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A COMPLETE ANALOGUE OF HARDY'S THEOREM ON SEMISIMPLE LIE GROUPS

BY

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Abstract. A result by G. H. Hardy ([11]) says that if f and its Fourier transform $\widehat{f}$ are $O(|x|^m e^{-\alpha x^2})$ and $O(|x|^n e^{-x^2/(4\alpha)})$ respectively for some $m, n \geq 0$ and $\alpha > 0$, then f and $\widehat{f}$ are $P(x)e^{-\alpha x^2}$ and $P'(x)e^{-x^2/(4\alpha)}$ respectively for some polynomials P and P'. If in particular f is as above, but $\widehat{f}$ is $o(e^{-x^2/(4\alpha)})$, then $f = 0$. In this article we will prove a complete analogue of this result for connected noncompact semisimple Lie groups with finite center. Our proof can be carried over to the real reductive groups of the Harish-Chandra class.

1. Introduction. Let G be a connected noncompact semisimple Lie group with finite center and let K be a maximal compact subgroup of G. The aim of this article is to provide a complete analogue of the result of Hardy (described in the abstract) for the full group G of this class. For a sufficiently nice function f, let $F_{P_0}(f)(\xi,\lambda)$ be its Fourier transform with respect to the minimal principal series representation $(\pi_{P_0,\xi,\lambda}, H_{P_0,\xi,\lambda})$ where $P_0 = M_0A_0N_0$ is a minimal cuspidal parabolic subgroup, $\lambda \in \mathfrak{a}^*_{0\mathbb{C}}$ and $\xi \in \mathcal{E}_2(M_0)$, $\mathcal{E}_2(M_0)$ being the set of discrete series representations of M_0. Suppose $\|F_{P_0}(f)(\xi,\lambda)\|$ is the operator norm of $F_{P_0}(f)(\xi,\lambda)$ relative to the norm of $H_{P_0,\xi,\lambda}$. Let d be the distance on the symmetric space G/K induced by the Riemannian metric on it. We define $\sigma(x) = d(xK, o)$ where $o = eK$. All other notation is explained in the next section. The main result of this article is the following.

THEOREM 1.1. *Let f be a measurable function on G such that*

$$|f(x)| \leq Ce^{-\alpha\sigma(x)^2}\Xi(x)(1+\sigma(x))^M \quad \text{for all } x \in G, \tag{1.1}$$

$$\|F_{P_0}(f)(\xi,\lambda)\| \leq C_\xi e^{-\beta|\lambda|^2}(1+|\lambda|)^N \quad \text{for all } (\xi,\lambda) \in \mathcal{E}_2(M_0) \times i\mathfrak{a}^*_0, \tag{1.2}$$

where $M > 0$ and N are integers and α, β, C, C_ξ are positive constants with $\alpha\beta=1/4$. Then for any fixed $\xi \in \mathcal{E}_2(M_0)$, $F^{u,v}_{P_0}(f)(\xi,\lambda) = P_{\beta,u,v}(\lambda)e^{\beta(\lambda_1^2+\ldots+\lambda_n^2)}$ or 0 for all $\lambda = (\lambda_1,\ldots,\lambda_n) \in \mathfrak{a}^*_{0\mathbb{C}} = \mathbb{C}^n$ and $u, v \in H_{P_0,\xi,\lambda}$, where $P_{\beta,u,v}$ is

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a polynomial which depends on β, u, v *and* $\deg P_{\beta,u,v} \le \min(M+2|\Sigma_0^+|, N)$, $|\Sigma_0^+|$ *being the set of cardinality of the set of indivisible positive roots of* G.

In particular if f *satisfies* (1.1) *and* $F_{P_0}(f)(\xi,\lambda)$ *is* $o(e^{-\beta|\lambda|^2})$ *for* $(\xi,\lambda) \in \mathcal{E}_2(M_0) \times i\mathfrak{a}_0^*$ *and* $\alpha\beta = 1/4$ *then* $f = 0$.

It can be shown that the function f in the above theorem can be uniquely determined from only $F_{P_0}(f)$ on $\mathcal{E}_2(M_0) \times \mathfrak{a}_{0\mathbb{C}}^*$. The above theorem may therefore be considered as the completion of the effort to generalize Hardy's theorem to groups in the Harish-Chandra class.

We supplement Theorem 1.1 by showing the optimality of the estimates and thereby justify the appearance of Ξ there. We also illustrate by an example that if we have an adequate knowledge of the subquotients of the principal series representations, then it is possible to characterize the function more explicitly from similar estimates. This example also explains the role played by the polynomials in the estimates in accommodating nontrivial isotypic components of the function within the consideration of Hardy's theorem.

Our article follows the pattern of its predecessors, especially of [23] and [19].

We conclude with a brief review of related results. Hardy proved his theorem on $\mathbb{R}$ in [11]. A well known stronger version, which we have generalized (given in the abstract), can be found e.g. in [15]. For Lie groups Hardy's theorem was first taken up by Sitaram and Sundari in [23]. This has attracted considerable attention in recent years. Different versions of this theorem were proved for semisimple Lie groups, symmetric spaces, nilpotent groups, Heisenberg groups and solvable extensions of H-type groups. Among these many articles, [6], [7], [19], [21], [22] and [23] itself have dealt with semisimple Lie groups and Riemannian symmetric spaces. (See [8] for a comprehensive survey and references for Hardy's theorem on other groups.) Except for [19] and [21], all the results are analogues of the second half of Hardy's theorem where the estimates force the function to be zero. These results are usually viewed as *mathematical uncertainty principle*. The second part of Theorem 1.1 is stronger than these results and provides the best possible uncertainty (see Section 4).

2. Notation and preliminaries. We follow these standard practices:

(i) Lower case German letters denote the Lie algebras of the groups denoted by the corresponding upper case Roman letters.

(ii) For any Lie algebra $\mathfrak{a}$, $\mathfrak{a}^*$ is its real dual and $\mathfrak{a}_{\mathbb{C}}^*$ is the complexification of $\mathfrak{a}^*$.

(iii) For a group M, $\mathcal{E}_2(M)$ denotes the set of (equivalence classes of) discrete series representations of M.

(iv) C, C' etc. are used to denote constants (real or complex) whose value might change from line to line. Polynomials are denoted by P, P'. We use subscripts of C or P when needed to indicate their dependence on parameters of interest. We may not repeat mentioning these at the particular places.

Let G be a connected noncompact semisimple Lie group with finite center and let K be a fixed maximal compact subgroup of G.

