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ON SUPPORTS OF DYNAMICAL LAMINATIONS AND BIACCESSIBLE POINTS IN POLYNOMIAL JULIA SETS

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Abstract. We use Beurling estimates and Zdunik's theorem to prove that the support of a lamination of the circle corresponding to a connected polynomial Julia set has zero length, unless f is conjugate to a Chebyshev polynomial. Equivalently, except for the Chebyshev case, the biaccessible points in the connected polynomial Julia set have zero harmonic measure.

A connected, locally connected, full, compact subset K of the complex plane $\mathbb{C}$ can be topologically described by a lamination of a circle, which tells how to pinch the circle to obtain K (see, e.g., [Dou]).

A lamination is an equivalence relation on the unit circle $\mathbb{T}$, identifying points ζ and ζ' if they are mapped to one point in K by the Riemann uniformization map of the complement of K. To obtain a topological model of the compact set K, we glue together points of the unit circle, belonging to one equivalence class. The *support* of a lamination is defined as the union of all non-trivial (containing 2 or more points) equivalence classes, i.e. it includes those points which are identified with some other points.

The laminations so defined are topologically fully characterized (among all equivalence relations on $\mathbb{T}$, see [Dou]) by the following properties:

(1) the graph $\{(\zeta,\zeta') : \zeta \sim \zeta'\}$ is a closed set in $\mathbb{T} \times \mathbb{T}$,
(2) the convex hulls of different equivalence classes are disjoint,
(3) each equivalence class is totally disconnected.

There are also analytical properties (e.g. the logarithmic capacity of each equivalence class is zero) which are not fully understood, and it is a difficult open question how to characterize laminations analytically among all equivalence relations on $\mathbb{T}$. It also makes sense to consider laminations corresponding to not necessarily locally connected compacta, but those lamina-

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tions satisfy weaker topological conditions (property (2) is lost), and carry less information about the topological structure of K.

The lamination language turns out to be particularly useful in the study of polynomial Julia sets (see [Dou] and [Thu]). One reason is that the corresponding laminations are invariant under the dynamics $z \mapsto z^d$ on the unit circle. Therefore not every lamination realizable by some compact K can be realized by a filled-in polynomial Julia set. There is an extensive theory describing the combinatorial structure of the laminations arising from polynomial dynamics (see [Dou]).

The goal of this note is to estimate the size of the support of polynomial laminations. For the Chebyshev polynomials, whose Julia set is the interval $[-2, 2]$, the lamination is supported by the whole unit circle without the two points ± 1. The following theorem shows that this is the only case when a polynomial lamination has support of positive Lebesgue measure:

THEOREM. *Suppose that a polynomial f has connected Julia set. Then the support of the corresponding lamination λ_f has zero length*, *unless f is conjugate to a Chebyshev polynomial.*

Harmonic measure. To make our statements more precise, consider a polynomial f of degree d with connected Julia set J, denote by $\mathcal{F}_\infty$ its domain of attraction to infinity, and by ϕ the Riemann uniformization map

$$\phi : \{|z| < 1\} \to \mathcal{F}_\infty, \qquad 0 \mapsto \infty.$$

Note that ϕ can be chosen so that it conjugates the dynamics $T : z \mapsto z^d$ in the unit disk with the dynamics f in $\mathcal{F}_\infty$ (see [CG] for the basic facts from complex dynamics). Let m denote the normalized (multiplied by $(2\pi)^{-1}$) length on the unit circle $\mathbb{T}$. Then by the Beurling theorem (see [Pom], Thm. 9.19), ϕ has angular limits nearly everywhere on the unit circle (i.e. except for a set of logarithmic capacity zero), which is much stronger than m-almost everywhere. Therefore we can extend the domain of definition of ϕ to include m-almost all points on $\mathbb{T}$ and define *harmonic measure* ω as the image of m under ϕ:

$$\omega(X) := m(\phi^{-1}X).$$

If A_∞ is multiply connected (i.e. for disconnected Julia sets), harmonic measure can be defined as the equilibrium measure for the logarithmic potential, since the capacity of the Julia set J is 1 (see [Bro]).

In the dynamical context harmonic measure was first considered by H. Brolin [Bro], who proved that it is invariant under f, and moreover balanced, i.e. its Jacobian is equal to d: if f is 1-to-1 on X, then $\omega(f(X)) = d \cdot \omega(X)$. Brolin also showed that ω is ergodic, has the strong mixing property, and the preimages $f^{-n}z$ are distributed asymptotically uniformly with respect to ω as n tends to infinity. M. Lyubich [Lju] proved that for every rational function there exists a unique invariant measure maximizing the entropy; it coincides with the balanced measure, and has entropy $\log d$, which is the topological entropy of this dynamical system. There is an extensive literature on the properties of harmonic measure on Julia sets (see references in the introduction of [Smi]).

