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# HYERS–ULAM STABILITY FOR A NONLINEAR ITERATIVE EQUATION

BY

BING XU and WEINIAN ZHANG (Chengdu)

**Abstract.** We discuss the Hyers–Ulam stability of the nonlinear iterative equation $G(f^{n_1}(x), \ldots, f^{n_k}(x)) = F(x)$. By constructing uniformly convergent sequence of functions we prove that this equation has a unique solution near its approximate solution.

**1. Introduction.** When we consider a functional equation

$$E_1(h) = E_2(h) \tag{1.1}$$

and know a function $g$ which is an *approximate* solution of (1.1), i.e., $E_1(g)$ and $E_2(g)$ are *close* in some sense, we may ask whether a solution $f$ of (1.1) exists *near* $g$. As in [3], we say equation (1.1) satisfies *Hyers–Ulam stability* if for every function $g$ such that

$$\|E_1(g) - E_2(g)\| \leq \delta \tag{1.2}$$

for some constant $\delta \geq 0$, there exists a solution $f$ of (1.1) such that

$$\|f - g\| \leq \varepsilon \tag{1.3}$$

for some positive constant $\varepsilon$ depending only on $\delta$. Sometimes we say $g$ is a $\delta$-*approximate solution* of (1.1) and $f$ is $\varepsilon$-*close* to $g$.

Such a problem was raised first by S. M. Ulam in 1940 and solved for the Cauchy equation by D. H. Hyers [5] in 1941. Later, many papers on the Hyers–Ulam stability have been published, generalizing Ulam's problem and Hyer's theorem in various directions (see, e.g., [2], [3], [9] and [10]). For instance, the problem of Hyers–Ulam stability is studied by Borelli [1] for Hosszú's functional equation, Ger and Šemrl [4] for the exponential equation, Jun, Kim and Lee [6] for the gamma functional equation and beta functional equation, Nikodem [8] for the Pexider equations, and Székelyhidi [15] for the sine functional equation and cosine functional equation.

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The iterative equation

$$G(f^{n_1}(x), \ldots, f^{n_k}(x)) = F(x) \tag{1.4}$$

is an important functional equation where $x \in I$, a subset of a Banach space $X$, $F : I \to I$ is a given map, $f : I \to I$ is an unknown map, $f^i$ denotes the $i$th iterate of $f$, i.e., $f^0(x) = x$ and $f^{i+1}(x) = f(f^i(x))$ for all $x \in I$ and all $i = 0, 1, 2, \ldots$, and $n_i, i = 1, \ldots, k$, are positive integers. For linear $G$, i.e., $G(y_1, \ldots, y_k) = \sum_{i=1}^k \lambda_i y_i$, many results have been given (e.g. [7], [11], [16], [17], and [18]) on existence, uniqueness, continuous dependence, smoothness and analyticity of solutions on $I = [a, b]$. For general $G$, some results are given in [12]–[14] under the basic hypotheses:

(H1) $G: I^k = I \times \ldots \times I \to I$ is continuous, $G(a, \ldots, a) = a$, $G(b, \ldots, b) = b$;

(H2) there exist constants $B_i \geq 0$, $i = 1, \ldots, k$, such that

$$G(y_1, \ldots, y_k) - G(z_1, \ldots, z_k) \leq \sum_{i=1}^k B_i (y_i - z_i) \tag{1.5}$$

if $y_i \geq z_i, i = 1, \ldots, k$;

(H3) $n_1 = 1$ and there exist constants $C_1 > 0$, $C_i \geq 0$, $i = 2, \ldots, k$, such that

$$G(y_1, \ldots, y_k) - G(z_1, \ldots, z_k) \geq \sum_{i=1}^k C_i (y_i - z_i) \tag{1.6}$$

if $y_i \geq z_i$, $i = 1, \ldots, k$.