Let $G = KA_0N_0$ be an Iwasawa decomposition of G and let $P_0 = M_0A_0N_0$ be the minimal parabolic subgroup corresponding to this Iwasawa decomposition. Let $\Sigma(\mathfrak{g}, \mathfrak{a}_0)$ be the set of restricted roots, Σ^+ be the set of positive restricted roots which is chosen once for all and Σ_0^+ be the set of indivisible positive roots. Denote the underlying set of simple roots by Δ_0 and the corresponding positive Weyl chamber in $\mathfrak{a}_0$ by $\mathfrak{a}_0^+$. Then G has a decomposition $G = K\overline{A_0^+}K$, where $A_0^+ = \exp \mathfrak{a}_0^+$.

For each subset of Δ_0, we have a standard parabolic subgroup P containing P_0. The minimal parabolic subgroup P_0 corresponds to the null set and G itself corresponds to the full set Δ_0. A parabolic subgroup P with Langlands decomposition $P = M_PA_PN_P$ is called *cuspidal* when $\mathcal{E}_2(M_P)$ is nonempty.

For $\xi \in \mathcal{E}_2(M_0)$ and $\lambda \in \mathfrak{a}_{0\mathbb{C}}^* = \mathbb{C}^n$, the minimal principal series representation $\pi_{P_0,\xi,\lambda}$ defined on the Hilbert space $H_{P_0,\xi,\lambda}$ is induced from the representation $\xi \otimes \exp(\lambda) \otimes 1$ of P_0. The induction is normalized so that $\pi_{P,\xi,\lambda}$ is unitary if λ is pure imaginary. For a suitable function f, let $F_{P_0}(f)(\xi, \lambda)$ denote its Fourier transform with respect to $\pi_{P_0,\xi,\lambda}$.

For a standard cuspidal parabolic subgroup P containing P_0 with Langlands decomposition $P = M_PA_PN_P$, M_P is generally a noncompact and disconnected reductive subgroup of the Harish-Chandra class (see [12]), and $A_P \subset A_0$, $N_P \subset N_0$ and $M_P \supset M_0$. Since P is cuspidal, $\mathcal{E}_2(M_P)$ is nonempty. To each such P we associate a series of admissible representations $\pi_{P,\xi,\lambda} = \mathrm{ind}_P^G(\xi \otimes \exp(\lambda) \otimes 1)$ on the Hilbert space $H_{P,\xi,\lambda}$, inducing from the representation $\xi \otimes \exp(\lambda) \otimes 1$ of P, where $\xi \in \mathcal{E}_2(M_P)$ and $\lambda \in \mathfrak{a}_{P\mathbb{C}}^*$. Again, $\pi_{P,\xi,\lambda}$ is unitary if $\lambda \in i\mathfrak{a}_P^*$. As in the minimal case, for $\xi \in \mathcal{E}_2(M_P)$ and $\lambda \in \mathfrak{a}_{P\mathbb{C}}^*$, $F_P(f)(\xi, \lambda)$ denotes the Fourier transform of f, when it exists, with respect to the nonminimal cuspidal principal series representation $\pi_{P,\xi,\lambda}$. Discrete series representations in this set up can be thought of as induced from the parabolic subgroup G itself. However we denote the Fourier transform of a function f with respect to $\pi \in \mathcal{E}_2(G)$ by $F_D(f)(\pi)$. It is known that the elements in $\mathcal{E}_2(G)$, i.e. the discrete series, are explicitly embedded in the nonunitary minimal principal series (see [17]).

Let $K' = K \cap M_P = K \cap P$. Then K' is the maximal compact subgroup of M_P and $M_0 \subset K' \subset M_P$. Let $M_P = K'A_1N_1$ be an Iwasawa decomposition of M_P. Then $A_0 = A_P A_1$ and $N_0 = N_P N_1$ and M_0 is the centralizer of A_1 in K'. Therefore $P_1 = M_0 A_1 N_1$ is a minimal parabolic subgroup of M_P.

Now take $\sigma \in \mathcal{E}_2(M_P)$ and $\nu \in i\mathfrak{a}_P^*$. Then $\pi_{P,\sigma,\nu}$ is unitary. If ν is also regular, i.e. $\langle \beta, \nu\rangle \neq 0$ for all roots β of $(\mathfrak{g}, \mathfrak{a}_P)$, then it is irreducible. As noted above, the discrete series representations are explicitly embedded in the nonunitary principal series. Therefore there exist $\widetilde{\sigma} \in \mathcal{E}_2(M_0)$ and $\mu_1 \in \mathfrak{a}_{1\mathbb{C}}^*$ so that σ is infinitesimally embedded in $\omega_{\widetilde{\sigma}\mu_1} = \mathrm{ind}_{M_0A_1N_1}^{M_P}(\widetilde{\sigma} \otimes e^{\mu_1} \otimes 1)$, which is a nonunitary minimal principal series representation of M_P. Then it can be shown that $\pi_{P,\omega,\nu}$ is canonically equivalent to $\pi_{P_0,\widetilde{\sigma},i\nu\oplus\mu_1}$ and hence $\pi_{P,\sigma,\nu}$ is infinitesimally embedded in the nonunitary minimal principal series $\pi_{P_0,\widetilde{\sigma},i\nu\oplus\mu_1}$ of G. Thus embedding of the nonminimal principal series in the minimal ones is obtained from embedding of the discrete series through a *double induction* (see [16, p. 240]). For a detailed description of these representations we also refer to [26].

Let dk and da be respectively the Haar measures on K and A and $\int_K dk = 1$. The norm induced by the Killing form $\langle\, ,\rangle$ of $\mathfrak{g}$ on $\mathfrak{a}$ and on its dual $\mathfrak{a}^*$ are both denoted by $|\cdot|$. Then $\sigma(\exp H) = |H| = \langle H, H\rangle^{1/2}$ for $H \in \mathfrak{a}_0$. Let $\varrho_0 = \frac{1}{2}\sum_{\gamma\in\Sigma^+} m_\gamma\gamma$, m_γ being the multiplicity of the root γ, and let $\Xi(x)$ be ϕ_0, i.e. the elementary spherical function with parameter 0. The following estimate of Harish-Chandra ([13, Section 9]) will be useful for us:

$$e^{-\varrho_0(H)} \leq \Xi(\exp H) \leq Ce^{-\varrho_0(H)}(1 + |H|)^{|\Sigma_0^+|} \quad \text{for all } H \in \overline{\mathfrak{a}_0^+}. \tag{2.1}$$

This estimate of Ξ is sufficient for the proof of the main theorem. Nonetheless let us record the *exact* estimate of Ξ due to Anker ([2]) for future use:

$$\Xi(\exp H) \asymp \Big\{\prod_{\gamma\in\Sigma_0^+}(1 + \gamma(H))\Big\}e^{-\varrho_0(H)} \quad \text{for all } H \in \overline{\mathfrak{a}_0^+} \tag{2.2}$$

where $f_1(x) \asymp f_2(x)$ means there exist positive constants $C \leq C'$ such that $Cf_2(x) \leq f_1(x) \leq C'f_2(x)$.

