is Harish-Chandra's (L^2) Schwartz space for τ-radial functions on G:

$$\mathcal{S}(G,\tau,\tau) = \{F\in C^\infty(G,\tau,\tau) : \forall D_1,D_2\in U(\mathfrak{g}),\ \forall N\in\mathbb{N},$$
$$\sup_{t\ge 0}\|F(D_1:a_t:D_2)\|_{\operatorname{End}V_\tau}\,(1+t)^N e^{\varrho t}<\infty\}.$$

Proceeding as in [GV88, §6.1], we can endow $\mathcal{S}(G,\tau,\tau)$ with a Fréchet topology, so that

$$\overline{C_{\mathrm{c}}^\infty(G,\tau,\tau)} = \mathcal{S}(G,\tau,\tau), \tag{6.5}$$

$$\overline{\mathcal{S}(G,\tau,\tau)} = L^2(G,\tau,\tau). \tag{6.6}$$

Note that the formulas in the previous proposition remain valid in the Schwartz setting.

REMARK 6.2. When n is even, it is easily seen that the following equivalences hold:

$$\begin{aligned} F^{\pm} \in C_R^{\infty}(G, \tau_n^{\pm}, \tau_n^{\pm}) &\Leftrightarrow f^{\pm} \in C_R^{\infty}(\mathbb{R})_{\mathrm{even}}, \\ F^{\pm} \in \mathcal{S}(G, \tau_n^{\pm}, \tau_n^{\pm}) &\Leftrightarrow f^{\pm} \in (\operatorname{ch} t)^{-\varrho}\mathcal{S}(\mathbb{R})_{\mathrm{even}}, \\ F^{\pm} \in L^2(G, \tau_n^{\pm}, \tau_n^{\pm}) &\Leftrightarrow f^{\pm} \in L^2(\mathbb{R}; 2^{n/2-1}(2\operatorname{sh} t)^{n-1}dt)_{\mathrm{even}}. \end{aligned}$$

When n is odd, we have similarly:

$$\begin{aligned} F \in C_R^{\infty}(G, \tau_n, \tau_n) &\Leftrightarrow f^{\pm} \in C_R^{\infty}(\mathbb{R}) \text{ and } f^{\pm}(-t) = f^{\mp}(t), \\ F \in \mathcal{S}(G, \tau_n, \tau_n) &\Leftrightarrow f^{\pm} \in (\operatorname{ch} t)^{-\varrho}\mathcal{S}(\mathbb{R}) \text{ and } f^{\pm}(-t) = f^{\mp}(t), \\ F \in L^2(G, \tau_n, \tau_n) &\Leftrightarrow f^{\pm} \in L^2(\mathbb{R}; 2^{(n-3)/2}(2\operatorname{sh} t)^{n-1}dt) \text{ and} \\ & \qquad f^{\pm}(-t) = f^{\mp}(t). \end{aligned}$$

In what follows, "topological linear isomorphism" will mean "bicontinuous isomorphism between vector spaces endowed with a Fréchet topology".

THEOREM 6.3. (I) *Let n be even. Let $d\nu(\lambda)$ denote the measure on $\mathbb{R}$ having, with respect to the Lebesgue measure $d\lambda$, the density*

$$\nu(\lambda) := 2^{3-2n}[(n/2-1)!]^{-2}\lambda\coth(\pi\lambda)\prod_{j=1}^{n/2-1}[\lambda^2+j^2]. \tag{6.7}$$

(I.A) *For $F^{\pm} \in C_c^{\infty}(G, \tau_n^{\pm}, \tau_n^{\pm})$ or $F^{\pm} \in \mathcal{S}(G, \tau_n^{\pm}, \tau_n^{\pm})$, we have the following inversion and Plancherel formulas*:

$$F^{\pm}(x) = \int_0^{\infty} d\nu(\lambda)\, \mathcal{H}^{\pm}(F^{\pm})(\lambda)\, \Phi^{\pm}(\lambda, x), \tag{6.8}$$

$$\|F^{\pm}\|^2_{L^2(G,\tau_n^{\pm},\tau_n^{\pm})} = 2^{n/2-1}\int_0^{\infty} d\nu(\lambda)\, |\mathcal{H}^{\pm}(F^{\pm})(\lambda)|^2. \tag{6.9}$$

(I.B) *$\mathcal{H} \simeq \mathcal{H}^{\pm}$ is a topological linear isomorphism between*

(i) *$C_R^{\infty}(G, \tau_n^{\pm}, \tau_n^{\pm})$ and* $\mathrm{PW}_R(\mathbb{C})_{\mathrm{even}}$;
(ii) *$\mathcal{S}(G, \tau_n^{\pm}, \tau_n^{\pm})$ and $\mathcal{S}(\mathbb{R})_{\mathrm{even}}$.*

(I.C) *$\mathcal{H}$ extends to a bijective isometry between $L^2(G, \tau_n^{\pm}, \tau_n^{\pm})$ and $L^2(\mathbb{R}; 2^{n/2-2}d\nu(\lambda))_{\mathrm{even}}$.*

(II) *Let n be odd. Let $d\nu(\lambda)$ denote the measure on $\mathbb{R}$ having, with respect to the Lebesgue measure $d\lambda$, the (polynomial) density*

$$\nu(\lambda) := 2^{-2}\pi^{-1}\left[\frac{n-1}{2}\left(\frac{n-1}{2}+1\right)\ldots(n-2)\right]^{-2} \times \prod_{j=1}^{(n-1)/2}[\lambda^2+(j-1/2)^2]. \tag{6.10}$$

(II.A) *For $F \in C_c^\infty(G,\tau_n,\tau_n)$ or $F \in \mathcal{S}(G,\tau_n,\tau_n)$, we have the following inversion and Plancherel formulas*:

$$F(x) = \int_0^\infty d\nu(\lambda)[\mathcal{H}_+(F)(\lambda)\Phi_+(\lambda,x) + \mathcal{H}_-(F)(\lambda)\Phi_-(\lambda,x)], \tag{6.11}$$

$$\|F\|^2_{L^2(G,\tau_n,\tau_n)} = 2^{(n-1)/2}\int_0^\infty d\nu(\lambda)\left[|\mathcal{H}_+(F)(\lambda)|^2 + |\mathcal{H}_-(F)(\lambda)|^2\right]. \tag{6.12}$$

(II.B) $\mathcal{H} = (\mathcal{H}_+,\mathcal{H}_-)$ *is a topological linear isomorphism between*

(i) $C_R^\infty(G,\tau_n,\tau_n)$ *and* $\{(h_+,h_-) \in \mathrm{PW}_R(\mathbb{C})^2 : h_\pm(-\lambda) = h_\mp(\lambda)\}$;

(ii) $\mathcal{S}(G,\tau_n,\tau_n)$ *and* $\{(h_+,h_-) \in \mathcal{S}(\mathbb{R})^2 : h_\pm(-\lambda) = h_\mp(\lambda)\}$.

