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INDECOMPOSABLE MODULES IN COILS

BY

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Dedicated to Daniel Simson on the occasion of his 60th birthday

Abstract. We describe the structure of all indecomposable modules in standard coils of the Auslander–Reiten quivers of finite-dimensional algebras over an algebraically closed field. We prove that the supports of such modules are obtained from algebras with sincere standard stable tubes by adding braids of two linear quivers. As an application we obtain a complete classification of non-directing indecomposable modules over all strongly simply connected algebras of polynomial growth.

Introduction. Let K be an algebraically closed field, and A a basic, connected, finite-dimensional K-algebra. We denote by $\operatorname{mod} A$ the category of finite-dimensional right A-modules, by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ consisting of a complete set of non-isomorphic indecomposable A-modules, by Γ_A the Auslander–Reiten quiver of A and by $\tau = \tau_A$ the Auslander–Reiten translation in Γ_A. We identify the vertices of Γ_A with the corresponding modules in $\operatorname{ind} A$, and the components of Γ_A with the corresponding full subcategories of $\operatorname{ind} A$. A component Γ of Γ_A is called *standard* if Γ is equivalent to the mesh-category $K(\Gamma)$ of Γ (see [10], [24]). It was shown in [31] that every standard component of Γ_A with infinitely many τ-orbits and without projective and injective modules is a stable tube, that is, a translation quiver of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ for some $r \geq 1$. A module X in $\operatorname{ind} A$ is called *directing* if it does not lie on a cycle $X = X_0 \to X_1 \to \cdots \to X_r = X$, $r \geq 1$, of non-zero non-isomorphisms in $\operatorname{ind} A$. The structure of directing modules is fairly well understood (see [8], [11], [12], [20], [21], [25]) because as shown in [24] their supports are tilted algebras. On the other hand, the Auslander–Reiten quiver Γ_A of an algebra A admits at most finitely many τ_A-orbits containing directing modules (see [22], [29]).

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In this paper we are interested in the structure of (non-directing) indecomposable modules lying in standard Auslander–Reiten components with oriented cycles. Our knowledge of such components is still relatively poor. In general, we only know that if Γ is a standard component of the Auslander–Reiten quiver Γ_A of an algebra A, then all but finitely many τ-orbits in Γ are periodic, and hence Γ admits at most finitely many modules of any fixed dimension d [31].

Important examples of standard Auslander–Reiten components with oriented cycles are provided by all stable tubes of the Auslander–Reiten quivers of concealed-canonical algebras. Recall that an algebra A is called *concealed-canonical* [16] if it is of the form $\mathrm{End}_C(T)$ where C is a canonical algebra (in the sense of Ringel [24]) and T is a tilting C-module which is a direct sum of indecomposable C-modules of positive rank. Tame concealed algebras and tubular algebras, described completely in [9], [15], [24], form the class of all concealed-canonical algebras which are of tame representation type.

In [3], [4] Assem and the second named author introduced the notion of admissible operations which generalized that of branch extensions or coextensions of [14], [24]. These allowed one to define and describe components of an Auslander–Reiten quiver called coils, and then a class of algebras, called coil enlargements of tame concealed algebras. This class of algebras is of fundamental interest in the study of simply connected algebras of polynomial growth. Namely, it was shown in [34] (see also [33]) that if A is a strongly simply connected algebra of polynomial growth then any non-directing indecomposable A-module is a module over a full convex subcategory of A which is a coil enlargement of a tame concealed algebra. Moreover, tame coil enlargements of tame concealed algebras (called coil algebras) are crucial in the study of representation-infinite selfinjective algebras of polynomial growth (see [6], [19], [26]).