The Haar measure dx on G can be normalized so that $dx = J(a)dk_1 da dk_2$, where $J(a) = \prod_{\gamma\in\Sigma^+}(e^{\gamma(\log a)} - e^{-\gamma(\log a)})^{m_\gamma}$ is the Jacobian of the $K\overline{A_0^+}K$-decomposition of G. Clearly,

$$|J(a)| \leq Ce^{2\varrho_0(\log a)}. \tag{2.3}$$

We can choose an orthonormal basis of $H_{P,\xi,\lambda}$ consisting of K-finite vectors. Let u, v be two elements of this basis. Then $F_P^{u,v}(f)(\xi,\lambda)$ denotes the (u,v)th matrix coefficient of the operator $F_P(f)(\xi,\lambda)$. Precisely,

$$F_P^{u,v}(f)(\xi,\lambda) = \int_G f(x)\langle \pi_{P,\xi,\lambda}(x)u, v\rangle\, dx.$$

Let $(\delta_1,\delta_2) \in \widehat{K} \times \widehat{K}$. Assume that u and v transform according to δ_1 and δ_2 respectively. Then, using the arguments of Miličić ([18, p. 83], see also [22, 4.2]) we have

$$|\langle \pi_{P,\xi,\lambda}(x)u,v\rangle| \le C_{\delta_1,\delta_2}\phi_{\lambda_{\mathbb{R}}+\varrho_P-\varrho_0}(x) \quad \text{for all } x\in G \tag{2.4}$$

where ϕ_λ is the elementary spherical function with parameter λ, $\lambda_{\mathbb{R}}$ is the real part of λ and ϱ_P is the half sum of the positive roots of (P, A_P). Note that $\lambda_{\mathbb{R}}$ and ϱ_P are extended to $\mathfrak{a}_0$ by defining them trivially on $\mathfrak{a}_1 = \mathfrak{a}_P^{\perp}$ which is the orthocomplement of $\mathfrak{a}_P$ in $\mathfrak{a}_0$ with respect to the Killing form (see [18]). If f is a biinvariant function and ξ_0 is the trivial representation of M_0, then $F_{P_0}^{0,0}(f)(\xi_0,\lambda) = \int_G f(x)\phi_\lambda(x)\,dx$ will also be denoted by $\widehat{f}(\lambda)$.

3. Proof of the theorem. We shall use the following Phragmén–Lindelöf theorem, a proof of which can be found in [5].

THEOREM 3.1 (Phragmén–Lindelöf). *Let $D = \{re^{i\theta} \mid r\ge 0,\ |\theta|\le \pi/(2\alpha)\}$, $\alpha > 1/2$. Suppose f is an analytic function on D such that*

$$|f(z)| \le M < \infty \quad \text{for } r\ge 0,\ \theta = \pm\frac{\pi}{2\alpha},$$
$$|f(z)| \le Ce^{r^\beta}, \qquad \beta<\alpha,\ z\in\overline{D},$$

where $\overline{D}$ denotes the closure of D. Then $|f(z)| < M$ for $z\in D$.

Now we will prove a complex-analytic result using the above theorem.

LEMMA 3.2. *Let f be an entire function on $\mathbb{C}^n$ for some $n\ge 1$ such that*

$$|f(\mathbf{z})| \le C_1 e^{\beta|\mathbf{z}|^2}(1+|\mathbf{z}|)^r \qquad \text{for all } \mathbf{z}\in\mathbb{C}^n, \tag{3.1}$$
$$|f(x)| \le C_2 e^{-\beta|x|^2}(1+|x|)^s \qquad \text{for all } x\in\mathbb{R}^n, \tag{3.2}$$

for positive β and integers $r,s>0$. Then $f(\mathbf{z}) = P(\mathbf{z})e^{-\beta(z_1^2+\ldots+z_n^2)}$ for $\mathbf{z} = (z_1,\ldots,z_n)\in\mathbb{C}^n$ where P is a polynomial with $\deg P\le \min(r,s)$.

Proof. The lemma will be proved by induction on n. Assume that the result is true for $n=1,\ldots,m-1$. For $\mathbf{z}\in\mathbb{C}^m$ write $\mathbf{z} = (\widetilde{z}, z_m)$ where $\widetilde{z} = (z_1,\ldots,z_{m-1})\in\mathbb{C}^{m-1}$ and $z_m\in\mathbb{C}$.

Let f be an entire function on $\mathbb{C}^m$ which satisfies (3.1) and (3.2) for $n=m$. For $\widetilde{x}\in\mathbb{R}^{m-1}$ and $z\in\mathbb{C}$, let $f_{\widetilde{x}}(z) = f(\widetilde{x},z)$. Then for each $\widetilde{x}\in\mathbb{R}^{m-1}$, $f_{\widetilde{x}}$ is an entire function on $\mathbb{C}$ which satisfies (3.1) and (3.2) for $n=1$. Therefore by induction hypothesis, for each $\widetilde{x}\in\mathbb{R}^{m-1}$, $f(\widetilde{x},z) = f_{\widetilde{x}}(z) = h(\widetilde{x},z)e^{-\beta z^2}$ where $h(\widetilde{x},z)$ is a polynomial in z for each fixed $\widetilde{x}\in\mathbb{R}^{m-1}$. Then $h(\widetilde{x},z) = f(\widetilde{x},z)e^{\beta z^2}$. As f is analytic on $\mathbb{C}^m$, $h(\widetilde{x},z)$ is a restriction of an analytic function on $\mathbb{R}^{m-1}\times\mathbb{C}$ and hence $f(\mathbf{z}) = f(\widetilde{z},z) = h(\widetilde{z},z)e^{-\beta z^2}$.

