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$L^p(\mathbb{R}^n)$ *BOUNDS FOR* *COMMUTATORS OF CONVOLUTION OPERATORS*

BY

GUOEN HU (Zhengzhou), QIYU SUN (Singapore)
and XIN WANG (Zhengzhou)

Abstract. The $L^p(\mathbb{R}^n)$ boundedness is established for commutators generated by $\mathrm{BMO}(\mathbb{R}^n)$ functions and convolution operators whose kernels satisfy certain Fourier transform estimates. As an application, a new result about the $L^p(\mathbb{R}^n)$ boundedness is obtained for commutators of homogeneous singular integral operators whose kernels satisfy the Grafakos–Stefanov condition.

1. Introduction. We will work in $\mathbb{R}^n$, $n \geq 1$. Let T be a standard Calderón–Zygmund operator and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Define the first order commutator of T and b by

$$T_b f(x) = b(x)Tf(x) - T(bf)(x).$$

In the remarkable work [3], Coifman and Meyer observed that the $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ boundedness of T_b can be obtained from the weighted $L^p(\mathbb{R}^n)$ estimates with A_p weights for the operator T, where A_p denotes the weight function class of Muckenhoupt (see [7, Chapter V] for definition and properties of A_p). Alvarez, Bagby, Kurtz and Pérez [2] developed the idea of Coifman and Meyer, and established a generalized boundedness result for commutators of linear operators. They showed that if $1 < p, q < \infty$, and the linear operator T is bounded on $L^p(\mathbb{R}^n, w(x)dx)$ with bound independent of w for any $w \in A_q$, then for any positive integer k and $b \in \mathrm{BMO}(\mathbb{R}^n)$, the kth order commutator of T defined by

$$T_{b,k} f(x) = b(x)T_{b,k-1}f(x) - T_{b,k-1}(bf)(x), \quad T_{b,0}f(x) = Tf(x)$$

is bounded on $L^p(\mathbb{R}^n)$ with bound $C(n,k,p)\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$. In [5], Hu considered the $L^2(\mathbb{R}^n)$ boundedness for commutators of convolution operators and proved the following result.

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THEOREM H. *Let k be a positive integer, $K(x)$ be a function on $\mathbb{R}^n\setminus\{0\}$ and $K(x)=\sum_{j\in\mathbb{Z}}K_j(x)$. Suppose that there are some constants $C>0$, $0<A\le 1/2$ and $\alpha>k+1$ such that for each $j\in\mathbb{Z}$,*

$$\|K_j\|_1\le C,\qquad \|\nabla\widehat{K}_j\|_\infty\le C2^j,$$
$$|\widehat{K}_j(\xi)|\le C\min\{A|2^j\xi|,\log^{-\alpha}(2+|2^j\xi|)\}.$$

Then for $b\in \mathrm{BMO}(\mathbb{R}^n)$ and $0<\nu<1$ such that $\alpha\nu>k+1$, the commutator

$$T_{b,k}f(x)=\int_{\mathbb{R}^n}(b(x)-b(y))^kK(x-y)f(y)\,dy,\qquad f\in C_0^\infty(\mathbb{R}^n),$$

is bounded on $L^2(\mathbb{R}^n)$ with bound $C(n,k,\alpha,\nu)\log^{-\alpha\nu+k+1}(1/A)\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$.

In this paper, we will continue the study begun in [5]. By Fourier transform estimates and approximation of the identity, we will establish the $L^p(\mathbb{R}^n)$ boundedness for commutators of convolution operators. We remark that in this paper, we are very much motivated by the work of Watson [8]; some ideas are from Pérez's paper [6]. For a function f on $\mathbb{R}^n$, denote by $\widehat{f}$ the Fourier transform of f. For a nonnegative integer m, let $\Phi_m(t)=t\log^m(2+t)$. For a locally integrable function f and a bounded measurable set E with Lebesgue measure $|E|$, define

$$\|f\|_{L(\log L)^m,E}=\inf\left\{\lambda>0:\frac{1}{|E|}\int_E\Phi_m\left(\frac{|f(y)|}{\lambda}\right)dy\le 1\right\}$$

and

$$\|f\|_{\exp(L^{1/m}),E}=\inf\left\{\lambda>0:\frac{1}{|E|}\int_E\exp\left(\frac{|f(y)|}{\lambda}\right)^{1/m}dy\le 2\right\}.$$