Further, it was shown in [1] that all representation-infinite algebras which are tilting-cotilting equivalent to tame concealed and tubular algebras are special types (branch enlargements) of coil enlargements of tame concealed algebras. In [7] coil enlargements of arbitrary algebras with weakly separating families of (standard) stable tubes were introduced and investigated. This class of (usually wild) algebras contains the class of quasi-tilted algebras of canonical type [18]. The sincere standard stable tubes have been extensively investigated (see [16], [17], [23], [24], [30]–[32], [35], [36]). In particular, it has been shown in [23] that an algebra A is concealed-canonical if and only if Γ_A admits a sincere (standard) stable tube without external short cycles. Moreover, as shown in [36], there are many algebras of arbitrary (finite or infinite) global dimension whose Auslander–Reiten quivers admit sincere (even faithful) stable tubes of arbitrarily large rank.

The main aim of this paper is to describe the structure of all indecomposable modules in standard coils of the Auslander–Reiten quivers of coil enlargements of algebras whose Auslander–Reiten quiver admits a sincere standard stable tube. As an application we get a complete classification of non-directing indecomposable modules over all strongly simply connected algebras of polynomial growth.

The paper is organized as follows. In Section 1 we fix the notation and recall the notions of admissible operations and coils. Section 2 is devoted to the coil enlargements of algebras with a sincere standard stable tube and to the structure of indecomposable modules in the coils obtained in this enlargement process. In Section 3 we formulate our main result on the structure of algebras which admit (weakly) sincere indecomposable modules lying in coils. In Section 4 we prove the implications (i)⇒(ii) and (i)⇒(iii) of Theorem 3.3, by describing all (weakly) sincere indecomposable modules over braid algebras. Sections 5–7 are devoted to the proof of the implication (ii)⇒(i) of Theorem 3.3, which completes the proof of our main result (since the implication (iii)⇒(ii) is trivial). As an application of our main results we give in Section 8 (Theorem 8.4) a complete classification of non-directing indecomposable modules over strongly simply connected algebras of polynomial growth. In Section 9 we present some examples.

1. Admissible operations and coils

1.1. Throughout this paper, K will denote a fixed algebraically closed field. An *algebra* A will always mean a basic, connected, associative, finite-dimensional K-algebra with an identity. Thus there exists a connected bound quiver (Q_A, I_A) and an isomorphism $A \cong KQ_A/I_A$. Equivalently, $A = KQ_A/I_A$ may be considered as a K-linear category whose object class A_0 is the set of points of Q_A, and the set of morphisms $A(x,y)$ from x to y is the quotient of the K-vector space $KQ_A(x,y)$ of all formal linear combinations of paths in Q_A from x to y by the subspace $I_A(x,y) = KQ_A(x,y) \cap I_A$ (see [10]). A full subcategory C of A is called *convex* (in A) if any path in A with source and target in C lies entirely in C. It is called *triangular* if Q_C contains no oriented cycle.

By an A-*module* is meant a finitely generated right A-module. We denote by $\operatorname{mod} A$ the category of A-modules and by $\operatorname{ind} A$ the full subcategory consisting of a complete set of representatives of the isomorphism classes of indecomposable A-modules. For a point i in Q_A, we denote by $S(i)$ the corresponding simple A-module and by $P(i)$ (or $I(i)$) the projective cover (or injective envelope, respectively) of $S(i)$. The *dimension-vector* of a module M is the vector $\underline{\dim}\, M = (\dim_K \operatorname{Hom}_A(P(i), M))_{i\in A_0}$. The *support* of an A-module M is the full subcategory $\operatorname{Supp} M$ of A with object class $\{i \in A_0 \mid \operatorname{Hom}_A(P(i), M) \neq 0\}$.

1.2. Given a standard component Γ of Γ_A, and an indecomposable module X in Γ, the *support* $\mathcal{S}(X)$ of the functor $\mathrm{Hom}_A(X,-)|_\Gamma$ is the K-linear category defined as follows [5]. Let $\mathcal{H}_X$ denote the full subcategory of Γ consisting of the indecomposable modules M in Γ such that $\mathrm{Hom}_A(X,M)\neq 0$, and $\mathcal{I}_X$ denote the ideal of $\mathcal{H}_X$ consisting of the morphisms $f: M\to N$ (with M,N in $\mathcal{H}_X$) such that $\mathrm{Hom}_A(X,f)=0$. We define $\mathcal{S}(X)$ to be the quotient category $\mathcal{H}_X/\mathcal{I}_X$. Following the above convention, we usually identify the K-linear category $\mathcal{S}(X)$ with its quiver.