Now for $\widetilde{z}\in\mathbb{C}^{m-1}$, $x\in\mathbb{R}$, consider the function $h(\widetilde{z},x) = f(\widetilde{z},x)e^{\beta x^2}$. For fixed $x\in\mathbb{R}$, $h(\widetilde{z},x)$ is entire on $\mathbb{C}^{m-1}$ and satisfies (3.1) and (3.2) for $n=m-1$. Hence again by induction hypothesis, $h(\widetilde{z},x) = P_x(\widetilde{z})e^{-\beta(z_1^2+\ldots+z_{m-1}^2)}$ where for each fixed $x\in\mathbb{R}$, P_x is a polynomial in $\widetilde{z}$. But as $P_x(\widetilde{z}) =$

$h(\widetilde{z},x)e^{\beta(z_1^2+\ldots+z_{m-1}^2)}$, $P(\widetilde{z},x)=P_x(\widetilde{z})$ is the restriction of an analytic map on $\mathbb{C}^{m-1}\times\mathbb{R}$. Therefore $h(\mathbf{z})=h(\widetilde{z},z)=P_z(\widetilde{z})e^{-\beta(z_1^2+\ldots+z_{m-1}^2)}$. Hence

$$\begin{aligned} f(\mathbf{z}) &= f_{\widetilde{z}}(z)=h(\widetilde{z},z)e^{-\beta z^2}=P_z(\widetilde{z})e^{-\beta(z_1^2+\ldots+z_{m-1}^2)}e^{-\beta z_m^2}\\ &= P_z(\widetilde{z})e^{-\beta(z_1^2+\ldots+z_m^2)}. \end{aligned}$$

Note that $P_z(\widetilde{z})$ is also a polynomial in z for fixed $\widetilde{z}$, as $e^{-\beta(z_1^2+\ldots+z_{m-1}^2)}$ is independent of z and $h(\widetilde{z},z)$ is a polynomial in z for fixed $\widetilde{z}$. Now as $P(\mathbf{z})=P_z(\widetilde{z})$ is separately polynomial in z and in $\widetilde{z}$, P is a polynomial in $\mathbf{z}=(\widetilde{z},z)$ (see [20]). Therefore, $f(\mathbf{z})=P(\mathbf{z})e^{-\beta(z_1^2+\ldots+z_m^2)}$ for some polynomial P.

Comparing f with (3.1) and (3.2), we have $\deg P\le\min(r,s)$.

The argument will be complete if we prove the case $n=1$.

Proof for $n=1$. Let $f=f_{\mathrm{e}}+f_{\mathrm{o}}$ where f_{e} and f_{o} are respectively the even and odd parts of f.

Consider $\psi(z)=f_{\mathrm{e}}(\sqrt{z})$. Then ψ is also entire and by (3.1) and (3.2) respectively, it satisfies the inequalities

$$|\psi(z)|\le Ce^{\beta|z|}(1+|z|)^{r'}\quad\text{for } z\in\mathbb{C},\tag{3.3}$$

$$|\psi(x)|\le C'e^{-\beta x}(1+x)^{s'}\quad\text{for } x\in\mathbb{R}^+,\tag{3.4}$$

for some positive r' and s'.

We first show that for $s''\ge\max\{r',s'\}$,

$$|\psi(z)e^{\beta z}|\le C(1+|z|)^{s''}\quad\text{for } z\in\mathbb{C}.\tag{3.5}$$

Let $D_\Theta=\{z=re^{i\theta}\mid 0\le\theta\le\Theta\}$ for $\Theta\in(0,\pi)$. Define

$$w(z,\Theta)=w(r,\theta,\Theta)=\exp\left[\frac{\beta ize^{-i\Theta/2}}{\sin(\Theta/2)}\right]$$

(see [5]). Then

$$|w(r,0,\Theta)|=e^{\beta r},\tag{3.6}$$

$$|w(r,\Theta,\Theta)|=e^{-\beta r},\tag{3.7}$$

$$w(z,\Theta)\to e^{\beta z}\quad\text{as }\Theta\to\pi.\tag{3.8}$$

Let $F'(z)=w(z,\Theta)\psi(z)/(z+i)^{s''}$ for $z\in D_\Theta$. From (3.6) and (3.4) we have

$$|F'(x)|=|F'(r)|\le C\,\frac{(1+r)^{s'}}{(i+r)^{s''}}\le M\quad\text{for } x\in\mathbb{R}^+.$$

Again from (3.7) and (3.3), we get

$$|F'(re^{i\Theta})|\le C\,\frac{(1+r)^{r'}}{(i+r)^{s''}}\le M$$

for some $M>0$. Therefore, by Theorem 3.1, we have $|F'(z)|\le M$ on D_Θ. Now from (3.8), we conclude that on $\{z\in\mathbb{C}\mid\Im z\ge0\}$, $|\psi(z)e^{\beta z}/(z+i)^{s''}|$

$\le M$. Similarly we can prove that on the lower half plane $|\psi(z)e^{\beta z}/(z-i)^{s''}| \le M$. Combining them we get (3.5).

From (3.5) we get, for $z \in \mathbb{C}$, $\psi(z) = P_1(z)e^{-\beta z}$ for some polynomial P_1. Therefore,

$$f_{\mathrm{e}}(z) = P_1(z^2)e^{-\beta z^2}. \tag{3.9}$$

Now let $g(z) = f_{\mathrm{o}}(z)/z$. Then $g(z)$ is an even entire function which satisfies (3.1) and (3.2) for $n = 1$ and hence,

$$f_{\mathrm{o}}(z) = zP_2(z^2)e^{-\beta z^2} \tag{3.10}$$

for some polynomial P_2. Therefore $f(z) = P(z)e^{-\beta z^2}$ for some polynomial P. ∎

Proof of Theorem 1.1. We can choose an orthonormal basis of $H_{P_0,\xi,\lambda}$ adapted to the decomposition of $\pi_{P_0,\xi,\lambda}$ into different K-types. Let $(\delta_1,\delta_2) \in \widehat{K} \times \widehat{K}$ and let u and v be two elements of this basis of $H_{P_0,\xi,\lambda}$ which transform according to δ_1 and δ_2 respectively.

From (2.4), taking $P = P_0$, we have

$$|\langle \pi_{P_0,\xi,\lambda}(x)u, v\rangle| \le C_{\delta_1,\delta_2}\phi_{\lambda_{\mathbb{R}}}(x) \quad \text{for all } x \in G. \tag{3.11}$$

Now as $|\phi_\lambda(a)| \le e^{\lambda_{\mathbb{R}}^+(\log a)}\Xi(a)$ (see [10, Prop. 4.6.1]), we have

$$|\langle \pi_{P_0,\xi,\lambda}(a)u, v\rangle| \le Ce^{\lambda_{\mathbb{R}}^+(\log a)}\Xi(a). \tag{3.12}$$

Here $\lambda_{\mathbb{R}}^+$ is the Weyl translate of $\lambda_{\mathbb{R}}$ which is dominant, i.e. belongs to the positive Weyl chamber.