(II.C) $\mathcal{H}$ *extends to a bijective isometry between* $L^2(G,\tau_n,\tau_n)$ *and* $\{(h_+,h_-) \in L^2(\mathbb{R}; 2^{(n-3)/2}d\nu(\lambda))^2 : h_\pm(-\lambda) = h_\mp(\lambda)\}$.

Proof. Consider first Case (I). Because of (6.3), from [Koo84, Theorem 2.3], we get immediately, for any $F^\pm \in C_c^\infty$,

$$f^\pm(t) = \pi^{-1}\left(\operatorname{ch}\frac{t}{2}\right)\int_0^\infty \frac{d(2\lambda)}{|\mathbf{c}(2\lambda)|^2}\,\mathcal{H}^\pm(F^\pm)(\lambda)\,\phi_{2\lambda}^{(n/2-1,n/2)}\left(\frac{t}{2}\right), \tag{6.13}$$

where $\mathbf{c}$ (defined for $\lambda \in \mathbb{C}$) has the following expression in terms of Euler's Gamma function:

$$\mathbf{c}(\lambda) := \mathbf{c}^{(n/2-1,n/2)}(\lambda) = 2^{n-i\lambda}\frac{\Gamma(n/2)\Gamma(i\lambda)}{\Gamma((i\lambda+n)/2)\Gamma(i\lambda/2)} \tag{6.14}$$

([Koo84, (2.18)]) and $d\lambda$ is the Lebesgue measure on $\mathbb{R}$.

The following lemma is easy. (Note that $|\mathbf{c}(\lambda)|^2 = \mathbf{c}(\lambda)\mathbf{c}(-\lambda)$ when λ is real and use standard functional equations for the Gamma function.)

LEMMA 6.4. *Let n be even. With $\mathbf{c}$ as in* (6.14), *define, for $\lambda \in \mathbb{R}$,*

$$d\nu(\lambda) = \frac{d(2\lambda)}{\pi\,|\mathbf{c}(2\lambda)|^2}.$$

Then $d\nu(\lambda)$ has density (6.7) *with respect to $d\lambda$.*

Then (6.8) follows trivially from (6.13). The Plancherel formula (6.9) is obtained from the inversion formula by standard arguments (see e.g. [Ped97, Theorem 6.3] in the similar case of differential forms).

Statement (I.B)(i) is Theorem 2.1 of [Koo84]. For (I.B)(ii), we use the decomposition $\mathcal{J}^{(\alpha,\beta)} = \mathcal{F}\circ\mathcal{A}^{(\alpha,\beta)}$, where $\mathcal{F}$ is the Euclidean Fourier transform and $\mathcal{A}^{(\alpha,\beta)}$ is the Abel transform defined in [Koo84, §5] (we shall define and handle this transform in our Section 8), which is a topological isomorphism

$$\mathcal{A}^{(\alpha,\beta)} : (\operatorname{ch} t)^{-s}\mathcal{S}(\mathbb{R})_{\mathrm{even}} \xrightarrow{\simeq} (\operatorname{ch} t)^{-s+\alpha+\beta+1}\mathcal{S}(\mathbb{R})_{\mathrm{even}} \tag{6.15}$$

for all $s \geq \alpha+\beta+1$ (use [Koo84, (6.8)] plus analytic continuation). In our case, $f^{\pm} \in (\operatorname{ch} t)^{-\varrho}\mathcal{S}(\mathbb{R})_{\text{even}}$, so that $f^{\pm}(2\cdot)\operatorname{ch}^{-1} \in (\operatorname{ch} t)^{-n}\mathcal{S}(\mathbb{R})_{\text{even}}$. Thus

$$\mathcal{J}^{(n/2-1,n/2)} : (\operatorname{ch} t)^{-n}\mathcal{S}(\mathbb{R})_{\text{even}} \xrightarrow[\mathcal{A}^{(n/2-1,n/2)}]{\simeq} \mathcal{S}(\mathbb{R})_{\text{even}} \xrightarrow[\mathcal{F}]{\simeq} \mathcal{S}(\mathbb{R})_{\text{even}},$$

and the result follows. Statement (I.C) is obtained by standard density arguments.

Consider now Case (II). Proceeding as in the even case, we easily get the following inversion formulas for the (modified) scalar components:

$$f(t) = \frac{1}{2\pi}\left(\operatorname{ch}\frac{t}{2}\right)\int_0^\infty \frac{d(2\lambda)}{|\mathbf{c}(2\lambda)|^2} \times [\mathcal{H}_+(F)(\lambda) + \mathcal{H}_-(F)(\lambda)]\,\phi_{2\lambda}^{(n/2-1,n/2)}\left(\frac{t}{2}\right), \tag{6.16}$$

where $\mathbf{c} = \mathbf{c}^{(n/2-1,n/2)}$ is defined as in the even case, and

$$\widetilde{f}(t) = \frac{1}{2\pi}\left(\operatorname{sh}\frac{t}{2}\right)\int_0^\infty \frac{d(2\lambda)}{|\widetilde{\mathbf{c}}(2\lambda)|^2} \times [\mathcal{H}_+(F)(\lambda) - \mathcal{H}_-(F)(\lambda)]\,\frac{-in}{2\lambda}\,\phi_{2\lambda}^{(n/2,n/2-1)}\left(\frac{t}{2}\right), \tag{6.17}$$

where

$$\widetilde{\mathbf{c}}(\lambda) := \mathbf{c}^{(n/2,n/2-1)}(\lambda) = 2^{n-i\lambda}\frac{\Gamma(n/2+1)\Gamma(i\lambda)}{\Gamma(i\lambda+n/2)\Gamma(i\lambda/2+1)}. \tag{6.18}$$

Now we have the following easy result.

LEMMA 6.5. *Let n be odd. With $\mathbf{c}$ as in* (6.14) *and $\widetilde{\mathbf{c}}$ as in* (6.18), *define, for $\lambda \in \mathbb{R}$,*

$$d\nu(\lambda) = \frac{d(2\lambda)}{2\pi\,|\mathbf{c}(2\lambda)|^2}, \qquad d\widetilde{\nu}(\lambda) = \frac{d(2\lambda)}{2\pi\,|\widetilde{\mathbf{c}}(2\lambda)|^2}.$$

Then:

(i) *$d\nu(\lambda)$ has density* (6.10) *with respect to $d\lambda$.*
(ii) *$d\widetilde{\nu}(\lambda)$ has density*

$$\widetilde{\nu}(\lambda) := 2^2\pi^{-1}\left[\frac{n+1}{2}\left(\frac{n+1}{2}+1\right)\cdots n\right]^{-2}\lambda^2\prod_{j=1}^{(n-1)/2}[\lambda^2+(j-1/2)^2]$$

with respect to $d\lambda$.