External rays, accessibility, and laminations. The image R_θ, $\theta \in [0, 2\pi)$, of a ray $\{re^{i\theta} : 0 < r < 1\}$ under the Riemann map ϕ is called the *external ray with angle* θ. The above-mentioned theorem of A. Beurling implies that for nearly every (and hence almost every) θ the limit $\phi(e^{i\theta}) := \lim_{r\to 1-} \phi(re^{i\theta})$ exists, in which case one says that the corresponding ray R_θ *lands*, and its endpoint $\phi(e^{i\theta})$ is *accessible*. We call a point $z \in J$ *biaccessible* if it is a landing point of more than one external ray. By the Beurling theorem ω-almost all points are accessible, and, moreover, for nearly every θ the bounded parts $\phi(\{re^{i\theta} : 1/2 < r < 1\})$ of external rays have finite length. The *lamination* λ_f is defined by identifying angles for which external rays land at the same point. Clearly, the *support* of λ_f is the set of angles of rays which land at biaccessible points. Hence we can reformulate our Theorem as follows:

THEOREM. *Suppose that a polynomial f has connected Julia set. Then either f is conjugate to a Chebyshev polynomial, or the set of biaccessible points has zero harmonic measure.*

The set of points with three or more accesses is clearly at most countable (because the convex hulls of the corresponding equivalence classes on the unit circle are disjoint with non-empty interiors). Also points z and $f(z)$ have the same number of accesses unless z is a critical point, so, except for an at most countable number of points, the point z and all points in its grand orbit have the same number of accesses. Since countable sets have zero harmonic measure, we infer that ω-almost all points have either one or two accesses, and, up to zero harmonic measure, the sets of points with one and two accesses are each invariant under f. They are also clearly F_σ, and hence measurable. In fact, to prove that, it is sufficient to consider the rays with the length of $\{re^{i\theta} : 1/2 < r < 1\}$ bounded by M and then let M tend to infinity; a similar argument easily solves all measurability problems which can arise below. Since ω is ergodic, we deduce that the set of biaccessible points has either zero or full measure. As already mentioned, the latter case occurs for the Chebyshev polynomials, whose Julia set is the interval $[-2, 2]$, and we are going to show that this is the only possibility.

The proof, based on a geometric analysis of harmonic measure, has three steps. First, we assume that the biaccessible points have full harmonic measure and prove that for many of them (positive harmonic measure), roughly speaking, ω is equally concentrated on opposite approaches. Here we use the dynamical structure, and it is essential: for a non-trivial piece of a quasicircle (say, von Koch snowflake or a piece of the Julia set of $z^2+\varepsilon$), all points except two are biaccessible, but our proof would not work since harmonic measures on two opposite sides are mutually singular. Second, we use the Beurling-type estimate $\omega_+(B_r)\cdot\omega_-(B_r)\lesssim r^2$, where $\omega_\pm$ are the "harmonic measures on two opposite accesses," and deduce that $\omega_+(B_r)\asymp\omega_-(B_r)\lesssim r$. Therefore harmonic measure is supported on a subset of J of positive length (1-dimensional Hausdorff measure). Third, we employ the Zdunik dichotomy, which states that the latter is possible only when the Julia set is a circle or an interval.

Note that if harmonic measures corresponding to two different Fatou components of a polynomial with connected Julia set are not mutually singular, one of these components is $\mathcal{F}_\infty$. Otherwise, there are plenty of points accessible from both components, which implies that the rays from the centers of the two components to two of those points bound some domain, containing a part of the Julia set, and separate it from infinity, which contradicts $J=\partial\mathcal{F}_\infty$. Thus we can repeat the last two steps of the proof, skipping the unnecessary first one, and arrive at the following

PROPOSITION. *Suppose that a polynomial f has connected Julia set. Then either it is conjugate to z^d, or harmonic measures for any two different Fatou components are mutually singular.*

It might be interesting to further investigate the analytical structure of laminations of the unit circle, arising from polynomial dynamics. Of particular interest might be to learn whether the support of every polynomial lamination is of zero logarithmic capacity, or, more generally, to better estimate its size. Indeed, any lamination with closed support of zero capacity can be realized by a conformal map to a domain with locally connected boundary (it is particularly easy, for a given zero capacity closed subset of the circle, to construct a conformal map of the disc carrying it to one point), whereas for a lamination with a larger support the question of realization is more delicate (and hence there is more to investigate in such laminations, arising from polynomial dynamics).