For $(\xi,\lambda) \in \mathcal{E}_2(M_0) \times \mathfrak{a}_{0\mathbb{C}}^*$, the (u,v)th matrix coefficient of $F_{P_0}(f)(\xi,\lambda)$ is

$$F_{P_0}^{u,v}(f)(\xi,\lambda) = \int_G f(x)\langle \pi_{P_0,\xi,\lambda}(x)u, v\rangle\,dx. \tag{3.13}$$

Therefore,

$$|F_{P_0}^{u,v}(f)(\xi,\lambda)| = \Big|\int_{A_0^+} f(a)\langle \pi_{P_0,\xi,\lambda}(a)u, v\rangle J(a)\,da\Big|. \tag{3.14}$$

We will show that for each $\xi \in \mathcal{E}_2(M_0)$, $F_{P_0}^{u,v}(f)(\xi,\lambda)$ is an entire function in $\lambda \in \mathfrak{a}_{0\mathbb{C}}^*$ and

$$|F_{P_0}^{u,v}(f)(\xi,\lambda)| \le Ce^{\beta|\lambda|^2}(1+|\lambda|)^{M'} \quad \text{for all } \lambda \in \mathbb{C}^n \tag{3.15}$$

where $M' = M + 2|\Sigma_0^+|$.

We rewrite the condition (1.1) as

$$\begin{aligned} &|f(k_1ak_2)| \\ &\quad \le Ce^{-\alpha|\log a|^2}\Xi(a)(1+|\log a|)^M \quad \text{for all } k_1,k_2 \in K,\ a \in A_0^+. \end{aligned} \tag{3.16}$$

Let $|F_{P_0}^{u,v}(f)(\xi,\lambda)| = I$. Then from (3.14), using (3.16), (2.3) and (3.12), for all $\xi \in \mathcal{E}_2(M_0)$ and $\lambda \in \mathfrak{a}_{0\mathbb{C}}^* = \mathbb{C}^n$, we have

$$I \le C \int\limits_{\overline{A_0^+}} e^{-\alpha|\log a|^2} e^{\lambda_{\mathbb{R}}^+(\log a)} \Xi(a)^2 (1+|\log a|)^M e^{2\varrho_0(\log a)} da.$$

Now applying (2.1), we get, for $M' = M + 2|\Sigma_0^+|$,

$$\begin{aligned} I &\le C \int\limits_{\overline{A_0^+}} e^{-\alpha|\log a|^2} e^{\lambda_{\mathbb{R}}^+(\log a)} e^{-2\varrho_0(\log a)} (1+|\log a|)^{M'} e^{2\varrho_0(\log a)}\, da \\ &= C \int\limits_{\overline{A_0^+}} e^{-\alpha|\log a|^2} e^{\lambda_{\mathbb{R}}^+(\log a)} (1+|\log a|)^{M'}\, da \\ &\le C \int\limits_{\mathfrak{a}_0} e^{-\alpha|H|^2} e^{\lambda_{\mathbb{R}}^+(H)} (1+|H|)^{M'} dH, \end{aligned}$$

where $H = \log a$, i.e. $H \in \mathfrak{a}_0$ such that $\exp H = a$, dH is the Lebesgue measure on $\mathfrak{a}_0$. Suppose $H_{\lambda_{\mathbb{R}}}$ corresponds to $\lambda_{\mathbb{R}}^+$ via the isomorphism of $\mathfrak{a}_0^*$ with $\mathfrak{a}_0$ through the Killing form (i.e. $\lambda_{\mathbb{R}}^+(H) = \langle H, H_{\lambda_{\mathbb{R}}}\rangle$ for all H) so that $|\lambda_{\mathbb{R}}^+| = |H_{\lambda_{\mathbb{R}}}|$. Then

$$I \le C e^{|H_{\lambda_{\mathbb{R}}}|^2/(4\alpha)} \int\limits_{\mathfrak{a}_0} e^{-\alpha\langle H - H_{\lambda_{\mathbb{R}}}/(2\alpha), H - H_{\lambda_{\mathbb{R}}}/(2\alpha)\rangle} (1+|H|)^{M'} dH$$

From this, by translation invariance of Lebesgue measure, we have

$$\begin{aligned} I &\le C e^{|H_{\lambda_{\mathbb{R}}}|^2/(4\alpha)} (1+|H_{\lambda_{\mathbb{R}}}|)^{M'} \int\limits_{\mathfrak{a}_0} e^{-\alpha|H|^2} (1+|H|)^{M'} dH \\ &\le C(1+|\lambda|)^{M'} e^{|\lambda|^2/(4\alpha)} \int\limits_{\mathfrak{a}_0} e^{-\alpha|H|^2} (1+|H|)^{M'}\, dH \quad (\text{as } |\lambda_{\mathbb{R}}^+| = |\lambda_{\mathbb{R}}| \le |\lambda|) \\ &= C'(1+|\lambda|)^{M'} e^{\beta|\lambda|^2} \qquad (\text{as } \beta = 1/(4\alpha)), \end{aligned}$$

for $\xi \in \mathcal{E}_2(M_0)$ and $\lambda \in \mathfrak{a}_{0\mathbb{C}}^* = \mathbb{C}^n$. This proves (3.15).

Let $F_\xi(\lambda) = F_{P_0}^{u,v}(f)(\xi,\lambda)$ and let $H(\lambda) = F_\xi(i\lambda)$. By (1.2) every matrix entry of $F_{P_0}^{u,v}(f)(\xi,\lambda)$ satisfies

$$|F_{P_0}(f)^{u,v}(\xi,\lambda)| \le C_\xi e^{-\beta|\lambda|^2} (1+|\lambda|)^N \quad \text{ for all } (\xi,\lambda) \in \mathcal{E}_2(M_0) \times i\mathfrak{a}_0^*.$$

From this and (3.15), by Lemma 3.2, we have

$$F_\xi(i\lambda) = H(\lambda) = P'(\lambda) e^{-\beta(z_1^2+\ldots+z_n^2)} = P(i\lambda) e^{\beta((iz_1)^2+\ldots+(iz_n)^2)},$$

where $P(\lambda) = P'(-i\lambda)$ and $\deg P \le \{M', N\}$. Hence

$$F_{P_0}^{u,v}(\xi,\lambda) = F_\xi(\lambda) = P(\lambda) e^{\beta(z_1^2+\ldots+z_n^2)}, \tag{3.17}$$

with P as above. This proves the first part of the theorem.