(iii) *These densities are related by $\widetilde{\nu}(\lambda) = (4\lambda^2/n^2)\nu(\lambda)$.*

As a consequence, with our usual notations, we can write the following formulas:

$$
\begin{aligned}
f(t) &= \int_0^\infty d\nu(\lambda)\,[\mathcal{H}_+(F)(\lambda)+\mathcal{H}_-(F)(\lambda)]\varphi_\pm(\lambda,t),\\
\widetilde{f}(t) &= \int_0^\infty d\widetilde{\nu}(\lambda)\,[\mathcal{H}_+(F)(\lambda)-\mathcal{H}_-(F)(\lambda)]\bigg(\frac{-in}{2\lambda}\operatorname{sh}\frac{t}{2}\bigg)\phi_{2\lambda}^{(n/2,n/2-1)}\bigg(\frac{t}{2}\bigg)\\
&= \int_0^\infty d\nu(\lambda)\,[\mathcal{H}_+(F)(\lambda)-\mathcal{H}_-(F)(\lambda)]\widetilde{\varphi}_+(\lambda,t),
\end{aligned}
$$

from which we get after some manipulations:

$$
\begin{aligned}
f^+(t) &= \int_0^\infty d\nu(\lambda)\,[\mathcal{H}_+(F)(\lambda)\varphi_+^+(\lambda,t)+\mathcal{H}_-(F)(\lambda)\varphi_-^+(\lambda,t)],\\
f^-(t) &= \int_0^\infty d\nu(\lambda)\,[\mathcal{H}_+(F)(\lambda)\varphi_+^-(\lambda,t)+\mathcal{H}_-(F)(\lambda)\varphi_-^-(\lambda,t)].
\end{aligned}
$$

These identities clearly imply the inversion formula (and the Plancherel formula) in (II.A).

The statements in (II.B) are not so easy as in the previous case, since we have two “interlaced” Jacobi transforms.

We first examine Case (i). Suppose $F \in C_R^\infty$, i.e. $f^\pm \in C_R^\infty(\mathbb{R})$. Then it is clear that $f(2\cdot)/\mathrm{ch}$ and $\widetilde{f}(2\cdot)/\mathrm{sh}$ both belong to $C_R^\infty(\mathbb{R})_{\mathrm{even}}$. By (6.4) and Theorem 2.1 of [Koo84], $\mathcal{H}_\pm$ maps continuously C_R^∞ into $\mathrm{PW}_R(\mathbb{C})$. Recall also (5.11), which yields the relation

$$\mathcal{H}_\pm(F)(-\lambda) = \mathcal{H}_\mp(F)(\lambda). \tag{6.19}$$

Conversely, let $h = (h_+, h_-) \in \mathrm{PW}_R(\mathbb{C})^2$ be such that $h_\pm(-\lambda) = h_\mp(\lambda)$, and define its *wave packet transform*:

$$F_h(x) = \int_0^\infty d\nu(\lambda)\,[h_+(\lambda)\Phi_+(\lambda,x)+h_-(\lambda)\Phi_-(\lambda,x)].$$

Clearly, F_h is a τ_n-radial function and its scalar components $f^\pm$ belong to $C^\infty(\mathbb{R})$ since $h_\pm$ is entire. Let us now check the support preserving property. With the usual notations, $f^\pm = f \pm \widetilde{f}$, where

$$f(t) = \bigg(\operatorname{ch}\frac{t}{2}\bigg)(\mathcal{J}^{(n/2-1,n/2)})^{-1}(h_+(\cdot/2)+h_-(\cdot/2))\bigg(\frac{t}{2}\bigg), \tag{6.20}$$

$$\widetilde{f}(t) = -\frac{in}{2}\bigg(\operatorname{sh}\frac{t}{2}\bigg)(\mathcal{J}^{(n/2,n/2-1)})^{-1}\bigg(\frac{h_+(\cdot/2)-h_-(\cdot/2)}{\cdot/2}\bigg)\bigg(\frac{t}{2}\bigg). \tag{6.21}$$

Since $h_+ + h_- \in \mathrm{PW}_R(\mathbb{C})_{\mathrm{even}}$, $f \in C_R^\infty(\mathbb{R})_{\mathrm{even}}$ by Theorem 2.1 of [Koo84]. On the other hand, since $h_+ - h_- \in \mathrm{PW}_R(\mathbb{C})_{\mathrm{odd}}$ and $h_+(0) = h_-(0)$, we see that $\lambda \mapsto (h_+(\lambda)-h_-(\lambda))/\lambda$ is in $\mathrm{PW}_R(\mathbb{C})_{\mathrm{even}}$ and this time we deduce that $\widetilde{f} \in C_R^\infty(\mathbb{R})_{\mathrm{odd}}$. Hence $f^\pm = f \pm \widetilde{f} \in C_R^\infty(\mathbb{R})$, as required.

In Case (ii), the proof is similar, based on the isomorphism (6.15). ∎

REMARK 6.6. The Plancherel measures defined in the theorem were calculated first in [Cam92] and can also be derived by using [Mia79].

7. Fourier transform. As a first application of Theorem 6.3, we give the inversion and Plancherel formulas for the Fourier transform of spinors on $H^n(\mathbb{R})$.

Given $f \in C_c^\infty(G,\tau)$ and $\sigma \in \widehat{M}(\tau)$, we define for $\lambda \in \mathbb{C}$ the *partial Fourier transforms*

$$\mathcal{H}_\sigma^\tau(f)(\lambda) := \int_G dx\, \pi_{\sigma,\lambda}(x) \circ J_\sigma^\tau f(x). \tag{7.1}$$

(The integration can be performed on G/K.) The collection

$$\lambda \mapsto \{\mathcal{H}_\sigma^\tau(f)(\lambda)\}_{\sigma\in\widehat{M}(\tau)}$$

of these $C^\infty(K,\sigma)$-valued functions is called the *Fourier transform* $\mathcal{H}(f)$ of f.

REMARK 7.1. In terms of the Helgason–Fourier transform $\widetilde{f}(\lambda,k)$ defined in [Cam97a], we have

$$\mathcal{H}_\sigma^\tau(f)(\lambda)(k) = \sqrt{\frac{\dim\tau}{\dim\sigma}}\, P_\sigma\, \widetilde{f}(\lambda,k) \qquad (k\in K),$$

where

$$\widetilde{f}(\lambda,k) = \int_G dx\, e^{-(i\lambda+\varrho)H(x^{-1}k)}\tau(\underline{k}(x^{-1}k)^{-1})f(x).$$

REMARK 7.2. The relation between the Fourier transform and the spherical Fourier transform defined earlier is as follows. Let $F \in C_c^\infty(G,\tau,\tau)$ and $v\in V_\tau$, so that $f = F(\cdot)v \in C_c^\infty(G,\tau)$. Then

$$\mathcal{H}_\sigma^\tau(f)(\lambda) = \mathcal{H}_\sigma^\tau(F)(\lambda)\, J_\sigma^\tau v.$$

(See e.g. the proof of Proposition 8.2 in [Ped97].)