Moreover, as it surfaced in the discussion of the author with P. Jones, the situation turns out to be non-trivial, as is shown by the following example (which can be generalized):

REMARK. *For a real non-hyperbolic Collet–Eckmann quadratic polynomial the biaccessible angles have positive Hausdorff dimension.*

In fact, such a polynomial has only one Fatou component $\mathcal{F}_\infty$ and a locally connected Julia set by [GS]. Thus every point in the Julia set is accessible, and except for two endpoints all points in the intersection of the Julia set with the real line have at least two accesses (from above and below). By [GS] the domain of attraction to infinity $\mathcal{F}_\infty$ is a Hölder domain, meaning that for any arc $I \subset \mathbb{T}$ one has

$$\operatorname{diam} \phi(I) \le \operatorname{const} |I|^\alpha,$$

where $\alpha > 0$ is the Hölder exponent. If a collection $\{I_j\}$ of arcs covers all biaccessible angles, then its image $\{\phi(I_j)\}$ covers the interval of intersection of the Julia set with the real line, so

$$\operatorname{const} < \sum_j \operatorname{diam} \phi(I_j) \le \operatorname{const} \sum_j |I_j|^\alpha,$$

implying that the Hausdorff dimension of the set of biaccessible angles is at least $\alpha > 0$.

Acknowledgments. The author has learned about this conjectured Theorem from the recent Stony Brook preprint [Zak1, SZ, Zak2] by S. Zakeri and D. Schleicher, who in turn have learned about it from J. Hubbard, M. Lyubich, and J. Milnor. In the preprint a similar theorem was established for locally connected, Siegel, and Cremer quadratics (with a stronger theorem in the latter cases, namely that only the preimages of the critical or Cremer point respectively can possibly be biaccessible). The author is grateful to Saeed Zakeri for helpful remarks on the first draft of this paper.

Some time after this paper was written, the author learned that a different proof of the theorem above was obtained by A. Zdunik [Zdu3].

Proof of the Theorem. We will arrive at a contradiction, assuming that there is a polynomial f not conjugate to a Chebyshev polynomial, such that the set of biaccessible points has positive (and hence full) harmonic measure.

Opposite accesses carry equal harmonic measure. Supposing that for a point x on the unit circle the point $\phi(x)$ is biaccessible, denote by $\widetilde{x}$ the point on the circle corresponding to the second approach to $\phi(x)$. Then the map $x \mapsto \widetilde{x}$ is measurable, defined almost everywhere on the unit circle, and commutes with T. Hence a measure $\widetilde{m}$ is well defined by $\widetilde{m}(X) := m(\widetilde{X})$, and has the same Jacobian with respect to T as m, namely d. But this implies that $m = \widetilde{m}$.

For points x, y on the circle denote by $|x, y|$ the length of the shortest arc joining them. Define E to be the set of all points x on the unit circle for which $\widetilde{x}$ is well defined and

$$|x, \widetilde{x}| \in \left[\frac{2\pi}{d+1}, \pi\right]. \tag{1}$$

Clearly, for any x and y there exists n such that $|T^n(x), T^n(y)| \in \left[\frac{2\pi}{d+1}, \pi\right]$.

Thus the set of points on the unit circle corresponding to biaccessible points in the Julia set is covered by preimages of E under T, and hence E has positive length.

If $d = 2$, there is only one topological way to put a few pairs of points satisfying (1) on the circle, since out of any three such pairs one must separate two others. Hence we can naturally split E into two sets E_-, E_+, so that x and $\widetilde{x}$ belong to different sets, and if $x, y \in E_+$, then x, $\widetilde{y}$ separate $\widetilde{x}$, y on the circle. If $d > 2$, then E will have at most d parts, admitting the same natural splitting, and we can just denote by E' one of them having positive length and work with it instead of E.

The intuitive meaning of this procedure is to distinguish between approaches to two different "sides" of $F := \phi(E)$. Define the corresponding "harmonic measures" $\omega_\pm$ by

$$\omega_\pm(X) := m(E_\pm \cap \phi^{-1}(X)),$$

where ϕ is defined m-almost everywhere on the unit circle by angular limits. Clearly $\widetilde{E}_- = E_+$, $\widetilde{\phi^{-1}(X)} = \phi^{-1}(X)$ and hence for any set X,

$$\omega_+(X) = \omega_-(X) = \tfrac{1}{2}\omega(X \cap E) \ . \tag{2}$$

Beurling estimate. The following inequality is essentially contained in the last section of A. Beurling's doctoral thesis (see [Beu], pp. 1–107); our notation is identical to that of [BCGJ], where one can find a self-contained proof. For any disjoint domains U and V, for any ball B_r of sufficiently small radius r one has the following inequality:

$$\omega_U(B_r) \cdot \omega_V(B_r) < A \cdot r^2, \tag{3}$$

where ω_U, ω_V denote the harmonic measures with respect to some fixed points in the corresponding domains, and the constant A depends on those domains and points only.