In this paper we further discuss the Hyers–Ulam stability of equation (1.4) on $I = [a, b]$ under the hypotheses (H1) and

(H2′) there exist constants $B_i \geq 0$, $i = 1, \ldots, k$, such that

$$|G(y_1, \ldots, y_k) - G(z_1, \ldots, z_k)| \leq \sum_{i=1}^k B_i |y_i - z_i| \tag{1.7}$$

if $y_i, z_i \in I, i = 1, \ldots, k$;

(H3′) there exist constants $C_1 > 0$, $C_i \geq 0$, $i = 2, \ldots, k$, such that

$$G(y_1, \ldots, y_k) - G(z_1, \ldots, z_k) \geq C_1(y_1 - z_1) - \sum_{i=2}^k C_i |y_i - z_i| \tag{1.8}$$

if $y_i, z_i \in I$, $i = 1, \ldots, k$ and $y_1 \geq z_1$.

Our requirements are much weaker than (H2)–(H3) in [12], because (H2′)–(H3′) allow $G$ not to be monotonic, for example, $G(y_1, y_2) = \frac{3}{2} y_1 - \frac{1}{2} y_2^2$. By constructing a uniformly convergent sequence of functions we prove that there is a unique solution of (1.4) near an approximate solution.

**2. Some lemmas.** Let $\mathcal{C}(I)$ consist of all continuous functions on $I$. Then $\mathcal{C}(I)$ is a Banach space equipped with the norm $\|f\| = \max_{x\in I} |f(x)|$. We can imitate [16] and [18] to prove the following lemma but do not need to require that $f$ and $g$ be both Lipschitzian as in [16] and [18].

LEMMA 2.1. *Suppose that $f, g : I \to I$ are continuous mappings and* $\mathrm{Lip}(f) \le M$ *where $M$ is a positive constant. Then*

$$\|f^k - g^k\| \le \sum_{j=0}^{k-1} M^j \|f-g\|, \quad \forall k = 1, 2, \ldots \tag{2.9}$$

Furthermore we need the following two lemmas to construct a certain convergent sequence of functions.

LEMMA 2.2. *Suppose that $P : I \to I$ is a Lipschitzian mapping fixing the end-points of $I$ with* $\mathrm{Lip}(P) \le M$ *where $M > 0$ is a constant. If the reals $C_j$, $j = 1, \ldots, k$, satisfy $C_1 > \sum_{i=2}^k C_i M^{n_i-1}$ then the function $LP$ defined by*

$$LP(x) = G(P^{n_1-1}(x), \ldots, P^{n_k-1}(x)) \tag{2.10}$$

*is an orientation-preserving homeomorphism of $I$ onto itself*, *and*

$$\mathrm{Lip}((LP)^{-1}) \le 1\Big/\Big(C_1 - \sum_{i=2}^k C_i M^{n_i-1}\Big). \tag{2.11}$$

*Proof.* Clearly, $P^i : I \to I$ is also a Lipschitzian mapping such that $P^i(a) = a$, $P^i(b) = b$ and $\mathrm{Lip}(P^i) \le M^i$, $i = 2, 3, \ldots$, so by hypothesis (H1), $LP(a) = a$, $LP(b) = b$. Let

$$\xi = C_1 - \sum_{i=2}^k C_i M^{n_i-1} \tag{2.12}$$

for short. For any $x_1, x_2 \in I$ with $x_2 > x_1$,

$$\begin{aligned} LP(x_2) - LP(x_1) &\\ = G(P^{n_1-1}(x_2), \ldots, P^{n_k-1}(x_2)) &- G(P^{n_1-1}(x_1), \ldots, P^{n_k-1}(x_1)) \\ \ge C_1(P^{n_1-1}(x_2) - P^{n_1-1}(x_1)) &- \sum_{i=2}^k C_i |P^{n_i-1}(x_2) - P^{n_i-1}(x_1)| \\ \ge C_1(x_2 - x_1) - (x_2 - x_1) &\sum_{i=2}^k C_i M^{n_i-1} \ge \xi(x_2 - x_1) > 0, \end{aligned} \tag{2.13}$$

since $n_1 = 1$. This implies that $LP$ is strictly increasing and invertible on $I$. Thus $LP$ is an orientation-preserving homeomorphism of $I$ onto itself. Moreover, (2.11) follows from (2.13) immediately. ∎