Our main result is

THEOREM 1. *Let k be a positive integer, $K(x)$ be a function on $\mathbb{R}^n\setminus\{0\}$ and $K_j(x)=K(x)\chi_{\{2^j\le|x|<2^{j+1}\}}(x)$ for $j\in\mathbb{Z}$, where χ_A is the characteristic function of the set A. Suppose that there exist constants $C>0$ and $\alpha>k+1$ such that for each $j\in\mathbb{Z}$,*

$$\|K_j\|_1\le C,\qquad |\widehat{K}_j(\xi)|\le C\min\{|2^j\xi|,\log^{-\alpha}(2+|2^j\xi|)\},\qquad \|\nabla\widehat{K}_j\|_\infty\le C2^j. \tag{1}$$

Then for $b\in\mathrm{BMO}(\mathbb{R}^n)$ and $2\alpha/(2\alpha-(k+1))<p<2\alpha/(k+1)$, the commutator

$$T_{b,k}f(x)=\int_{\mathbb{R}^n}(b(x)-b(y))^kK(x-y)f(y)\,dy,\qquad f\in C_0^\infty(\mathbb{R}^n), \tag{2}$$

is bounded on $L^p(\mathbb{R}^n)$ with bound $C(n,k,p,\alpha)\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$.

As an application of Theorem 1, we will obtain

THEOREM 2. *Let k be a positive integer and $b \in \mathrm{BMO}(\mathbb{R}^n)$, Ω be homogeneous of degree zero and have mean value zero. Suppose that for some $\alpha > k+1$,*

$$\sup_{\zeta \in S^{n-1}} \int\limits_{S^{n-1}} \Omega(\theta)\left(\log \frac{1}{|\theta\cdot\zeta|}\right)^{\alpha} d\theta < \infty. \tag{3}$$

Then for $2\alpha/(2\alpha-(k+1)) < p < 2\alpha/(k+1)$, the commutator defined by

$$\overline{T}_{b,k} f(x) = \int\limits_{\mathbb{R}^n} (b(x)-b(y))^k \frac{\Omega(x-y)}{|x-y|^n} f(y)\,dy$$

is bounded on $L^p(\mathbb{R}^n)$ with bound $C(n,k,p,\alpha)\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$.

REMARK. The size condition (3) for $\alpha \ge 1$ was introduced by Grafakos and Stefanov [4] in order to study the $L^p(\mathbb{R}^n)$ boundedness for the homogeneous singular integral operator defined by

$$\overline{T} f(x) = \int\limits_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\,dy.$$

It has been proved in [4] that there exist integrable functions on S^{n-1} which are not in $H^1(S^{n-1})$, but satisfy (3) for all $\alpha > 1$. Thus our Theorem 2 shows that there exists $\Omega \in L^1(S^{n-1}) \setminus H^1(S^{n-1})$ such that the corresponding commutator $\overline{T}_{b,k}$ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ and a positive integer k.

2. Proof of theorems. By the estimates used in [4], it is easy to see that Theorem 2 follows from Theorem 1 directly, so we only prove Theorem 1. We begin with some preliminary lemmas.

LEMMA 1. *Let m and k be integers such that $0 \le m \le k$. Suppose that f and g are functions on $\mathbb{R}^n$ with compact support. Then for any bounded measurable set E,*

$$\|f*g\|_{L(\log L)^k,E} \le C|E| \inf\left\{\lambda>0 : \frac{1}{|E|}\int\limits_{\mathbb{R}^n} \Phi_{k-m}\left(\frac{|f(x)|}{\lambda}\right)dx \le 1\right\}$$
$$\times \inf\left\{\lambda>0 : \frac{1}{|E|}\int\limits_{\mathbb{R}^n} \Phi_{m}\left(\frac{|g(x)|}{\lambda}\right)dx \le 1\right\}.$$

Proof. Without loss of generality, we may assume that

$$\inf\left\{\lambda>0 : \frac{1}{|E|}\int\limits_{\mathbb{R}^n} \Phi_{k-m}\left(\frac{|f(x)|}{\lambda}\right)dx \le 1\right\} = \frac{1}{2\|g\|_1}.$$

Thus, by homogeneity,

$$\inf\left\{\lambda>0:\frac{1}{|E|}\int_{\mathbb{R}^n}\Phi_{k-m}\left(\frac{|f(x)|\cdot\|g\|_1}{\lambda}\right)dx\le 1\right\}=\frac{1}{2}.$$