For the convenience of the reader we outline the proof, noting that the domains under consideration need not be locally connected. First one invokes the classical Beurling estimate

$$\omega_U(B_r(z)) < \mathrm{const}\exp\left(-\pi\int_r^1 \frac{dt}{\theta_U(t)}\right), \tag{4}$$

where $\theta_U(t)$ denotes the length of the longest arc in the intersection of U with the circle of radius t centered at z. It can be deduced by various methods, and perhaps the most powerful is Beurling's method of extremal length: harmonic measure of the ball B_r is comparable to $\exp(-\pi\lambda)$, where λ is the extremal length of the curve family joining B_r inside U to some fixed compact set. Considering a metric whose element is equal to $|dz|/\theta_U(t)$ on the longest arc in the intersection of U with the circle of radius t centered at

z, one obtains an estimate of λ from below, yielding (4). For more detailed discussion of extremal length consult the book [Ahl] (particularly pp. 76–78). Applying (4) to domains U and V, one obtains the desired inequality (3) after observing that $\theta_U(t)+\theta_V(t) \leq 2\pi t$ (the domains U and V are disjoint), and hence

$$-\frac{1}{\theta_U(t)} - \frac{1}{\theta_V(t)} \leq -\frac{2}{\pi t}.$$

We note that similar estimates have many applications. For example, Beurling techniques imply the Pommerenke–Levin–Yoccoz inequality on multipliers of cycles for connected polynomial Julia sets.

In our setting we can practically consider approaches to $\phi(E)$ via E_- and E_+ as two different domains. In fact, adding two slits from J to ∞, and choosing some reference points in the resulting domains $U_\pm$, we change the harmonic measure by a constant:

$$\omega_\pm(X) = \omega_\pm(X \cap E) \asymp \omega_{U_\pm}(X \cap E) \ \leq \ \omega_{U_\pm}(X).$$

Hence using the same Beurling estimate we arrive at the following version of (3): for any ball B_r of sufficiently small radius r,

$$\omega_+(B_r) \cdot \omega_-(B_r) \leq A' \cdot r^2.$$

But by (2) we know that $\omega_+(B_r) = \omega_-(B_r) = \frac{1}{2}\omega(B_r \cap E)$, and therefore we conclude that

$$\omega(B_r \cap E) < \frac{\sqrt{A'}}{2} \cdot r. \tag{5}$$

Zdunik's dichotomy. In [Zdu1] A. Zdunik proved the following theorem (in the connected case even finer analysis is possible, see [PUZ] and [Zdu2]):

For any rational function f either

(i) *f is critically finite with parabolic orbifold*, *or*

(ii) *the measure of maximal entropy ω is singular with respect to Hausdorff measure Λ_α, where $\alpha = \mathrm{HDim}(\omega)$.*

One can find the definition and classification of the critically finite rational functions with parabolic orbifold in [DH]; in particular, the only polynomials which fall into this category are conjugates of Chebyshev polynomials, or of z^d. In those cases the Julia sets are intervals and circles respectively, and ω is absolutely continuous with respect to length. Chebyshev polynomials are excluded by our assumptions, and circles do not have biaccessible points, hence the polynomial under consideration falls into the second category.

As mentioned before, for polynomials harmonic measure and the measure of maximal entropy coincide. The Hausdorff dimension $\mathrm{HDim}(\omega)$ of a measure ω is defined as the infimum of the Hausdorff dimensions of the Borel sets of full ω measure. A. Manning [Man] proved that for any connected polynomial Julia set the Hausdorff dimension of the harmonic measure is equal to 1. Moreover, N. Makarov [Mak] showed that this is the case for any domain bounded by a Jordan curve, which was improved later to any simply connected domain (with an estimate $\mathrm{HDim}(\omega) \le 1$ holding for any planar domain).

So for the polynomial under consideration harmonic measure is singular with respect to Hausdorff measure Λ_1 (i.e. length), meaning that there is a set of full harmonic measure and zero length. In particular, for any $\varepsilon > 0$ we can cover ω-almost all of J by a collection $\{B_{r_j}\}$ of small balls with $\sum r_j < \varepsilon$.

Then, using (5), we write

$$\omega(E) = \omega\Big(\bigcup_j B_{r_j} \cap E\Big) \le \sum_j \omega(B_{r_j} \cap E) < \sum_j \frac{\sqrt{A'}}{2} \cdot r_j < \frac{\sqrt{A'}}{2} \cdot \varepsilon.$$

Since $\omega(E) > 0$, whereas ε can be taken arbitrarily small, we arrive at a contradiction, which proves our theorem.

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