For brevity let us keep the notations $\mathcal{H}^\pm = \mathcal{H}_{\sigma_{n-1}}^{\tau_n^\pm}$ and $\mathcal{H}_\pm = \mathcal{H}_{\sigma_{n-1}^\pm}^{\tau_n}$.

The next result can be considered as a more concrete (analytic) version of Theorem 3.2.

THEOREM 7.3. (I) *Let n be even and define the Plancherel measure $d\nu$ as in* (6.7). *Then*:

(i) *The Fourier transform on $C_c^\infty(G,\tau_n^\pm)$ is inverted by the following formula*:

$$f^\pm(x) = \int_0^\infty d\nu(\lambda)\, P^\pm \circ \pi_{\sigma_{n-1},\lambda}(x^{-1})\, \mathcal{H}^\pm(f^\pm)(\lambda).$$

(ii) *For* $f^{\pm} \in C_c^\infty(G,\tau_n^{\pm})$, *we have the Plancherel formula*

$$\|f^{\pm}\|^2_{L^2(G,\tau_n^{\pm})} = \int_0^\infty d\nu(\lambda)\, \|\mathcal{H}^{\pm}(f^{\pm})(\lambda)\|^2_{L^2(K,\sigma_{n-1})}.$$

(iii) *The Fourier transform* $\mathcal{H}^{\pm}$ *extends to a bijective isometry from* $L^2(G,\tau_n^{\pm})$ *onto* $L^2(\mathbb{R}_+; d\nu(\lambda); L^2(K,\sigma_{n-1}))$.

(II) *Let* n *be odd and define the Plancherel measure* $d\nu$ *as in* (6.10). *Then*:

(i) *The Fourier transform on* $C_c^\infty(G,\tau_n)$ *is inverted by the following formula*:

$$f(x) = \int_0^\infty d\nu(\lambda)\, [P_+ \circ \pi_{\sigma^+_{n-1},\lambda}(x^{-1})\mathcal{H}_+(f)(\lambda) + P_- \circ \pi_{\sigma^-_{n-1},\lambda}(x^{-1})\mathcal{H}_-(f)(\lambda)].$$

(ii) *For* $f \in C_c^\infty(G,\tau_n)$, *we have the Plancherel formula*

$$\|f\|^2_{L^2(G,\tau_n)} = \int_0^\infty d\nu(\lambda)\, [\|\mathcal{H}_+(f)(\lambda)\|^2_{L^2(K,\sigma^+_{n-1})} + \|\mathcal{H}_-(f)(\lambda)\|^2_{L^2(K,\sigma^-_{n-1})}].$$

(iii) *The Fourier transform* $\mathcal{H} = (\mathcal{H}_+,\mathcal{H}_-)$ *extends to a bijective isometry from* $L^2(G,\tau_n)$ *onto*

$$L^2(\mathbb{R}_+; d\nu(\lambda); L^2(K,\sigma^+_{n-1})) \oplus L^2(\mathbb{R}_+, d\nu(\lambda); L^2(K,\sigma^-_{n-1})).$$

Proof. Since the proof is similar to the one given in [Ped97] for the case of differential forms, we just indicate the main ingredients. For simplicity, let us come back to generic notations τ and $\sigma \in \widehat{M}(\tau)$. For $f \in C_c^\infty(G,\tau)$ and $F \in C^\infty(G,\tau,\tau)$, define the convolution product $f * F \in C_c^\infty(G,\tau)$ by

$$f * F(x) = \int_G dy\, F(y^{-1}x)\, f(y). \tag{7.2}$$

Proposition 7.4. *For* $f \in C_c^\infty(G,\tau)$, *we have*

$$f(x) = \sum_{\sigma\in\widehat{M}(\tau)} \int_0^\infty d\nu_\sigma(\lambda)\, \{f * \Phi^\tau_\sigma(\lambda,\cdot)\}(x). \tag{7.3}$$

Proof. Fix $v \in V_\tau$ and consider the τ-radial function $x \mapsto F_y(x)$ defined for any $y \in G$ by

$$F_y(x) = \int_K dk\, (f(ykx) \otimes v^*)\tau(k),$$

where v^* is defined by $v^*(w) = (w, v)$ and $\xi \otimes v^*$ is considered as an element of $\operatorname{End} V_\tau$ by the usual rule $\xi \otimes v^*(w) = (w, v)\,\xi$. Computing $\operatorname{tr} F_y(e)$ first directly, and then with the inversion formulas of Theorem 6.3 yields the result after some manipulations. ■

The inversion formulas in the theorem are then an immediate consequence of (7.3) and of the following identity:

$$\{f * \Phi_\sigma^\tau(\lambda,\cdot)\}(x) = P_\sigma^\tau \circ \pi_{\sigma,\lambda}(x^{-1})\, \mathcal{H}_\sigma^\tau(f)(\lambda). \tag{7.4}$$

For part (ii) of the above theorem, we use (7.3) and (7.4) to prove that

$$(f * \Phi_\sigma^\tau(\lambda,\cdot), f) = (\mathcal{H}_\sigma^\tau(f)(\lambda), \mathcal{H}_\sigma^\tau(f)(\overline{\lambda})) = \|\mathcal{H}_\sigma^\tau(f)(\lambda)\|^2 \qquad (\lambda \in \mathbb{R}).$$

The proof of the surjectivity statement in assertion (iii), although rather technical, can be done by adapting the one given in [Hel94, §III.1.2], for the scalar case. ■

Remark 7.5. As regards the Paley–Wiener theorem for the Fourier transform, we formulate the following conjecture, which is analogous to Helgason's Paley–Wiener theorem in the scalar case ([Hel94, Ch. III, Theorem 5.1]).