Lemma 2.3. *Suppose that $P_0,\ F : I \to I$ are both Lipschitzian mappings fixing the end-points of $I$ such that* $\mathrm{Lip}(P_0) \le M$ *and* $\mathrm{Lip}(F) \le M_0$ *for positive constants $M$ and $M_0$. If the reals $C_j$, $j = 1, \ldots, k$, satisfy $C_1 \ge M_0/M + \sum_{i=2}^{k} C_i M^{n_i - 1}$, then both*

$$LP_{k-1}(x) := G(P_{k-1}^{n_1-1}(x), \ldots, P_{k-1}^{n_k-1}(x)) \tag{2.14}$$

*and*

$$P_k := (LP_{k-1})^{-1} \circ F \tag{2.15}$$

*are well defined and $P_k : I \to I$ is a Lipschitzian mapping fixing the end-points of $I$ with* $\mathrm{Lip}(P_k) \le M$, $k = 1, 2, \ldots$

*Proof.* By Lemma 2.2, $LP_0(x) := G(P_0^{n_1-1}(x), \ldots, P_0^{n_k-1}(x))$ is well defined and maps $I$ onto itself homeomorphically with $LP_0(a) = a$, $LP_0(b) = b$ and $\mathrm{Lip}((LP_0)^{-1}) \le 1/\xi$, where $\xi$ is defined in (2.12). Thus $P_1(x) := (LP_0)^{-1} \circ F(x)$ is meaningful and $P_1(a) = a$, $P_1(b) = b$. Moreover, $\mathrm{Lip}(P_1) \le \mathrm{Lip}((LP_0)^{-1}) \cdot \mathrm{Lip}(F) \le (1/\xi) M_0 \le M$, by the assumptions on $C_j$.

For the inductive proof we assume that the conclusion of Lemma 2.3 is true for the integer $k$. By Lemma 2.2, $LP_k(x) := G(P_k^{n_1-1}(x), \ldots, P_k^{n_k-1}(x))$ is also well defined and maps $I$ onto itself homeomorphically with $LP_k(a) = a$, $LP_k(b) = b$ and $\mathrm{Lip}((LP_k)^{-1}) \le 1/\xi$. Similarly we see that $P_{k+1}(x) := (LP_k)^{-1} \circ F(x)$ is also meaningful and $P_{k+1}(a) = a$, $P_{k+1}(b) = b$. Moreover, $\mathrm{Lip}(P_{k+1}) \le \mathrm{Lip}((LP_k)^{-1}) \cdot \mathrm{Lip}(F) \le (1/\xi) M_0 \le M$. This implies that the conclusion of Lemma 2.3 is also true for $k + 1$ and completes the proof of Lemma 2.3. ∎

## 3. Main result

Theorem. *Suppose that equation* (1.4) *satisfies the hypotheses* (H1), (H2′) *and* (H3′) *and that $F : I \to I$ is a Lipschitzian mapping fixing the end-points of $I$ with* $\mathrm{Lip}(F) \le M_0$ *for a positive constant $M_0$. If $g : I \to I$ is a Lipschitzian mapping fixing the end-points of $I$ with* $\mathrm{Lip}(g) \le M$ *such that*

$$|F(x) - G(g^{n_1}(x), \ldots, g^{n_k}(x))| \le \delta, \qquad \forall x \in I, \tag{3.16}$$