Now if $|F_{P_0}^{u,v}(f)(\xi,\lambda)|$ is $o(e^{-\beta(z_1^2+\ldots+z_n^2)})$ then (1.2) is satisfied and hence by (3.17), for $\lambda \in i\mathfrak{a}_0^*$ we have $F_{P_0}^{u,v}(f)(\xi,\lambda) = P(\lambda) e^{-\beta(r_1^2+\ldots+r_n^2)}$ where $\lambda =$

$(ir_1, \dots, ir_n)$, $r_i \in \mathbb{R}$, for $i = 1, \dots, n$. But then $|F^{u,v}_{P_0}(f)|$ cannot be $o(e^{-\beta|\lambda|^2})$ for $\lambda \in i\mathfrak{a}^*_0$ unless $P \equiv 0$, which would imply that $F^{u,v}_{P_0}(f)(\xi, \cdot) \equiv 0$ on $\mathfrak{a}^*_{0\mathbb{C}}$. As this is true for all basis elements $u, v \in H_{P_0,\xi,\lambda}$, for all $\xi \in \mathcal{E}_2(M_0)$, $F_{P_0}(f)$ is identically zero on $\mathcal{E}_2(M_0) \times \mathfrak{a}^*_{0\mathbb{C}}$.

We will now see that $F_P(f) \equiv 0$ on $\mathcal{E}_2(M_P) \times \mathfrak{a}^*_{P\mathbb{C}}$ for any standard cuspidal parabolic subgroup P containing P_0 and $F_D(f) \equiv 0$ on $\mathcal{E}_2(G)$.

Minimal and nonminimal principal series and discrete series representations are admissible. Unitary nonminimal principal series representations and discrete representations are infinitesimally equivalent to subrepresentations of minimal principal series representations (see Section 3). If two admissible representations are infinitesimally equivalent then they have the same set of K-finite matrix coefficients (see [16, p. 211]). Since $F_{P_0}(f) \equiv 0$ on $\mathcal{E}_2(M_0) \times \mathfrak{a}^*_{0\mathbb{C}}$, clearly all K-finite matrix coefficients of $F_{P_0}(f)(\xi, \lambda)$ are zero for all $(\xi, \lambda) \in \mathcal{E}_2(M_0) \times \mathfrak{a}^*_{0\mathbb{C}}$ and hence by the above argument, all K-finite matrix coefficients of $F_P(f)(\xi, \lambda)$ are also zero for all $(\xi, \lambda) \in \mathcal{E}_2(M_P) \times \mathfrak{a}^*_{P\mathbb{C}}$. Therefore $F_P(f) \equiv 0$ on $\mathcal{E}_2(M_P) \times \mathfrak{a}^*_{P\mathbb{C}}$. For the same reason $F_D(f) \equiv 0$ on $\mathcal{E}_2(G)$.

As Plancherel measure has support only on $\pi_{P,\xi\lambda}$ for standard cuspidal parabolic subgroups $P = M_P A_P N_P$, $\xi \in \mathcal{E}_2(M_P)$ and $\lambda \in i\mathfrak{a}^*_P$ and on the discrete series representations $\mathcal{E}_2(G)$, by the Plancherel Theorem $f = 0$ almost everywhere (see [27, II, p. 421]). ∎

4. Examples and remarks

1. In Theorem 1.1 if in particular $N < 0$ then clearly $f = 0$ (compare with Theorem 4.2(iv) below). Note that here $\alpha\beta = 1/4$ and hence it is stronger than the following theorem of [6]:

THEOREM 4.1 (Cowling–Sitaram–Sundari). *Let f be a measurable function on G such that*

$$|f(k_1 a k_2)| \le C e^{-\alpha(|\log a|)^2} \quad \text{for all } k_1, k_2 \in K,\ a \in A^+_0, \tag{4.1}$$

$$\|F_{P_0}(f)(\xi, \lambda)\| \le C_\xi e^{-\beta|\lambda|^2} \quad \text{for all } (\xi, \lambda) \in \mathcal{E}_2(M_0) \times i\mathfrak{a}^*_0, \tag{4.2}$$

where α, β, C, C_ξ are positive constants. If $\alpha\beta > 1/4$, then $f = 0$.

Proof. It is possible to choose $\alpha' < \alpha$ and $\beta' < \beta$ so that $\alpha'\beta' = 1/4$. Then f and $F_{P_0}(f)$ satisfy (1.1) and (1.2) with α, β replaced by this α', β' and for some $N < 0$. Therefore $f = 0$ by the last part of Theorem 1.1. ∎

The results obtained in [23], [22] and [7] follow from the above theorem in [6] and hence also follow from Theorem 1.1. Thus in semisimple Lie groups, Theorem 1.1 with $N < 0$ provides the strongest uncertainty principle in the form of Hardy's theorem. It is also the best in view of the first part of the theorem.

2. In the proof of Theorem 1.1, we have shown that if for a function f, $F_{P_0}(f)(\xi,\lambda)$ can be defined on $\mathfrak{a}^*_{0\mathbb{C}}$ for all $\xi \in \mathcal{E}_2(M_0)$ and is identically zero there, then $f = 0$. Now if two functions f_1 and f_2 satisfy (1.1), then for $i = 1, 2$, $F_{P_0}(f_i)(\xi,\cdot)$ is entire on $\mathfrak{a}^*_{0\mathbb{C}}$ for all $\xi \in \mathcal{E}_2(M_0)$. If $F_{P_0}(f_1)(\xi,\lambda) = F_{P_0}(f_2)(\xi,\lambda)$ for all $\xi \in \mathcal{E}_2(M_0)$ and $\lambda \in \mathfrak{a}^*_{0\mathbb{C}}$, then clearly, $f_1 = f_2$ as $F_{P_0}(f_1 - f_2) \equiv 0$ on $\mathcal{E}_2(M_0) \times \mathfrak{a}^*_{0\mathbb{C}}$. Thus the function f satisfying (1.1) is uniquely determined by its Fourier transform with respect to the minimal (unitary and nonunitary) principal series.

3. To complete the picture we should show that if we replace Ξ by Ξ^l for some $l \in [0, 1)$ in (1.1), then there are functions other than what we have characterized, satisfying this modified estimate while their Fourier transforms still satisfy (1.2). This will ensure that the estimates in Theorem 1.1 (and in Theorem 4.3 below) are optimal.

Example. Let $G = SL_2(\mathbb{C})$ and K be its maximal compact subgroup $SU(2)$. Then

$$A = \left\{ a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

Let α be the unique positive root given by $\alpha(\log a_t) = t$. Every $\lambda \in \mathbb{C}$ can then be identified with an element in $\mathfrak{a}_{\mathbb{C}}$ by $\lambda = \lambda\alpha$. In this identification, the unitary spherical principal series representations are given by elements of $\mathbb{R}$, the Plancherel measure is $|c(\lambda)|^{-2} = |\lambda|^2$ and the elementary spherical function $\phi_\lambda(a_t) = 2\sin(\lambda t)/(\lambda \sinh(2t))$. Also $|\lambda| = |\lambda|/4$ and $\sigma(a_t) = 4|t|$ and $\varrho = 2$ (see [14, p. 432], [25, p. 313] and [23]).