Therefore,

$$\frac{1}{|E|}\int_{\mathbb{R}^n}\Phi_{k-m}(|f(x)|\cdot\|g\|_1)\,dx\le 1.$$

Suppose that $\operatorname{supp} g$ is contained in some ball B. By the Jensen inequality,

$$\begin{aligned}\Phi_{k-m}(|f*g(x)|)&\le\Phi_{k-m}\left(\int_B|f(x-y)|\cdot\|g\|_1\frac{|g(y)|}{\|g\|_1}\,dy\right)\\&\le\int_B\Phi_{k-m}(|f(x-y)|\cdot\|g\|_1)\frac{|g(y)|}{\|g\|_1}\,dy.\end{aligned}$$

Let $\overline{B}$ be the support of f. Invoking the Jensen inequality again, we obtain

$$\begin{aligned}&\Phi_m(\Phi_{k-m}(|f*g(x)|))\\&\le\Phi_m\left(\int_{\overline{B}}\frac{|g(x-y)|\int_{\mathbb{R}^n}\Phi_{k-m}(|f(z)|\cdot\|g\|_1)\,dz}{\|g\|_1}\cdot\frac{\Phi_{k-m}(|f(y)|\cdot\|g\|_1)\,dy}{\int_{\mathbb{R}^n}\Phi_{k-m}(|f(z)|\cdot\|g\|_1)\,dz}\right)\\&\le\int_{\mathbb{R}^n}\Phi_m\left(\frac{|g(x-y)|\cdot|E|}{\|g\|_1}\right)\frac{\Phi_{k-m}(|f(y)|\cdot\|g\|_1)}{\int_{\mathbb{R}^n}\Phi_{k-m}(|f(z)|\cdot\|g\|_1)\,dz}\,dy,\end{aligned}$$

which via the Young inequality gives

$$\int_E\Phi_m(\Phi_{k-m}(|f*g(x)|))\,dx\le\int_{\mathbb{R}^n}\Phi_m\left(\frac{|g(y)|\cdot|E|}{\|g\|_1}\right)dy.$$

Note that for each $t>0$, $\Phi_k(t)\le\Phi_m(\Phi_{k-m}(t))$. Thus,

$$\begin{aligned}\|f*g\|_{L(\log L)^k,E}&\le\inf\left\{\lambda>0:\frac{1}{|E|}\int_{\mathbb{R}^n}\Phi_m\left(\frac{|g(x)|\cdot|E|}{\|g\|_1\lambda}\right)dx\le 1\right\}\\&=|E|\cdot\|g\|_1^{-1}\inf\left\{\lambda>0:\frac{1}{|E|}\int_{\mathbb{R}^n}\Phi_m\left(\frac{|g(x)|}{\lambda}\right)dx\le 1\right\}.\end{aligned}$$

This leads to our desired estimate. ∎

LEMMA 2. *Let k be a positive integer and $b\in\mathrm{BMO}(\mathbb{R}^n)$, $K(x)$ be a function on $\mathbb{R}^n\setminus\{0\}$ such that for all $R>0$ and $|y|<R/2$,*

$$\begin{aligned}&\sum_{d\ge1}d^k\int_{B(0,2^{d+1}R)\setminus B(0,2^dR)}|K(x-y)-K(x)|\,dx\\&\quad+\sum_{d\ge1}|B(0,2^dR)|\cdot\|K(\cdot-y)-K(\cdot)\|_{L(\log L)^k,\,B(0,2^{d+1}R)\setminus B(0,2^dR)}\le A.\end{aligned}$$

Suppose that for each $0 \le m \le k$, the operator

$$T_{b,m} f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))^m K(x-y) f(y)\, dy$$

is bounded on $L^2(\mathbb{R}^n)$ with bound $C_m \|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}$. Then the commutator $T_{b,k}$ is a bounded operator on $L^p(\mathbb{R}^n)$ with bound $C(A + \sum_{m=0}^k C_m)\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$ for all $1 < p < \infty$.