A translation quiver Γ is called a *tube* [14], [24] if it contains a cyclical path and if its underlying topological space is homeomorphic to $S^1\times\mathbb{R}^+$ (where S^1 is the unit circle, and $\mathbb{R}^+$ the non-negative real line). A tube has only two types of arrows: pointing to infinity and pointing to the mouth. This also applies to *sectional paths*, that is, paths $x_0\to x_1\to\cdots\to x_m$ in Γ such that $x_{i-1}\neq\tau x_{i+1}$ for all i, $0<i<m$. A maximal sectional path consisting of arrows pointing to infinity (or to the mouth) is called a *ray* (or a *coray*, respectively). Tubes containing neither projective vertices nor injective vertices are called *stable*. They are of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, $r\geq 1$. The *rank* of a stable tube Γ is the least positive integer r such that $\tau^r x=x$ for all x in Γ. A tube of rank $r=1$ is called *homogeneous*.

1.3. The *one-point extension* of the algebra A by an A-module X is the matrix algebra

$$A[X]=\begin{bmatrix}A & 0\\ X & K\end{bmatrix}$$

with the usual addition and multiplication of matrices. The quiver of $A[X]$ contains Q_A as a full subquiver and there is an additional (extension) point which is a source. The $A[X]$-modules are usually identified with the triples (V,M,φ), where V is a K-vector space, M an A-module and $\varphi: V\to\mathrm{Hom}_A(X,M)$ is a K-linear map. An $A[X]$-linear map $(V,M,\varphi)\to(V',M',\varphi')$ is then identified with a pair (f,g), where $f:V\to V'$ is K-linear, $g:M\to M'$ is A-linear and $\varphi' f=\mathrm{Hom}_A(X,g)\varphi$. One defines dually the *one-point coextension* $[X]A$ of A by X.

1.4. A coil is a translation quiver constructed inductively from a stable tube by a sequence of operations called admissible. Our first task is thus to define the latter. Throughout this section, let A be an algebra, and Γ be a standard component of Γ_A. For an indecomposable module X in Γ, called the *pivot*, the admissible operation to be applied to Γ depends on the shape of the support $\mathcal{S}(X)$ of $\mathrm{Hom}_A(X,-)|_\Gamma$.

(ad 1) Assume $\mathcal{S}(X)$ consists of an infinite sectional path starting at X:

$$X=X_0\to X_1\to X_2\to\cdots$$

Thus the component Γ may look as follows:

$$
\begin{tikzcd}[column sep=small]
X_0 \arrow[dr] & & \tau_A^- X_0 \arrow[dr] & & & \\
& X_1 \arrow[ur] \arrow[dr] & & \tau_A^- X_1 \arrow[dr] & & \\
& & X_2 \arrow[ur] \arrow[dr] & & \tau_A^- X_2 \arrow[dr] & \\
& & & X_3 \arrow[ur] \arrow[dr] & & \ddots \\
& & & & \ddots &
\end{tikzcd}
$$