For a suitable function f on $\mathbb{R}$, let $\widetilde{f}$ be its Euclidean Fourier transform and $\mathcal{A} : C_c^\infty(K/G/K) \to C_c^\infty(\mathbb{R})^{\mathrm{even}}$ be the Abel transform, defined by $\mathcal{A}(f)(a) = e^{\varrho(\log a)} \int_N f(an)\, dn$.

Define a function g in the Schwartz space $C^2(K/G/K)$ by $\widehat{g}(\lambda) = \widetilde{\psi}(\lambda)\widehat{h}(\lambda)P(\lambda)$ for $\lambda \in \mathbb{R}$, where ψ is an even function in $C_c^\infty(\mathbb{R})$ with support $[-s, s]$ for some $s > 0$, $\widehat{h}(\lambda) = e^{-\lambda^2/4}$ for $\lambda \in \mathbb{R}$ and $P(\lambda)$ is an even polynomial on $\mathbb{R}$.

By Fourier inversion, we get

$$\begin{aligned} g(a_t) &= C \int_{\mathbb{R}} \widetilde{\psi}(\lambda) e^{-\lambda^2/4} P(\lambda) \frac{\sin(\lambda t)}{\lambda \sinh(2t)}\, \lambda^2 \, d\lambda \\ &= \frac{C}{\sinh(2t)} \int_{\mathbb{R}} \widetilde{\psi}(\lambda) e^{-\lambda^2/4} \lambda P(\lambda) \sin(\lambda t)\, d\lambda = \frac{C}{\sinh(2t)} (\psi *_E h')(t), \end{aligned}$$

where $h(t) = e^{-t^2}$, $h'(t) = P'(t)h(t)$ for some polynomial P' and $*_E$ is the Euclidean convolution. Therefore,

$$\begin{aligned}|g(a_t)| &\le \frac{C}{|\sinh 2t|}\, e^{-t^2} e^{2st}(1+|t|)^M \\ &\le Ce^{-\sigma(a_t)^2/16} e^{-(1-s)2t}(1+|t|)^M \\ &= Ce^{-\sigma(a_t)^2/16} e^{-(1-s)2t}(1+|t|)^M \\ &\le Ce^{-\sigma(a_t)^2/16} \varXi(a_t)^{1-s}(1+|t|)^M,\end{aligned}$$

for some $M > 0$. Now if we choose s so that $1-s>0$, then g satisfies

$$|g(x)| \le Ce^{-\sigma(x)^2/16} \varXi(x)^l (1+\sigma(x))^M \quad \text{for some } l \in (0,1) \text{ and } M > 0. \tag{4.3}$$

Its Fourier transform is $\widehat{g}(\lambda) = \widetilde{\psi}(\lambda)\widehat{h}(\lambda)P(\lambda)$ for $\lambda \in \mathbb{R}$. As $\widetilde{\psi}$ is bounded on $\mathbb{R}$,

$$|\widehat{g}(\lambda)| \le C' e^{-4|\lambda|^2}(1+|\lambda|)^N \quad \text{on } \mathbb{R} \text{ for some } N > 0. \tag{4.4}$$

As the Euclidean Fourier transform of the Abel transform of a biinvariant C_c^∞-function is the same as its group Fourier transform, $g(x) = \mathcal{A}^{-1}(\psi) * h(D;\cdot)$ where D is a left G-invariant differential operator which is also right K-invariant and of even order. As ψ can be any even C^∞-function on $\mathbb{R}$ with support $\{t \mid |t| \le s\}$ for some $s \in (0,1)$, $g(x)$ can be $f * h(D;\cdot)(x)$ for some biinvariant C_c^∞-function f with support $\{x \mid \sigma(x) \le s\}$ for some $s \in (0,1)$ (see [1]).

Note that if we take f to be any biinvariant C_c^∞-function (i.e. s is any positive number), then these g's will serve as examples of functions which satisfy (1.1) and (1.2) for $\alpha\beta < 1/4$.

4. For a function f as in Theorem 1.1, an exact description of the matrix coefficient functions $F_{P_0}^{u,v}(f)$ requires a complete knowledge of the properties of the matrix coefficients of the representations $\langle \pi_{P_0,\xi,\lambda}(x)u, v\rangle$. Unfortunately this is not available in general, because it needs an exhaustive understanding of the subquotients of the principal series representations. For instance, for fixed $\xi \in \widehat{M_0}$, the λ's in $\mathfrak{a}_{0\mathbb{C}}^*$ for which the (u,v)th matrix coefficient $\langle \pi_{P_0,\xi,\lambda}(x)u, v\rangle$ is identically zero will show up in $F_{P_0}^{u,v}(f)(\xi,\lambda)$. The following example shows that we can have more explicit results where these are known completely.

Example. Let G be $SL_2(\mathbb{R})$. Then K is the circle $\mathbb{T} = \{k_\theta \mid \theta \in [0, 2\pi)\}$ and $M = M_0 = \{0, \pi\}$. Then $\widehat{M} = \{\xi_0, \xi_1\}$, of which ξ_0 is the trivial representation. K-types are parametrized by integers as $\widehat{K} = \{\chi_n \mid n \in \mathbb{Z}\}$, where $\chi_n(k_\theta) = e^{in\theta}$. The (m,n)th isotypic component of f is denoted by $f_{m,n}$. When restricted to M, every χ_n contains exactly one copy of either ξ_0 or ξ_1 according as n is even or odd. For $\xi \in \widehat{M}$ and $\lambda \in \mathfrak{a}_{0,\mathbb{C}}^* = \mathbb{C}$, let $(\pi_{\xi,\lambda}, H_\xi)$ be the principal series representation where H_ξ is a subspace of $L^2(K)$ (compact picture) and the action of $\pi_{\xi,\lambda}$ is given by 4.1 of [4]. For details on the parametrization of the representations and their realization on $L^2(K)$, we refer to [4]. The function $e_n : k_\theta \mapsto e^{in\theta}$ on K is in $L^2(K)$ and it transforms according to the K-type χ_n. Let $\mathbb{Z}^\xi$ be the set of even or odd integers according as $\xi = \xi_0$ or $\xi = \xi_1$. Then $\{e_n \mid n \in \mathbb{Z}^\xi\}$ is an orthonormal basis for H_ξ. It can be shown easily that $F_{P_0}^{m,n}(f) = F_{P_0}(f_{m,n})$. Also as m, n determines a $\xi \in \widehat{M}$ by $\chi_m, \chi_n|_M \supset \xi$, we may write $F_{P_0}^{m,n}(f)(\lambda)$ for $F_{P_0}^{m,n}(f)(\xi,\lambda)$, omitting the obvious ξ. It is known that for a fixed $\xi \in \widehat{M}$, there are exactly $|m-n|/2$ points λ in $\mathbb{C}$ where the matrix coefficient $\langle \pi_{\xi,\lambda}(x)e_m, e_n\rangle$ is zero for all $x \in G$ (see [4, Proposition 7.1]). These are precisely those λ's where $\pi_{\xi,\lambda}$ has an irreducible subrepresentation which contains e_m but not e_n. Therefore if for a function f, $F_{P_0}^{m,n}(f)(\lambda)$ can be defined on the whole of $\mathbb{C}$, then $\lambda \mapsto F_{P_0}^{m,n}(f)(\lambda)$ has at least $|m-n|/2$ zeros each of order one. This observation and Theorem 1.1 yield the following result on $SL_2(\mathbb{R})$.