Let τ be as above. For $R > 0$ and $\sigma \in \widehat{M}(\tau)$, let $\mathrm{PW}_R(\mathbb{C}; C^\infty(K,\sigma))$ denote the space of holomorphic functions $h : \mathbb{C} \to C^\infty(K,\sigma)$ satisfying

$$\forall D \in U(\mathfrak{k}),\ \forall N \in \mathbb{N},\ \exists C_{D,N} > 0,\ \forall(\lambda,k) \in \mathbb{C}\times K,$$
$$\|h(\lambda)(k : D)\|_{V_\sigma} \le C_{D,N}\,(1+|\lambda|)^{-N} e^{R|\mathrm{Im}\,\lambda|}.$$

When n is even, $\mathcal{H}^\pm$ should be a topological linear isomorphism between $C_R^\infty(G,\tau_n^\pm)$ and the subset of functions $h \in \mathrm{PW}_R(\mathbb{C}; C^\infty(K,\sigma_{n-1}))$ satisfying

$$P^\pm \circ \pi_{\sigma_{n-1},\lambda}(x^{-1})h(\lambda) = P^\pm \circ \pi_{\sigma_{n-1},-\lambda}(x^{-1})h(-\lambda) \qquad (\lambda \in \mathbb{C},\ x \in G).$$

This last condition is suggested by (7.3), (7.4) and by the Weyl group action on the spherical functions (recall (3.7)). Similarly, when n is odd, $\mathcal{H} = (\mathcal{H}_+, \mathcal{H}_-)$ should be a topological linear isomorphism between $C_R^\infty(G,\tau_n)$ and the subset of couples of functions (h_+,h_-), with $h_\pm \in \mathrm{PW}_R(\mathbb{C}; C^\infty(K,\sigma_{n-1}^\pm))$ that satisfy

$$P_+ \circ \pi_{\sigma_{n-1}^+,\lambda}(x^{-1})h_+(\lambda) = P_- \circ \pi_{\sigma_{n-1}^-,-\lambda}(x^{-1})\, h_-(-\lambda) \qquad (\lambda \in \mathbb{C},\ x \in G).$$

8. Abel transform and heat equation. In this section we give further applications of the spherical Fourier transform theory.

8.1. *The Abel transform and its inverse.* Let $\tau \in \widehat{K}$ as before. For $F \in C_c^\infty(G,\tau,\tau)$, define its *Abel transform* $\mathcal{A}(F)$ by

$$\mathcal{A}(F)(t) := e^{\varrho t}\int_N dn\, F(a_t n) = e^{-\varrho t}\int_N dn\, F(n a_t) \qquad (t \in \mathbb{R}).$$

Here, the measure dn on N is defined such that $dx = e^{2\varrho t} dk\, dt\, dn$ in the Iwasawa decomposition $G = KAN$, with dx, dk and dt as in (5.3). It is immediate that $\mathcal{A}(F)(t) \in \operatorname{End}_M V_\tau$ and is therefore scalar on each subspace $V_\sigma \subset V_\tau$ if $\sigma \in \widehat{M}(\tau)$. We denote by $\mathcal{A}^\tau_\sigma(F)(t)$ the scalar components of $\mathcal{A}(F)(t)$ and think of them as *partial Abel transforms*.

The following result exhibits the link between the partial Abel transforms $\mathcal{A}^\tau_\sigma$ and the partial spherical Fourier transforms $\mathcal{H}^\tau_\sigma$. For a proof, see [Ped97, §7.1].

LEMMA 8.1. *Let $\tau \in \widehat{K}$ and $\sigma \in \widehat{M}(\tau)$. Define the Euclidean Fourier transform $\mathcal{F}$ by $\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}} dt\, e^{i\lambda t} f(t)$. Then we have the factorization*

$$\mathcal{H}^\tau_\sigma = \mathcal{F} \circ \mathcal{A}^\tau_\sigma,$$

where $\mathcal{H}^\tau_\sigma$ was defined in (6.1).

As always, we define for simplicity

$$\mathcal{A}^\pm = \mathcal{A}^{\tau_n^\pm}_{\sigma_{n-1}} \quad (n \text{ even}), \qquad \mathcal{A}_\pm = \mathcal{A}^{\tau_n}_{\sigma^\pm_{n-1}} \quad (n \text{ odd}),$$

so that we have the consistent relations

$$\mathcal{H}^\pm = \mathcal{F} \circ \mathcal{A}^\pm, \qquad \mathcal{H}_\pm = \mathcal{F} \circ \mathcal{A}_\pm.$$

Note that we then have the parity relations

$$\mathcal{A}^\pm(F^\pm)(-t) = \mathcal{A}^\pm(F^\pm)(t), \qquad \mathcal{A}_\pm(F)(-t) = \mathcal{A}_\mp(F)(t).$$

The expressions of $\mathcal{A}^\pm$ and $\mathcal{A}_\pm$ will be made very explicit in a moment. Let us introduce first some more notation, and recall some basic facts about *Weyl fractional integral operators* taken from [Koo84, §§5–6].

For $\operatorname{Re} m > 0$ and $\varepsilon > 0$, define the Weyl operator $\mathcal{W}^\varepsilon_m$ by

$$\mathcal{W}^\varepsilon_m(f)(t) = \frac{1}{\Gamma(m)} \int_t^\infty d(\operatorname{ch} \varepsilon s)\, (\operatorname{ch} \varepsilon s - \operatorname{ch} \varepsilon t)^{m-1} f(s). \tag{8.1}$$

Then $\mathcal{W}^\varepsilon_m \circ \mathcal{W}^\varepsilon_{m'} = \mathcal{W}^\varepsilon_{m+m'}$. Moreover, since

$$\mathcal{W}^\varepsilon_m = \left(-\frac{d}{d(\operatorname{ch} \varepsilon t)}\right)^j \circ \mathcal{W}^\varepsilon_{m+j} = \mathcal{W}^\varepsilon_{m+j} \circ \left(-\frac{d}{d(\operatorname{ch} \varepsilon t)}\right)^j$$

for $j \in \mathbb{N}$ and $\operatorname{Re} m > -j$, the Weyl transform has an analytic continuation to all complex m, $\mathcal{W}^\varepsilon_m$ has inverse $\mathcal{W}^\varepsilon_{-m}$ and is a topological linear isomorphism:

$$\mathcal{W}^\varepsilon_m : C^\infty_R(\mathbb{R})_{\text{even}} \xrightarrow{\simeq} C^\infty_R(\mathbb{R})_{\text{even}},$$

$$(\operatorname{ch} t)^{-s} \mathcal{S}(\mathbb{R})_{\text{even}} \xrightarrow{\simeq} (\operatorname{ch} t)^{-s+\varepsilon \operatorname{Re}(m)} \mathcal{S}(\mathbb{R})_{\text{even}} \qquad (s \geq \varepsilon \operatorname{Re}(m) \geq 0).$$

Now, for complex α, β with $\alpha \neq -1, -2, \ldots$, define an operator on $C^\infty_c(\mathbb{R})_{\text{even}}$ by

$$\mathcal{A}^{(\alpha,\beta)} = 2^{3\alpha+1/2}\pi^{-1/2}\,\Gamma(\alpha+1)\,\mathcal{W}^1_{\alpha-\beta}\circ\mathcal{W}^2_{\beta+1/2}. \tag{8.2}$$

We call this operator the *scalar Abel transform*. Then the Jacobi transform defined in (6.2) is the composition

$$\mathcal{J}^{(\alpha,\beta)} = \mathcal{F}\circ\mathcal{A}^{(\alpha,\beta)}. \tag{8.3}$$

We have the following expressions for the partial Abel transforms.