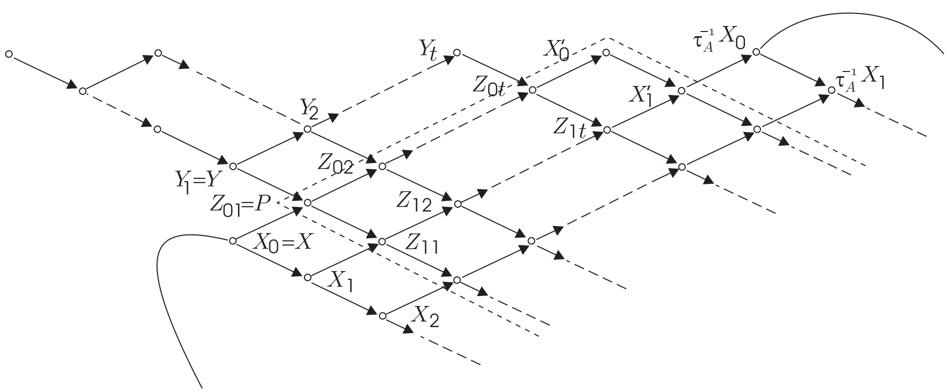
In this case, we let $t \geq 1$ be a positive integer,

$$
D = T_t(K) = \begin{bmatrix} K & 0 & \dots & 0 \\ K & K & \dots & 0 \\ \vdots & & \ddots & \vdots \\ K & \dots & \dots & K \end{bmatrix}
$$

the full $t \times t$ lower triangular matrix algebra, and $Y_1, \dots, Y_t$ the indecomposable injective D-modules with $Y = Y_1$ being the unique indecomposable projective-injective. We define the *modified algebra* A' of A to be the one-point extension

$$
A' = (A \times D)[X \oplus Y],
$$

and the *modified component* Γ' of Γ to be



where $Z_{ij} = (K, X_i \oplus Y_j, \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right))$ for $i \geq 0$ and $1 \leq j \leq t$, and $X_i' = (K, X_i, 1)$ for $i \geq 0$. The morphisms are defined in the obvious way. The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 1$, $j \geq 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1, \tau' Z_{0j} = Y_{j-1}$ if $j \geq 2, Z_{01} = P$ is projective, $\tau' X_0' = Y_t, \tau' X_i' = Z_{i-1,t}$ if $i \geq 1, \tau'(\tau^{-1} X_i) = X_i'$ provided X_i is not an injective A-module, otherwise X_i' is injective in Γ'. For the remaining vertices of Γ', the translation τ' coincides with the translation of Γ, or Γ_D, respectively.

If $t = 0$, we define the modified algebra A' to be the one-point extension $A' = A[X]$ and the modified component Γ' to be the component obtained from Γ by inserting only the sectional path consisting of the vertices X'_i, $i \geq 0$.

Intuitively, this operation amounts to "opening" the component Γ along the arrows $X_i \to \tau_A^{-1} X_{i-1}$, and then "glueing" Γ with Γ_D by inserting the infinite rectangle (indicated by the dashed lines in the figure above) consisting of the vertices Z_{ij} and X'_i. This rectangle is equal to the support $\mathcal{S}(P)$ in Γ' of the functor $\mathrm{Hom}_{A'}(P,-)\,|_{\Gamma'}$, where P is the new projective. We say that Γ' is obtained from Γ and Γ_D by *inserting the rectangle determined by P.*

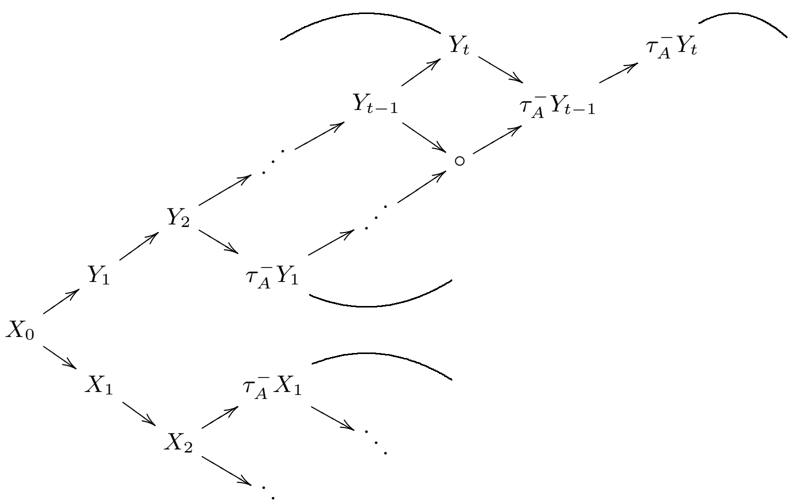
The non-negative integer t is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangle equals $t+1$. We call t the *parameter* of the operation.