Proof. Without loss of generality, we may assume that $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = 1$. By duality, it suffices to prove that for $0 \le m \le k$ and $1 < p < 2$,

$$\|T_{b,m} f\|_p \le C\Big(A + \sum_{l=0}^{m} C_l\Big)\|f\|_p. \tag{4}$$

We shall carry out the argument by induction on the order m. For $m = 0$, it is obvious that the operator $T_{b,0}$ is bounded from $L^1(\mathbb{R}^n)$ to weak $L^1(\mathbb{R}^n)$ with bound $C(C_0 + A)$, and the estimate (4) holds for $m = 0$. Now let m be a positive integer and $m \le k$. We assume that (4) holds for all $0 \le l \le m-1$. By the Marcinkiewicz interpolation theorem, it is enough to show that for each $1 < p < 2$ and $\lambda > 0$,

$$|\{x : T_{b,m} f(x) > \lambda\}| \le C\lambda^{-p}\Big(A + \sum_{l=0}^{m} C_l\Big)\|f\|_p^p. \tag{5}$$

For given $f \in L^p(\mathbb{R}^n)$ and $\lambda > 0$, applying the Calderón–Zygmund decomposition of $|f|^p$ at the level λ^p, we can write $f(x) = g(x) + h(x)$, where $\|g\|_\infty \le C\lambda$, $\|g\|_p \le C\|f\|_p$; $h(x) = \sum_j h_j(x)$, h_j is supported on Q_j, $\int_{\mathbb{R}^n} h_j(x)\, dx = 0$, $\int |h_j(x)|^p\, dx \le C\lambda^p |Q_j|$ and $\sum_j |Q_j| \le C\lambda^{-p}\|f\|_p^p$. The $L^2(\mathbb{R}^n)$ boundedness of $T_{b,m}$ states that

$$|\{x : T_{b,m} g(x) > \lambda\}| \le \lambda^{-2}\|T_{b,m} g\|_2^2 \le C\lambda^{-p}\|f\|_p^p.$$

For each fixed j, let y_j^0 and r_j be the center and the side length of Q_j. Set $B_j = B(y_j^0, 4nr_j)$ and $E = \bigcup_j B_j$. It is obvious that

$$|E| \le C\sum_j |Q_j| \le C\lambda^{-p}\|f\|_p^p.$$

Thus, the proof of (5) can be reduced to proving that for $\lambda > 0$,

$$|\{x \in \mathbb{R}^n \setminus E : |T_{b,m} h(x)| > \lambda\}| \le C\lambda^{-p}\|f\|_p^p.$$

For each fixed j, denote by $m_{B_j}(b)$ the mean value of b on B_j. With the aid of the formula

$$(b(x) - b(y))^m = (b(x) - m_{B_j}(b))^m - \sum_{l=0}^{m-1} C_m^l (b(x) - b(y))^l (b(y) - m_{B_j}(b))^{m-l}, \quad x, y \in \mathbb{R}^n,$$

we have

$$T_{b,m}h(x) = \sum_j (b(x) - m_{B_j}(b))^m Th_j(x) - \sum_{l=0}^{m-1} C_m^l T_{b,l}\Big(\sum_j (b(\cdot) - m_{B_j}(b))^{m-l} h_j\Big)(x).$$

Let $1 < p_0 < p$ and $r = p/p_0$. For each fixed $0 \le l \le m-1$, our inductive hypothesis together with the Hölder inequality tells us that

$$\begin{aligned}
&\Big|\Big\{x \in \mathbb{R}^n : \Big|T_{b,l}\Big(\sum_j (b(\cdot) - m_{B_j}(b))^{m-l} h_j\Big)(x)\Big| > \lambda\Big\}\Big| \\
&\le \lambda^{-p_0}\Big\|T_{b,l}\Big(\sum_j (b(\cdot) - m_{B_j}(b))^{m-l} h_j\Big)\Big\|_{p_0}^{p_0} \\
&\le C\Big(A + \sum_{i=0}^{l-1} C_i\Big)\lambda^{-p_0} \sum_j \int_{B_j} |b(y) - m_{B_j}(b)|^{(m-l)p_0} |h_j(y)|^{p_0}\, dy \\
&\le C\Big(A + \sum_{i=0}^{l-1} C_i\Big)\lambda^{-p_0} \\
&\quad \times \sum_j \Big(\int_{B_j} |b(y) - m_{B_j}(b)|^{(m-l)p_0 r'}\, dy\Big)^{1/r'} \Big(\int_{B_j} |h_j|^p\, dy\Big)^{1/r} \\
&\le C\Big(A + \sum_{i=0}^{l-1} C_i\Big) \sum_j |B_j| \le C\Big(A + \sum_{i=0}^{l-1} C_i\Big)\lambda^{-p}\|f\|_p^p.
\end{aligned}$$