*for a constant $\delta > 0$, then there exists a unique continuous solution $f : I \to I$ of equation* (1.4) *such that*

$$|f(x) - g(x)| \le \gamma\delta, \quad \forall x \in I, \tag{3.17}$$

*where*

$$\gamma = \Big(C_1 - \sum_{i=2}^{k} C_i M^{n_i-1} - \max\Big\{\sum_{i=2}^{k} C_i \sum_{j=0}^{n_i-2} M^j, \sum_{i=2}^{k} B_i \sum_{j=0}^{n_i-2} M^j\Big\}\Big)^{-1},$$

*provided*

$$C_1 > \sum_{i=2}^{k} C_i M^{n_i-1} + \max\Big\{M_0/M, \sum_{i=2}^{k} C_i \sum_{j=0}^{n_i-2} M^j,\ \sum_{i=2}^{k} B_i \sum_{j=0}^{n_i-2} M^j\Big\}. \tag{3.18}$$

This Theorem implies that equation (1.4) satisfies Hyers–Ulam stability if the constants in (1.7) and (1.8) satisfy (3.18).

In this Theorem we free both $F$ and $f$ from the requirement of increasing monotonicity, which were imposed in [12]–[14]. So the form of equation in this paper is more general. Additionally, unlike [16] and [12] we do not restrict our discussion to the subset

$$\begin{aligned} X(I;0,M) := \{f : I \to I \mid f(a) = a,\ f(b) = b,\\ 0 \le f(x_2) - f(x_1) \le M(x_2 - x_1),\ \forall x_1, x_2 \in I \text{ with } x_2 > x_1\} \end{aligned} \tag{3.19}$$

For example, for

$$F(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{4}, \\ -\frac{1}{2}x + \frac{5}{8}, & \frac{1}{4} < x \le \frac{5}{8}, \\ \frac{11}{6}x - \frac{5}{6}, & \frac{5}{8} < x \le 1, \end{cases}$$

consider the equation

$$\frac{21}{20} f(x) - \frac{1}{20}(f^2(x))^2 = F(x), \quad x \in I = [0,1]. \tag{3.20}$$

Let $G(y_1, y_2) = \frac{21}{20}y_1 - \frac{1}{20}y_2^2$. So, $G(0,0) = 0$, $G(1,1) = 1$. If $y_i, z_i \in I$, $i = 1, 2$, we have

$$\begin{aligned} |G(y_1, y_2) - G(z_1, z_2)| &\le \frac{21}{20}|y_1 - z_1| + \frac{1}{20}|y_2 + z_2| \cdot |y_2 - z_2| \\ &\le \frac{21}{20}|y_1 - z_1| + \frac{1}{10}|y_2 - z_2|. \end{aligned} \tag{3.21}$$

Moveover, if $y_1 \ge z_1$, we have

$$G(y_1, y_2) - G(z_1, z_2) \ge \frac{21}{20}(y_1 - z_1) - \frac{1}{10}|y_2 - z_2|. \tag{3.22}$$

Thus the hypotheses (H1)–(H2′) and (H3′) are satisfied, where $C_1 = \frac{21}{20}$ and $C_2 = \frac{1}{10}$. Clearly, the function

$$g(x) = \begin{cases} \frac{3}{5}x, & 0 \le x \le \frac{5}{6}, \\ 3x - 2, & \frac{5}{6} < x \le 1, \end{cases}$$

satisfies the inequality

$$\left| F(x) - \left(\frac{21}{20} g(x) - \frac{1}{20}(g^2(x))^2\right)\right| \le 0.343,$$

i.e., $g$ is a $\delta$-approximate solution where $\delta = 0.343$. Clearly $\mathrm{Lip}(F) = 2$ and $\mathrm{Lip}(g) = 3$. We can check that condition (3.18) is satisfied. By our Theorem, equation (3.20) satisfies Hyers–Ulam stability.