In case Γ is a stable tube, it is clear that any module on the mouth of Γ satisfies the condition for being a pivot for the above operation. Actually, the above operation is, in this case, the tube insertion as considered in [14].

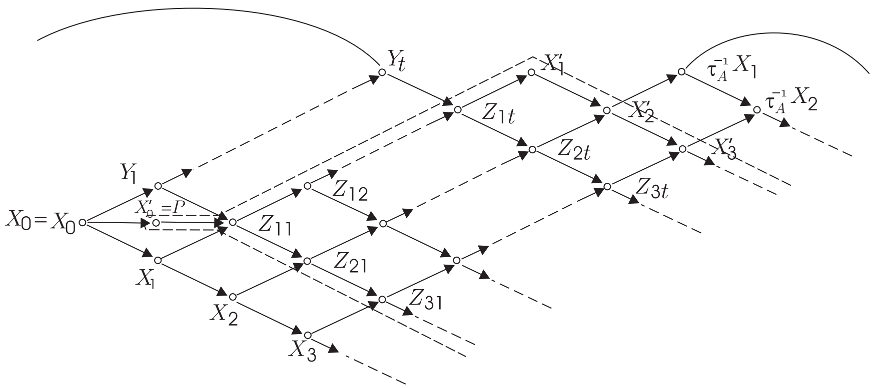
(ad 2) Assume $\mathcal{S}(X)$ to consist of two sectional paths starting at X, the first infinite and the second finite with at least one arrow:

$$Y_t \leftarrow \cdots \leftarrow Y_2 \leftarrow Y_1 \leftarrow X = X_0 \to X_1 \to X_2 \to \cdots$$

where $t \geq 1$. In particular, X is necessarily injective. The component Γ may then look as follows:



We define the *modified algebra* A' of A to be the one-point extension $A' = A[X]$ and the *modified component* Γ' of Γ to be



where $Z_{ij} = (K, X_i \oplus Y_j, \binom{1}{1})$ for $i \geq 1$ and $1 \leq j \leq t$, and $X'_i = (K, X_i, 1)$ for $i \geq 0$. The morphisms are the obvious ones. The translation τ' of Γ' is defined as follows: $P = X'_0$ is projective-injective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2, j \geq 2, \tau' Z_{i1} = X_{i-1}$ if $i \geq 1, \tau' Z_{1j} = Y_{j-1}$ if $j \geq 2, \tau' X'_i = Z_{i-1,t}$ if $i \geq 2, \tau' X'_1 = Y_t, \tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not an injective A-module, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', the translation τ' coincides with the translation τ of Γ.