PROPOSITION 8.2. (I) *Let n be even. If $F^\pm \in C^\infty_{\mathrm{c}}(G,\tau^\pm_n,\tau^\pm_n)$ or $F^\pm \in \mathcal{S}(G,\tau^\pm_n,\tau^\pm_n)$ has scalar component $f^\pm$, then*

$$\mathcal{A}^\pm(F^\pm)(t) = \frac{1}{4}\mathcal{A}^{(n/2-1,n/2)}\left(\frac{f^\pm(2\cdot)}{\mathrm{ch}}\right)\left(\frac{t}{2}\right), \tag{8.4}$$

where $\mathcal{A}^{(n/2-1,n/2)} = 2^{(3n-5)/2}\pi^{-1/2}\Gamma(n/2)\mathcal{W}^1_{-1}\circ\mathcal{W}^2_{(n+1)/2}$.

(II) *Let n be odd. If $F\in C^\infty_{\mathrm{c}}(G,\tau_n,\tau_n)$ or $F\in\mathcal{S}(G,\tau_n,\tau_n)$ has scalar components f^+, f^-, let as usual $f=\frac12(f^+ + f^-)$ and $\widetilde{f}=\frac12(f^+ - f^-)$. Then*

$$\begin{aligned}\mathcal{A}_\pm(F)(t) &= \frac{1}{4}\mathcal{A}^{(n/2-1,n/2)}\left(\frac{f(2\cdot)}{\mathrm{ch}}\right)\left(\frac{t}{2}\right)\\ &\quad\mp\frac{1}{2}\frac{d}{dt}\mathcal{A}^{(n/2,n/2-1)}\left(\frac{\widetilde{f}(2\cdot)}{\mathrm{sh}}\right)\left(\frac{t}{2}\right),\end{aligned} \tag{8.5}$$

where $\mathcal{A}^{(n/2-1,n/2)}$ is defined as in (I) *and*

$$\mathcal{A}^{(n/2,n/2-1)} = 2^{(3n+1)/2}\pi^{-1/2}\Gamma(n/2+1)\mathcal{W}^1_1\circ\mathcal{W}^2_{(n-1)/2}.$$

Proof. Compare Lemma 8.1 with (8.3) and use Proposition 6.1. ∎

As in the case of the spherical Fourier transform, the reduction to scalar Abel analysis allows the following statement.

THEOREM 8.3. (I) *Let n be even.*

(I.A) *If $F^\pm\in C^\infty_{\mathrm{c}}(G,\tau^\pm_n,\tau^\pm_n)$ or $F^\pm\in\mathcal{S}(G,\tau^\pm_n,\tau^\pm_n)$ has scalar component $f^\pm$, then for all $t\in\mathbb{R}$, we have the equivalent inversion formulas:*

$$\begin{aligned}f^\pm(t) &= 2^{(5-3n)/2}\Gamma\left(\frac{n}{2}\right)^{-1}\left(\mathrm{ch}\,\frac{t}{2}\right)\\ &\qquad\times\int_t^\infty\frac{d(\mathrm{ch}\,s)}{\sqrt{\mathrm{ch}\,s-\mathrm{ch}\,t}}\left(-\frac{d}{d(\mathrm{ch}\,s)}\right)^{n/2}\left(\mathrm{ch}\,\frac{s}{2}\right)^{-1}\mathcal{A}^\pm(F^\pm)(s)\\ &= 2^{(5-3n)/2}\Gamma\left(\frac{n}{2}\right)^{-1}\left(\mathrm{ch}\,\frac{t}{2}\right)\\ &\qquad\times\left(-\frac{d}{d(\mathrm{ch}\,t)}\right)^{n/2}\int_t^\infty\frac{d(\mathrm{ch}\,s)}{\sqrt{\mathrm{ch}\,s-\mathrm{ch}\,t}}\left(\mathrm{ch}\,\frac{s}{2}\right)^{-1}\mathcal{A}^\pm(F^\pm)(s).\end{aligned}$$

(I.B) *$\mathcal{A} \simeq \mathcal{A}^{\pm}$ is a topological linear isomorphism between*

(i) *$C_R^\infty(G, \tau_n^\pm, \tau_n^\pm)$ and $C_R^\infty(\mathbb{R}; \mathrm{End}_M V_{\tau_n^\pm}) \simeq C_R^\infty(\mathbb{R})_{\mathrm{even}}$ for all $R > 0$;*

(ii) *$\mathcal{S}(G, \tau_n^\pm, \tau_n^\pm)$ and $\mathcal{S}(\mathbb{R}; \mathrm{End}_M V_{\tau_n^\pm}) \simeq \mathcal{S}(\mathbb{R})_{\mathrm{even}}$.*

(II) *Let n be odd.*

(II.A) *If $F \in C_c^\infty(G, \tau_n, \tau_n)$ or $F \in \mathcal{S}(G, \tau_n, \tau_n)$ has scalar components f^+, f^-, then $f^\pm = f \pm \widetilde{f}$ where, for all $t \in \mathbb{R}$,*

$$f(t) = 2^{(3-3n)/2}\pi^{1/2}\Gamma(n/2)^{-1}\left(\operatorname{ch}\frac{t}{2}\right) \times \left(-\frac{d}{d(\operatorname{ch} t)}\right)^{(n-1)/2}\left(\operatorname{ch}\frac{t}{2}\right)^{-1}[\mathcal{A}_+(F)(t) + \mathcal{A}_-(F)(t)],$$

$$\widetilde{f}(t) = 2^{(1-3n)/2}\pi^{1/2}\Gamma(n/2+1)^{-1}\left(\operatorname{sh}\frac{t}{2}\right) \times \left(-\frac{d}{d(\operatorname{ch} t)}\right)^{(n-1)/2}\left(\operatorname{sh}\frac{t}{2}\right)^{-1}[\mathcal{A}_+(F)(t) - \mathcal{A}_-(F)(t)].$$

(II.B) *$\mathcal{A} = (\mathcal{A}_+, \mathcal{A}_-)$ is a topological linear isomorphism between*

(i) *$C_R^\infty(G, \tau_n, \tau_n)$ and $C_R^\infty(\mathbb{R}; \mathrm{End}_M V_{\tau_n}) \simeq \{(g_+, g_-) \in C_R^\infty(\mathbb{R})^2 : g_\pm(-t) = g_\mp(t)\}$ for all $R > 0$;*

(ii) *$\mathcal{S}(G, \tau_n, \tau_n)$ and $\mathcal{S}(\mathbb{R}; \mathrm{End}_M V_{\tau_n}) \simeq \{(g_+, g_-) \in \mathcal{S}(\mathbb{R})^2 : g_\pm(-t) = g_\mp(t)\}$.*

Proof. We begin with an intermediate result.