**4. Proof of Theorem.** For simplicity, we apply the notation $\xi$ as in (2.12) and

$$\eta = \max\Big\{ \sum_{i=2}^{k} C_i \sum_{j=0}^{n_i-2} M^j, \sum_{i=2}^{k} B_i \sum_{j=0}^{n_i-2} M^j \Big\}. \tag{4.23}$$

Construct a sequence $\{P_k(x)\}$ of functions as follows. Take $P_0(x) = g(x)$ first and then define $P_k(x)$ by (2.15) inductively. By Lemma 2.3, both $LP_{k-1}(x)$ and $P_k(x)$ are well defined for $k > 1$. Lemmas 2.2 and 2.3 also imply that $P_k(a) = a,\ P_k(b) = b$, $\mathrm{Lip}(P_k) \le M$ and that $LP_k$ is an orientation-preserving homeomorphism of $I$ onto itself with $\mathrm{Lip}((LP_k)^{-1}) \le 1/\xi$.

Now we claim that

$$|P_k(x) - P_{k-1}(x)| \le \frac{1}{\xi}\left(\frac{\eta}{\xi}\right)^{k-1}\delta, \tag{4.24}$$

$$|F(x) - LP_k \circ P_k(x)| \le \left(\frac{\eta}{\xi}\right)^{k}\delta, \tag{4.25}$$

for all $x \in I$ and $k = 1, 2, \ldots$

First (4.24) and (4.25) are obvious when $k = 1$. Assume that they are true for the integer $k$. Then

$$\begin{aligned} |P_{k+1}(x) - P_k(x)| &= |(LP_k)^{-1} \circ F(x) - (LP_k)^{-1} \circ (LP_k) \circ P_k(x)| \\ &\le \frac{1}{\xi}\,|F(x) - (LP_k) \circ P_k(x)| \\ &\le \frac{1}{\xi}\left(\frac{\eta}{\xi}\right)^{k}\delta, \end{aligned} \tag{4.26}$$

by (4.25). Moreover,

$$\begin{aligned} &F(x) - LP_{k+1} \circ P_{k+1}(x) \\ &\quad = G(P_{k+1}(x), P_k^{n_2-1} \circ P_{k+1}(x), \ldots, P_k^{n_k-1} \circ P_{k+1}(x)) \\ &\qquad - G(P_{k+1}(x), P_{k+1}^{n_2}(x), \ldots, P_{k+1}^{n_k}(x)) \\ &\quad \ge\ -\sum_{i=2}^{k} C_i |P_k^{n_i-1} - P_k^{n_i-1}| \ge -\sum_{i=2}^{k} C_i M^{n_i-1} |P_k - P_{k+1}|, \end{aligned} \tag{4.27}$$

by (1.8), and

$$\begin{aligned}(4.28)\quad & F(x) - LP_{k+1} \circ P_{k+1}(x) \\ &= G(P_{k+1}(x), P_k^{n_2-1} \circ P_{k+1}(x), \dots, P_k^{n_k-1} \circ P_{k+1}(x)) \\ &\quad - G(P_{k+1}(x), P_{k+1}^{n_2}(x), \dots, P_{k+1}^{n_k}(x)) \\ &\le \sum_{i=2}^{k} B_i |P_k^{n_i-1} - P_k^{n_i-1}| \le \sum_{i=2}^{k} B_i M^{n_i-1} |P_k - P_{k+1}|,\end{aligned}$$

by (1.7). It follows that

$$\begin{aligned}(4.29)\quad & |F(x) - LP_{k+1} \circ P_{k+1}(x)| \\ &\le |P_k - P_{k+1}| \max\Big\{ \sum_{i=2}^{k} C_i \sum_{j=0}^{n_i-2} M^j,\ \sum_{i=2}^{k} B_i \sum_{j=0}^{n_i-2} M^j \Big\} \\ &\le \eta \left( \frac{1}{\xi} \left( \frac{\eta}{\xi} \right)^k \delta \right) = \left( \frac{\eta}{\xi} \right)^{k+1} \delta,\end{aligned}$$

by hypotheses (H2′)–(H3′) and (4.26). Thus (4.24) and (4.25) are proved by induction.