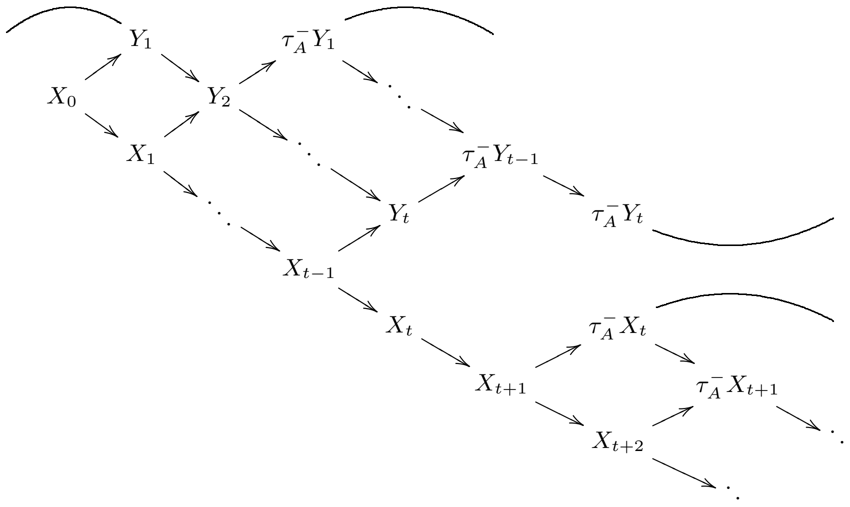
Intuitively, the above operation amounts to "opening" the component Γ along the arrows $X_i \to \tau_A^{-1} X_{i-1}$, "plugging in" a new projective-injective P and inserting the infinite rectangle (indicated by the dashed lines in the figure above) consisting of the vertices Z_{ij} and X'_i. On the other hand, those modules M such that there is a walk from M to $\tau_A^{-1} Y_{j-1}$ for some j, $2 < j \leq t$, not factoring through one of the arrows $Y_j \to \tau_A^{-1} Y_{j-1}$ are "removed" from the component. The inserted rectangle is equal to the support $\mathcal{S}(P)$ in Γ' of the functor $\mathrm{Hom}_{A'}(P, -)\,|_{\Gamma'}$, where P is the new projective-injective. We say that Γ' is obtained from Γ by *inserting the rectangle determined by* P.

The integer $t \geq 1$ is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangle equals $t+1$. We call t the *parameter* of the operation.

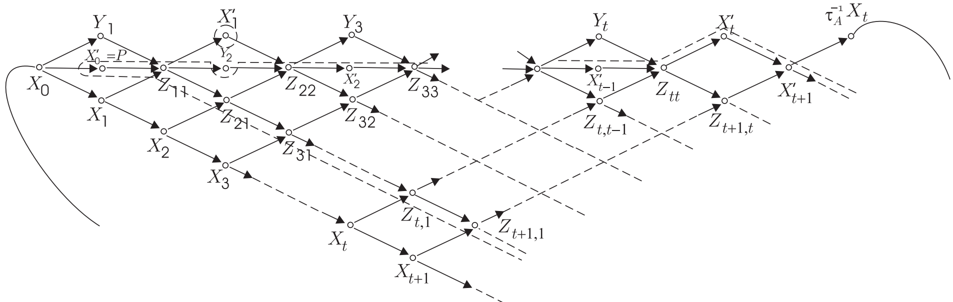
(ad 3) Assume $\mathcal{S}(X)$ to consist of two parallel sectional paths, the first infinite and starting at X, the second finite with at least one arrow

$$\begin{array}{ccccccccccc}
Y_1 & \to & Y_2 & \to & \cdots & \to & Y_t & & & & \\
\uparrow & & \uparrow & & & & \uparrow & & & & \\
X = X_0 & \to & X_1 & \to & \cdots & \to & X_{t-1} & \to & X_t & \to & \cdots
\end{array}$$

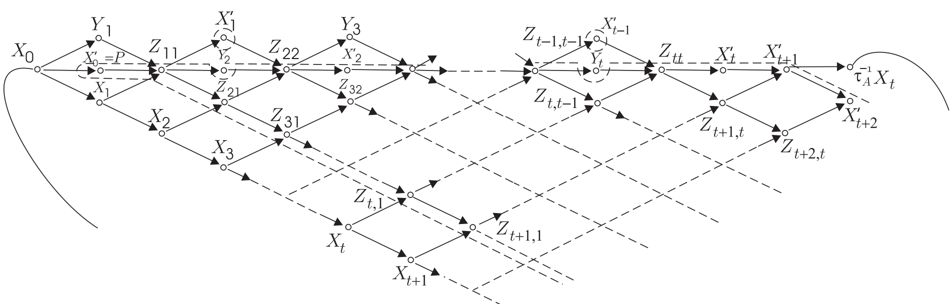
where $t \geq 2$. In particular, X_{t-1} is necessarily injective. The component Γ may then look as follows:



We define the *modified algebra* A' of A to be the one-point extension $A' = A[X]$ and the *modified component* Γ' of Γ to be



if t is odd, while



if t is even, where $Z_{ij} = (K, X_i \oplus Y_j, \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right))$ for $i \geq 1$, $1 \leq j \leq i$ and $j \leq t$, and $X_i' = (K, X_i, 1)$ for $i \geq 1$. The morphisms are the obvious ones. The translation τ' of Γ' is defined as follows: $P = X_0'$ is projective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2$, $2 \leq j \leq t$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau' X_i' = Y_i$ if $1 \leq i \leq t$, $\tau' X_i' = Z_{i-1,t}$ if $i \geq t+1, \tau' Y_j = X_{j-2}'$ if $2 \leq j \leq t$, $\tau'(\tau^{-1} X_i) = X_i'$ if

$i \geq t$ provided X_i is not an injective A-module, otherwise X'_i is injective in Γ'. In both cases, X'_{t-1} is injective. For the remaining vertices of Γ', the translation τ' coincides with the translation τ of Γ.

Intuitively, the above operation amounts to "opening" the component Γ along the arrows $X_i \to \tau_A^{-1} X_{i-1}$, "plugging in" a new projective P and inserting the infinite rectangle (indicated by the dashed lines in the figure above) consisting of the vertices Z_{ij} and X'_i. On the other hand, those modules M such that there is a walk from M to $\tau_A^{-1} Y_{j-1}$ for some j, $2 \leq j \leq t$, not factoring through one of the arrows $Y_j \to \tau_A^{-1} Y_{j-1}$ are "removed" from the component. The reason for the appearance of the two cases depending on the parity of t follows from easy combinatorial considerations involving the length functions [4, (4.4)]. The inserted rectangle is equal to the support $\mathcal{S}(P)$ in Γ' of the functor $\mathrm{Hom}_{A'}(P, -)\,|_{\Gamma'}$, where P is the new projective. We say that Γ' is obtained from Γ by *inserting the rectangle determined by P.*

The integer $t \geq 2$ is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \cdots$ in the inserted rectangle equals $t + 1$. We call t the *parameter* of the operation.

Finally, together with each (ad 1), (ad 2) or (ad 3) operation, we consider its dual, denoted by (ad 1*), (ad 2*) and (ad 3*), respectively. These six operations are called the *admissible operations.*

Clearly, the admissible operations can be defined as operations on translation quivers rather than on Auslander–Reiten components. The definitions are obvious (see [3] or [30] for the details).

Definition. A translation quiver Γ is called a *coil* if there exists a sequence of translation quivers $\Gamma_0, \Gamma_1, \ldots, \Gamma_m = \Gamma$ such that Γ_0 is a stable tube and, for each i $(0 \leq i < m)$, Γ_{i+1} is obtained from Γ_i by an admissible operation.

Observe that the present notion of coil is clearly a natural generalization of the notion of coherent tube: indeed, any stable tube is (trivially) a coil, and a tube can be characterized to be a coil such that each admissible operation in the sequence defining it is of type (ad 1) or (ad 1*). Also, a coil without injectives (or without projectives) is a tube. A *quasi-tube* (in the sense of [27]) is a coil having the property that each admissible operation in the sequence defining it is of type (ad 1), (ad 2), (ad 1*) or (ad 2*).