Observe that $\Phi_m(t) = t\log^m(2+t)$ is a Young function and its complementary Young function is $\Psi_m(t) \approx e^{t^{1/m}}$. For $y \in Q_j$ and positive integer d, it follows from the generalized Hölder inequality (see [1, Chapter 8] or [6, p. 168]) that

$$\begin{aligned}
&\int_{2^d B_j \setminus 2^{d-1} B_j} |K(x-y) - k(x-y_0^j)| \cdot |b(x) - m_{B_j}(b)|^m\, dx \\
&\le C|m_{B_j} - m_{2^d B_j}(b)|^m \int_{2^d B_j \setminus 2^{d-1} B_j} |K(x-y) - K(x-y_0^j)|\, dx \\
&\quad + C\int_{2^d B_j \setminus 2^{d-1} B_j} |b(x) - m_{2^d B_j}(b)|^m |K(x-y) - K(x-y_0^j)|\, dx \\
&\le Cd^m \int_{2^d B_j \setminus 2^{d-1} B_j} |K(x-y) - K(x-y_0^j)|\, dx
\end{aligned}$$

$$
\begin{aligned}
&+ C|2^dB_j| \cdot \|(b(x)-m_{2^dB_j}(b))^m\|_{\exp(L^{1/m}),2^dB_j}\\
&\times \|K(\cdot-y)-K(\cdot-y_0^j)\|_{L(\log L)^m,2^dB_j\setminus 2^{d-1}B_j}\\
\le{}& Cd^m \int\limits_{2^dB_j\setminus 2^{d-1}B_j} |K(x-y)-K(x-y_0^j)|\,dx\\
&+ C|2^dB_j| \cdot \|K(\cdot-y)-K(\cdot-y_0^j)\|_{L(\log L)^m,2^dB_j\setminus 2^{d-1}B_j},
\end{aligned}
$$

where in the last but one inequality, we have invoked the fact that

$$|m_{B_j}(b)-m_{2^dB_j}(b)| \le Cd\|b\|_{\mathrm{BMO}(\mathbb{R}^n)},$$

and in the last inequality, we have used the John–Nirenberg inequality which states that for some positive constants λ_1, λ_2,

$$\frac{1}{|2^dB_j|}\int\limits_{2^dB_j} \exp\left(\frac{|b(z)-m_{2^dB_j}(b)|}{\lambda_1\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\right)dz \le \lambda_2.$$

By the vanishing mean value of h_j, we see that for each fixed j,

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^n\setminus B_j} |b(x)-m_{B_j}(b)|^m|Th_j(x)|\,dx\\
&\quad= \int\limits_{\mathbb{R}^n\setminus B_j} |b(x)-m_{B_j}(b)|^m\Big|\int\limits_{\mathbb{R}^n}[K(x-y)-K(x-y_0)]h_j(y)\,dy\Big|\,dx\\
&\quad\le C\sum_{d=1}^{\infty}\int\limits_{B_j}|h_j(y)|\\
&\qquad\times \int\limits_{2^dB_j\setminus 2^{d-1}B_j} |b(x)-m_{B_j}(b)|^m|K(x-y)-K(x-y_0)|\,dx\,dy\\
&\quad\le CA\int\limits_{B_j}|h_j(y)|\,dy \le CA|B_j|^{1-1/p}\|h_j\|_p \le CA\lambda|Q_j|,
\end{aligned}
$$

which in turn implies

$$
\begin{aligned}
&\Big|\Big\{x\in\mathbb{R}^n\setminus E : \sum_j |b(x)-m_{B_j}(b)|^m|Th_j(x)| > \lambda\Big\}\Big|\\
&\le \lambda^{-1}\sum_j\int\limits_{\mathbb{R}^n\setminus E}|b(x)-m_{B_j}(b)|^m|Th_j(x)|\,dx \le CA\sum_j|Q_j| \le CA\lambda^{-p}\|f\|_p^p.
\end{aligned}
$$

Combining the estimates above yields the desired estimate. ■

Proof of Theorem 1. By duality, it suffices to consider the case $2 < p < 2\alpha/(k+1)$. As in the proof of [8, Theorem 1], let $\phi \in C_0^\infty(\mathbb{R}^n)$ be a radial nonnegative function such that $\int_{\mathbb{R}^n}\phi(x)\,dx = 1$, $\operatorname{supp}\phi \subset \{x : |x| \le 1/4\}$.