For any positive integers $k$ and $s$ with $k > s$,

$$\begin{aligned}(4.30)\quad |P_k(x) - P_s(x)| &\le |P_k(x) - P_{k-1}(x)| + |P_{k-1}(x) - P_{k-2}(x)| \\ &\quad + \dots + |P_{s+1}(x) - P_s(x)| \\ &\le \frac{1}{\xi}\left(\frac{\eta}{\xi}\right)^{k-1} \delta + \frac{1}{\xi}\left(\frac{\eta}{\xi}\right)^{k-2} \delta + \dots + \frac{1}{\xi}\left(\frac{\eta}{\xi}\right)^{s} \delta \\ &= \frac{\delta}{\xi} \cdot \frac{(\eta/\xi)^s - (\eta/\xi)^k}{1 - \eta/\xi},\end{aligned}$$

by (4.24). Note from (3.18) that $\xi > \eta$. It follows from (4.30) that

$$(4.31)\qquad |P_k(x) - P_s(x)| \to 0 \quad \text{as } k > s \to \infty.$$

As a Cauchy sequence, $\{P_k(x)\}$ converges uniformly in the Banach space $\mathcal{C}(I)$. Let

$$(4.32)\qquad \lim_{k\to\infty} P_k(x) = f(x).$$

Clearly, $f : I \to I$ is also a Lipschitzian mapping with $\mathrm{Lip}(f) \le M$. From (4.25),

$$\begin{aligned}(4.33)\quad |F(x) - Lf \circ f(x)| &= \lim_{k\to\infty} |F(x) - LP_k \circ P_k(x)| \\ &\le \lim_{k\to\infty} (\eta/\xi)^k \delta = 0,\end{aligned}$$

i.e., $f$ is a solution of equation (1.4). Furthermore, from (4.24),

$$
\begin{aligned}
(4.34)\qquad |f(x)-g(x)| &= \lim_{k\to\infty}|P_k(x)-P_0(x)| \\
&\le \lim_{k\to\infty}\{|P_k(x)-P_{k-1}(x)|+|P_{k-1}(x)-P_{k-2}(x)| \\
&\qquad\qquad +\ldots+|P_1(x)-P_0(x)|\} \\
&\le \lim_{k\to\infty}\left\{\frac{1}{\xi}\left(\frac{\eta}{\xi}\right)^{k-1}\delta+\frac{1}{\xi}\left(\frac{\eta}{\xi}\right)^{k-2}\delta+\ldots+\frac{1}{\xi}\,\delta\right\}=\frac{1}{\xi-\eta}\,\delta.
\end{aligned}
$$

This proves (3.17).

Concerning uniqueness, we assume that there is another continuous solution $\phi : I \to I$ for equation (1.4), which may not be Lipschitzian, such that

$$|\phi(x)-g(x)|\le\varepsilon$$

where $\varepsilon>0$ only depends on $\delta$. Then

$$G(f^{n_1}(x),\ldots,f^{n_k}(x))=G(\phi^{n_1}(x),\ldots,\phi^{n_k}(x)). \tag{4.35}$$

It follows from Lemma 2.1 and hypothesis (H3′) that

$$C_1(f^{n_1}(x)-\phi^{n_1}(x))-\sum_{i=2}^{k}C_i|f^{n_i}(x)-\phi^{n_i}(x)|\le 0 \tag{4.36}$$

that is,

$$\Bigl(C_1-\sum_{i=2}^{k}C_i\sum_{j=0}^{n_i-1}M^j\Bigr)\|f-\phi\|\le 0. \tag{4.37}$$

However, $C_1>\sum_{i=2}^{k}C_i\sum_{j=0}^{n_i-1}M^j$ by (3.18). This implies that $\|f-\phi\|=0$, i.e., $f\equiv\phi$. ∎

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Department of Mathematics
Sichuan University
Chengdu, Sichuan 610064
P.R. China
E-mail: wnzhang@scu.edu.cn

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