LEMMA 8.4. (I) *Let n be even. If $g \in C_c^\infty(\mathbb{R})_{\mathrm{even}}$, resp. $g \in \mathcal{S}(\mathbb{R})_{\mathrm{even}}$, then*

$$(\mathcal{A}^{(n/2-1,n/2)})^{-1}(g) \in C_c^\infty(\mathbb{R})_{\mathrm{even}},$$
$$\textit{resp.}\quad (\mathcal{A}^{(n/2-1,n/2)})^{-1}(g) \in (\operatorname{ch} t)^{-n}\mathcal{S}(\mathbb{R})_{\mathrm{even}},$$

and

$$\begin{aligned}(\mathcal{A}^{(n/2-1,n/2)})^{-1}(g)(t) &= 2^{(1-3n)/2}\Gamma(n/2)^{-1}\int_t^\infty \frac{d(\operatorname{ch} 2s)}{\sqrt{\operatorname{ch} 2s - \operatorname{ch} 2t}}\left(-\frac{d}{d(\operatorname{ch} 2s)}\right)^{n/2}(\operatorname{ch} s)^{-1}g(s)\\ &= 2^{(1-3n)/2}\Gamma(n/2)^{-1}\left(-\frac{d}{d(\operatorname{ch} 2t)}\right)^{n/2}\int_t^\infty \frac{d(\operatorname{ch} 2s)}{\sqrt{\operatorname{ch} 2s - \operatorname{ch} 2t}}(\operatorname{ch} s)^{-1}g(s).\end{aligned}$$

(II) *Let n be odd. If $g \in C_c^\infty(\mathbb{R})_{\mathrm{even}}$, resp. $g \in \mathcal{S}(\mathbb{R})_{\mathrm{even}}$, then*

$$(\mathcal{A}^{(n/2-1,n/2)})^{-1}(g),\ (\mathcal{A}^{(n/2,n/2-1)})^{-1}(g) \in C_c^\infty(\mathbb{R})_{\mathrm{even}},$$
$$\textit{resp.}\quad (\mathcal{A}^{(n/2-1,n/2)})^{-1}(g),\ \mathcal{A}^{(n/2,n/2-1)})^{-1}(g) \in (\operatorname{ch} t)^{-n}\mathcal{S}(\mathbb{R})_{\mathrm{even}},$$

and

$$\begin{aligned}\mathcal{A}^{(n/2-1,n/2)})^{-1}(g)(t) &= 2^{(1-3n)/2}\,\pi^{1/2}\,\Gamma(n/2)^{-1}\\ &\quad\times\left(-\frac{d}{d(\operatorname{ch}2t)}\right)^{(n-1)/2}(\operatorname{ch}t)^{-1}g(t),\\ (\mathcal{A}^{(n/2,n/2-1)})^{-1}(g)(t) &= 2^{-(1+3n)/2}\,\pi^{1/2}\Gamma(n/2+1)^{-1}\\ &\quad\times\left(-\frac{d}{d(\operatorname{ch}2t)}\right)^{(n-1)/2}\left(-\frac{d}{d(\operatorname{ch}t)}\right)g(t).\end{aligned}$$

Proof. The first assertions in both parts follow from (6.7) and (6.8) in [Koo84]. So let us prove the formulas.

Suppose first n is even. From the definition (8.2) and the inversion rule for the Weyl operators we have

$$(\mathcal{A}^{(n/2-1,n/2)})^{-1} = 2^{(5-3n)/2}\pi^{1/2}\Gamma(n/2)^{-1}\mathcal{W}^2_{-(n+1)/2}\circ\mathcal{W}^1_1.$$

Rewriting $\mathcal{W}^2_{-(n+1)/2}$ as $\mathcal{W}^2_{1/2}\circ\mathcal{W}^2_{-n/2-1}$, we get

$$\begin{aligned}&(\mathcal{A}^{(n/2-1,n/2)})^{-1}(g)(t)\\ &= 2^{(5-3n)/2}\Gamma(n/2)^{-1}\int_t^\infty\frac{d(\operatorname{ch}2s)}{\sqrt{\operatorname{ch}2s-\operatorname{ch}2t}}\left(-\frac{d}{d(\operatorname{ch}2s)}\right)^{n/2+1}\int_s^\infty d(\operatorname{ch}r)\,g(r).\end{aligned}$$

But it is easily seen that

$$-\frac{d}{d(\operatorname{ch}2s)}\int_s^\infty d(\operatorname{ch}r)\,g(r) = \tfrac14(\operatorname{ch}s)^{-1}g(s),$$

hence the first formula. The second one is obtained by writing the alternative decomposition of $\mathcal{W}^2_{-(n+1)/2}$ as $\mathcal{W}^2_{-n/2}\circ\mathcal{W}^2_{1/2}\circ\mathcal{W}^2_{-1}$.

Suppose now n is odd. In this case we write

$$(\mathcal{A}^{(n/2-1,n/2)})^{-1} = 2^{(5-3n)/2}\pi^{1/2}\Gamma(n/2)^{-1}\mathcal{W}^2_{-(n-1)/2}\circ\mathcal{W}^2_{-1}\circ\mathcal{W}^1_1$$

and proceed as in the even case to get the first formula. Similarly,

$$(\mathcal{A}^{(n/2,n/2-1)})^{-1} = 2^{-(1+3n)/2}\pi^{1/2}\Gamma(n/2+1)^{-1}\mathcal{W}^2_{-(n-1)/2}\circ\mathcal{W}^1_{-1}$$

yields the second formula. ∎

We come back to the proof of the theorem. Suppose n is even. By inverting (8.4) we immediately get

$$f^\pm(t) = 4\left(\operatorname{ch}\frac{t}{2}\right)(\mathcal{A}^{(n/2-1,n/2)})^{-1}(\mathcal{A}^\pm(F^\pm)(2\cdot))\left(\frac{t}{2}\right).$$

Thus part (I) of the above lemma implies all statements in part (I) of the theorem.

Suppose now n is odd. By inverting (8.5), we get

$$f(t) = 2\left(\operatorname{ch}\frac{t}{2}\right)(\mathcal{A}^{(n/2-1,n/2)})^{-1}[\mathcal{A}_+(F)(2\cdot) + \mathcal{A}_-(F)(2\cdot)]\left(\frac{t}{2}\right)$$

$$\widetilde{f}(t) = \left(\operatorname{sh}\frac{t}{2}\right)(\mathcal{A}^{(n/2,n/2-1)})^{-1}\Big\{\int_{2\cdot}^{\infty} dr\,[\mathcal{A}_+(F)(r) - \mathcal{A}_-(F)(r)]\Big\}\left(\frac{t}{2}\right)$$

and the statements follow again easily from the lemma. Note that in (II.B) we use the trivial fact that the operator d/dt is an isomorphism from $C_{\mathrm{c}}(\mathbb{R})_{\mathrm{even}}$ (resp. $\mathcal{S}(\mathbb{R})_{\mathrm{even}}$) onto $C_{\mathrm{c}}(\mathbb{R})_{\mathrm{odd}}$ (resp. $\mathcal{S}(\mathbb{R})_{\mathrm{odd}}$). ∎

8.2. *The heat kernel and the heat equation.* The spinor heat kernel was already calculated in [Cam92]. Since its determination is also a nice and very quick application of our previous theorem, we give here the result for the sake of completeness. Moreover, our present method is more general and can be applied to other classical invariant P.D.E.'s, as in the case of functions on hyperbolic spaces.