For $l \in \mathbb{Z}$, set $\phi_l(x) = 2^{-nl}\phi(2^{-l}x)$. For a positive integer j, define

$$\widetilde{K}_j(x) = \sum_{l=-\infty}^{\infty} K_l * \phi_{l-j}(x).$$

Let S_j be the convolution operator whose kernel is $\widetilde{K}_j$. Note that

$$|\widehat{\phi}_{l-j}(\xi) - 1| \le C\min\{|2^{l-j}\xi|, 1\}, \qquad \|\nabla\widehat{\phi}_{l-j}(\xi)\|_\infty \le C2^{l-j}.$$

Now the Fourier transform estimate of K_l gives

$$|\widehat{K}_l(\xi)\widehat{\phi}_{l-j}(\xi) - \widehat{K}_l(\xi)| \le C\min\{2^{-j}|2^l\xi|, \log^{-\alpha}(2+|2^l\xi|)\},$$

and

$$\|\nabla(\widehat{K}_l\widehat{\phi}_{l-j}) - \nabla\widehat{K}_l\|_\infty \le \|\nabla\widehat{K}_l\|_\infty\|\widehat{\phi}_{l-j} - 1\|_\infty + \|\widehat{K}_l\|_\infty\|\nabla\widehat{\phi}_{l-j}\|_\infty \le C2^l.$$

This together with Theorem H says that for $0 \le m \le k$, $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $0 < \nu < 1$ such that $\alpha\nu > k+1$,

$$\|T_{b,m}f - S_{j;b,m}f\|_2 \le C(n,m)\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)} j^{-\alpha\nu+m+1}\|f\|_2, \qquad 0 \le m \le k.$$

By the $L^2(\mathbb{R}^n)$ boundedness of $T_{b,m}$, we know that for all positive integers j and $0 \le m \le k$, $S_{j;b,m}$ is also bounded on $L^2(\mathbb{R}^n)$ with bound $C\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}$. Note that

$$\|S_{2^{j+1};b,k}f - S_{2^j;b,k}f\|_2 \le C2^{(-\alpha\nu+k+1)j}\|f\|_2. \tag{6}$$

Therefore, the series

$$T_{b,k} = S_{1;b,k} + \sum_{j=0}^{\infty}(S_{2^{j+1};b,k} - S_{2^j;b,k}) \tag{7}$$

converges in the $L^2(\mathbb{R}^n)$ operator norm.

Now we turn to the $L^p(\mathbb{R}^n)$ boundedness of $S_{2^j;b,k}$. For $y \in \mathbb{R}^n$, it is easy to verify that

$$\|\phi_{l-j}(\cdot - y) - \phi_{l-j}(\cdot)\|_1 \le C\min\{1, 2^{j-l}|y|\},$$
$$\|\phi_{l-j}(\cdot - y) - \phi_{l-j}(\cdot)\|_\infty \le C2^{-(n+1)(l-j)}|y|.$$

Set $\lambda_0 = R^{-n}j^k\min\{1, 2^j|y|/R\}$. Straightforward computation shows that if $|y| < R/2$ and $R \approx 2^l$, then

$$\begin{aligned}
&CR^{-n}\int_{\mathbb{R}^n}\frac{|\phi_{l-j}(z-y) - \phi_{l-j}(z)|}{\lambda_0}\log^k\left(2 + \frac{|\phi_{l-j}(z-y) - \phi_{l-j}(z)|}{\lambda_0}\right)dz\\
&\quad\le Cj^{-k}\log^k\left(2 + \frac{2^{-(n+1)(l-j)}|y|}{R^{-n}\min\{1, 2^j|y|/R\}}\right)\\
&\quad\le Cj^{-k}\max\left\{\log^k\left(2 + \frac{2^{(n+1)j}|y|}{2^{(n+1)l}R^{-n}}\right), \log^k\left(2 + \frac{2^{(n+1)j}R}{2^{(n+1)l}2^jR^{-n}}\right)\right\} \le C.
\end{aligned}$$