THEOREM 4.2. *Let f be a measurable function on $SL_2(\mathbb{R})$ which satisfies* (1.1) *and* (1.2) *and let $\alpha\beta = 1/4$. Then*:

(i) *if* $|m-n|/2 \le \min(M+2, N)$, *then* $F_{P_0}^{m,n}(f)(\lambda) = P_\beta(m,n,\lambda)e^{\beta\lambda^2}$, $\lambda \in \mathbb{C}$,

(ii) *if* $|m-n|/2 > \min(M+2, N)$, *then* $f_{m,n} = 0$,

(iii) *if* $N = 0$ *then* $f_{m,n} = 0$ *for* $m \ne n$ *and* $F_{P_0}^{n,n}(f)(\lambda) = C_{\beta,n}e^{\beta\lambda^2}$, $\lambda \in \mathbb{C}$,

(iv) *if* $N < 0$ *then* $f = 0$.

Proof. Theorem 1.1 yields (i) with $\deg P_\beta(m,n,\lambda) \le \min(M+2, N)$ as $|\Sigma_0^+| = 1$ for $SL_2(\mathbb{R})$. Now by the above observation $F_{P_0}^{m,n}(f)(\lambda)$ has at least $|m-n|/2$ zeros. Therefore if $|m-n|/2 > \min(M+2, N)$, then $F_{P_0}^{m,n}(f)(\lambda) = 0$ for $\lambda \in \mathbb{C}$. Since $F_{P_0}^{m,n}(f) = F_{P_0}(f_{m,n})$, (ii) is proved. The first part of (iii) follows from (ii) while $\deg P_\beta(m,n,\lambda) \le \min(M+2, N)$ implies the second part. Lastly, (iv) follows from (ii). ∎

For a detailed account on Hardy's theorem on $SL_2(\mathbb{R})$ we refer to [21].

5. This concluding discussion aims at explaining the analogy of our result with the original theorem of Hardy on $\mathbb{R}$ which characterizes polynomial times the Gauss kernel ($c_t e^{-x^2/(4t)}$). The Gauss kernel is the fundamental solution of the heat equation of the Laplace operator. For semisimple Lie groups its analogue is the heat kernel $\{h_t \mid t > 0\}$, the fundamental solution of the heat equation $\Delta u = \frac{\partial}{\partial t} u$ for the Laplace–Beltrami operator Δ on G/K. Details on this topic can be found for example in [24]. It is well known that the heat kernel h_t on G/K is given by (see [9])

$$h_t(x) = \frac{1}{|W|} \int_{i\mathfrak{a}_0^*} e^{t(|\lambda|^2+|\varrho_0|^2)} \phi_\lambda(x) |c(\lambda)|^{-2}\, d\lambda, \tag{4.5}$$

where $|c(\lambda)|^{-2}$ is the Plancherel measure and ϕ_λ is the elementary spherical function. Thus h_t is biinvariant and $\widehat{h}_t(\xi_0,\lambda) = e^{t(|\lambda|^2+|\varrho_0|^2)}$, where ξ_0 is the trivial representation of M_0.

A number of authors studied the estimates of the heat kernel. See for instance [3] and the references therein. In [3] Anker and Ji have finally proved a sharp estimate for h_t. In view of (2.2) that estimate reduces to

$$\begin{aligned} (4.6) \quad & h_t(\exp H) \\ & \asymp t^{-n/2} e^{-|\varrho_0|^2 t}\, \Xi(\exp H) \Big\{ \prod_{\gamma\in\Sigma_0^+} (1+t+\langle\gamma,H\rangle)^{(m_\gamma+m_{2\gamma})/2-1} \Big\} e^{-|H|^2/(4t)} \end{aligned}$$

for $t>0$ and $H\in\overline{\mathfrak{a}_0^+}$. It follows that $h_t(x)$ satisfies (1.1) for $\alpha = 1/(4t)$. It is also clear that $\widehat{h}_t(\xi_0,\lambda)$ satisfies (1.2) for $N=0$. The analogy of our result with Hardy's original theorem will be perhaps more apparent if we use the heat kernel to re-state it as follows.

Theorem 4.3. *Let f be a measurable function on G such that*

$$(4.7) \qquad |f(x)| \le C(1+\sigma(x))^M h_{t_1}(x) \quad \text{for all } x\in G$$

$$(4.8) \qquad \|F_{P_0}(f)(\xi,\lambda)\| \le C_\xi(1+|\lambda|)^N \widehat{h}_{t_2}(\xi_0,\lambda) \text{ for all } (\xi,\lambda)\in\mathcal{E}_2(M_0)\times i\mathfrak{a}_0^*,$$

for integers $M>0$ and N, positive constants C, C_ξ, and $t_1,t_2>0$. If $t_1 = t_2 = t$, then $F_{P_0}^{u,v}(f)(\xi,\lambda) = \widehat{h}_t(\xi_0,\lambda)P_{\xi,u,v}(\lambda)$ for all $(\xi,\lambda)\in\mathcal{E}_2(M_0)\times\mathfrak{a}_{0\mathbb{C}}^*$ such that $u,v\in H_{P_0,\xi,\lambda}$, $P_{\xi,u,v}(\lambda)$ being a polynomial which depends on ξ,u,v. If $t_1<t_2$ then $f=0$.

The example on $SL_2(\mathbb{C})$ points out that if $t_1>t_2$, then there are *other* functions satisfying the estimates.

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