We keep the notations of the previous sections. Let us introduce the Schwartz space for functions of type τ on G:

$$\mathcal{S}(G,\tau) = \{f \in C^\infty(G,\tau) : \forall D_1, D_2 \in U(\mathfrak{g}),\ \forall N \in \mathbb{N},\\ \sup_{x\in G} \|f(D_1 : x : D_2)\|_{V_\tau}\,(1 + d(o,x))^N e^{\varrho d(o,x)} < \infty\},$$

which is similar to the one given for τ-radial functions on G (see §6).

Let $f \in \mathcal{S}(G,\tau)$. The heat equation associated with the spinor Laplacian D^2 is the differential problem

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = -D_x^2\,u(t,x), \\ \lim\limits_{t\to 0} u(t,x) = f(x), \end{cases} \tag{8.6}$$

for all $t > 0$ and $x \in G$, with $u(t,\cdot) \in \mathcal{S}(G,\tau)$. As in e.g. [Ped98b, §7], and with notation (7.2), a solution for (8.6) is given by $u(t,x) = f * P_t(x)$, where the *heat kernel* P_t is the element in $\mathcal{S}(G,\tau,\tau)$ such that $\mathcal{H}_\sigma^\tau(P_t)(\lambda) = e^{-t\lambda^2}$ for $\sigma \in \widehat{M}(\tau)$. By the inversion formula for τ-radial functions (Theorem 6.3), the heat kernel is exactly

$$P_t(x) = \sum_{\sigma\in\widehat{M}(\tau)} \int_0^\infty d\nu_\sigma(\lambda)\, e^{-t\lambda^2} \varPhi_\sigma^\tau(\lambda,x) \quad (x \in G,\ t > 0). \tag{8.7}$$

A better expression is obtained by inverting the spherical transform via Lemma 8.1, and by applying Theorem 8.3. More precisely, we have the following result in our setting.

THEOREM 8.5. (I) *Let n be even. Denote by $p_t^{\pm}$ the scalar component of the heat kernel $P_t^{\pm} \in \mathcal{S}(G, \tau_n^{\pm}, \tau_n^{\pm})$. Then, for all $s \in \mathbb{R}$,*

$$\begin{aligned} p_t^{\pm}(s) = {} & 2^{(5-3n)/2} \Gamma(n/2)^{-1} \frac{1}{\sqrt{4\pi t}} \left(\operatorname{ch} \frac{s}{2}\right) \\ & \times \int_s^{\infty} \frac{d(\operatorname{ch} r)}{\sqrt{\operatorname{ch} r - \operatorname{ch} s}} \left(-\frac{d}{d(\operatorname{ch} r)}\right)^{n/2} \left(\operatorname{ch} \frac{r}{2}\right)^{-1} e^{-r^2/(4t)}. \end{aligned}$$

(II) *Let n be odd. Denote by p_t^+ and p_t^- the scalar components of the heat kernel $P_t \in \mathcal{S}(G, \tau_n, \tau_n)$. Then, for all $s \in \mathbb{R}$,*

$$\begin{aligned} p_t^+(s) &= p_t^-(s) \\ &= 2^{(5-3n)/2} \pi^{1/2} \Gamma(n/2)^{-1} \frac{1}{\sqrt{4\pi t}} \left(\operatorname{ch} \frac{s}{2}\right) \\ & \quad \times \left(-\frac{d}{d(\operatorname{ch} s)}\right)^{(n-1)/2} \left(\operatorname{ch} \frac{s}{2}\right)^{-1} e^{-s^2/(4t)}. \end{aligned}$$

Proof. Straightforward from the inversion formulas of Theorem 8.3 and from the very classical identity $\mathcal{F}^{-1}(e^{-t\lambda^2})(r) = (1/\sqrt{4\pi t}) e^{-r^2/(4t)}$. Note that in the odd case the "odd" component $\widetilde{p}_t = \frac{1}{2}(p_t^+ - p_t^-)$ vanishes since $\mathcal{A}_+(P_t) = \mathcal{A}_-(P_t)$. ∎

REMARK 8.6. The result given in [Cam92] for the case of n even looks a bit different, but a simple calculation shows that it actually coincides with the one given in the previous theorem.

We end this section with a result that gives information about the decay at infinity of the spinor heat kernel $P_t(x)$ when $x = e$ is the identity in G.

PROPOSITION 8.7. *Let P_t be the spinor heat kernel on $H^n(\mathbb{R})$. Then*

$$\operatorname{tr} P_t(e) \underset{t \to \infty}{\sim} c t^{-1/2},$$

where $c > 0$ is some constant.

Proof. We use the inversion formula (8.7) and make the change of variables $\mu = t^{1/2}\lambda$ in the integral. Then, observing the asymptotic behaviour of $\nu(\mu t^{-1/2})$, where ν is the Plancherel density given by (6.7) and (6.10), easily implies the result (which does not depend on the parity of n, for once). ∎

REMARK 8.8. Let M be a closed oriented topological manifold, with universal covering $\widetilde{M}$. Then one can define certain *Novikov–Shubin numbers* $\alpha_p(M)$ related to the decay at infinity of the trace of the heat kernel operator $e^{-t\widetilde{\Delta}_p}$ associated with the Laplacian $\widetilde{\Delta}_p$ acting on p-forms on $\widetilde{M}$ (see e.g. [Lot92, §II]). These numbers are topological invariants of M (see *ibid.*, §V). It seems to be an open question whether the analogous Novikov–Shubin number associated with the spinor heat kernel on $\widetilde{M}$ can also have topological (or, at least, geometrical) invariance properties. In any case, our previous proposition shows that the following "spin Novikov–Shubin number"

$$\alpha'(M) := \sup\{\beta : \operatorname{tr} \widetilde{P}_t(e) = O(t^{-\beta/2}) \text{ as } t \to \infty\}$$

equals 1 for all (closed oriented) real hyperbolic manifolds M with universal covering $H^n(\mathbb{R})$.

REMARK 8.9. Other invariant P.D.E.'s have been considered previously on the spinor bundle over $H^n(\mathbb{R})$. Namely, by using a Radon transform on the spinor bundle, it was shown in [BOS94] that in odd dimension (for n) Huyghens' principle and equipartition of charge hold for the Dirac equation, and the spinor wave equation has an equipartitioned energy.

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