Thus,

$$\inf\left\{\lambda>0:\frac{1}{|B(0,2R)\setminus B(0,R)|}\int\limits_{\mathbb{R}^n}\Phi_k\left(\frac{|\phi_{l-j}(z-y)-\phi_{l-j}(z)|}{\lambda}\right)dz\le 1\right\}$$
$$\le CR^{-n}j^k\min\{1,2^j|y|/R\}.$$

Note that for $R>0$ and $|y|<R/2$,

$$\|\widetilde{K}_j(\cdot-y)-\widetilde{K}_j(\cdot)\|_{L(\log L)^k,B(0,2R)\setminus B(0,R)}$$
$$\le\sum_{2^l\approx R}\|K_l*\phi_{l-j}(\cdot-y)-K_l*\phi_{l-j}(\cdot)\|_{L(\log L)^k,B(0,2R)\setminus B(0,R)}.$$

Applying Lemma 1, we find that for $l\in\mathbb{Z}$ such that $2^l\approx R$,

$$\|K_l*\phi_{l-j}(\cdot-y)-K_l*\phi_{l-j}(\cdot)\|_{L(\log L)^k,B(0,2R)\setminus B(0,R)}$$
$$\le C\inf\left\{\lambda>0:\frac{1}{|B(0,2R)\setminus B(0,R)|}\right.$$
$$\left.\times\int\limits_{\mathbb{R}^n}\Phi_k\left(\frac{|\phi_{l-j}(z-y)-\phi_{l-j}(z)|}{\lambda}\right)dz\le 1\right\}\|K_l\|_1$$
$$\le CR^{-n}j^k\min\{1,2^j|y|/R\}.$$

On the other hand, it is easy to verify that

$$\int\limits_{R<|x|\le 2R}|\widetilde{K}_j(x-y)-\widetilde{K}_j(x)|\,dx\le\sum_{2^l\approx R}\|K_l*\phi_{l-j}(\cdot-y)-K_l*\phi_{l-j}(\cdot)\|_1$$
$$\le\sum_{2^l\approx R}\|K_l\|_1\|\phi_{l-j}(\cdot-y)-\phi_{l-j}(\cdot)\|_1$$
$$\le C\min\{1,2^j|y|/R\}.$$

This leads to

$$\sum_{d=1}^{\infty}d^k\int\limits_{B(0,2^{d+1}R)\setminus B(0,2^dR)}|\widetilde{K}_j(x-y)-\widetilde{K}_j(x)|\,dx$$
$$+\sum_{d=1}^{\infty}|B(0,2^dR)|\cdot\|\widetilde{K}_j(\cdot-y)-\widetilde{K}_j(\cdot)\|_{L(\log L)^k,B(0,2^{d+1}R)\setminus B(0,2^dR)}$$
$$\le Cj^k\sum_{d=1}^{\infty}(2^dR)^n\min\{1,2^{j-d}|y|/R\}(2^dR)^{-n}+\sum_{d=1}^{\infty}d^k\min\{1,2^{j-d}|y|/R\}$$
$$\le Cj^{k+1}.$$

Lemma 2 now shows that for $1<p<\infty$,

$$\|S_{2^{j+1};b,k}f-S_{2^j;b,k}f\|_p\le C2^{(k+1)j}\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_p. \tag{8}$$

By the Riesz–Thorin interpolation theorem, it follows from the inequalities (6) and (8) that for $2 < p < \infty$ and any $\theta_0 > 0$,

$$(9)\quad \|S_{2^{j+1};b,k}f - S_{2^j;b,k}f\|_p \le C(n,k,p,\alpha,\nu,\theta_0)2^{(-2\alpha\nu/p+k+1+\theta_0)j}\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_p.$$

For each fixed $2 < p < 2\alpha/(k+1)$, we can choose $\theta_0 > 0$ small enough and $0 < \nu < 1$ such that $-2\alpha\nu/p+k+1+\theta_0 < 0$. So summing up the inequalities (9) over all nonnegative integers j shows that the series (7) converges in the $L^p(\mathbb{R}^n)$ operator norm. This completes the proof of Theorem 1. ∎

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Department of Applied Mathematics
University of Information Engineering
P.O. Box 1001-747
Zhengzhou 450002, China
E-mail: huguoen@371.net
wangxinwsj@sina.com

Department of Mathematics
The National University of Singapore
Lower Kent Ridge Road
119260 Singapore
E-mail: matsunqy@nus.edu.sg

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