It follows from the definition that coils share many properties with tubes. For instance, all but finitely many vertices in a coil belong to a cyclical path. A vertex x in a coil Γ is said to belong to the *mouth* of Γ if x is the starting, or ending, vertex of a mesh in Γ with a unique middle term. Also, Γ contains a (maximal) tube as a cofinite full translation subquiver. Arrows of this tube point either to the mouth or to infinity. An infinite sectional path in Γ

$$x = x_1 \xrightarrow{\alpha_1} x_2 \xrightarrow{\alpha_2} \cdots \to x_i \xrightarrow{\alpha_i} x_{i+1} \to \cdots$$

is called a *ray* if there exists $i_0 \geq 1$ such that, for all $i \geq i_0$, the arrow α_i points to infinity. *Corays* are defined dually. Thus the parameter of an (ad 1), (ad 2) or (ad 3) operation (or (ad 1^*), (ad 2^*) or (ad 3^*)) is used to measure the number of rays (or corays, respectively) inserted in the coil by the operation.

2. Coil enlargements of sincere standard stable tubes

2.1. Throughout the paper, C denotes an algebra with a fixed standard stable tube $\mathcal{T} = \mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ of Γ_C containing an indecomposable sincere module. It is well known that then all indecomposable modules in $\mathcal{T}$ whose distance to the mouth is at least r are sincere, and hence all but finitely many modules in $\mathcal{T}$ are sincere (see [24, (3.1)]). We note that the class of such algebras C is wide and contains all concealed canonical algebras (hence all tame concealed and tubular algebras) [16], [17], [32], or more generally, all generalized canonical algebras [36].

Following [7], an algebra B is said to be a *coil enlargement of C using modules from $\mathcal{T}$* if there is a sequence of algebras $C = C_0, C_1, \ldots, C_m = B$ such that, for each $0 \leq j < m$, C_{j+1} is obtained from C_j by an admissible operation with the pivot either in the stable tube $\Gamma_0 = \mathcal{T}$ (if $j = 0$) or in the coil Γ_j of Γ_{C_j} obtained from Γ_0 by means of the sequence of admissible operations done so far. The sequence $C = C_0, C_1, \ldots, C_m = B$ is then called an *admissible sequence*. In this process we get a standard coil $\mathcal{C} = \Gamma_m$ in Γ_B (see [3], [4], [5] or [7]). We note that in general $\mathcal{C}$ does not contain sincere indecomposable modules.

The aim of this paper is to describe the structure of all indecomposable modules lying in the coil $\mathcal{C}$ of a coil enlargement B of C using modules from $\mathcal{T}$. The indecomposable modules in $\mathcal{C}$ may be divided into two disjoint classes: the indecomposable modules whose restriction to C is zero, and the remaining ones. The supports of indecomposable modules of the first class are representation-finite simply connected algebras [10], and consequently, those indecomposable modules are known (see [8], [11], [12], [25]). Therefore, it remains to describe the indecomposable modules in $\mathcal{C}$ having non-zero restriction to C.

2.2. Definition. A module M in $\mathcal{C}$ is called *weakly sincere* if M has non-zero restriction to C and $\operatorname{Supp} M$ contains any object of B which is not in C.

The following theorem will play a crucial role in our investigation of coils which admit weakly sincere indecomposable modules.

2.3. THEOREM. *Let B be a coil enlargement of C using modules from $\mathcal{T}$ and $\mathcal{C}$ be the standard coil of Γ_B obtained by the corresponding sequence of admissible operations. Then*:

(i) *There is a unique maximal branch coextension B^- of B which is a* (*full*) *convex subcategory of B, and B can be obtained from B^- by a sequence of admissible operations of types* (ad 1), (ad 2), (ad 3).

(ii) *There is a unique maximal branch extension B^+ of B which is a* (*full*) *convex subcategory of B, and B can be obtained from B^+ by a sequence of admissible operations of types* (ad 1*), (ad 2*), (ad 3*).

Proof. (i) and (ii) are direct consequences of the proof of [7, Theorem 3.5]. ∎

2.4. Let C be an algebra and Γ_0 be a standard stable tube of Γ_C. If the pair (A, Γ), formed by an algebra A and a component Γ of Γ_A, is obtained from the pair (C, Γ_0) by a sequence of admissible operations, then the corresponding sequence of modified algebras and components

$$(C, \Gamma_0) = (C_0, \Gamma_0), (C_1, \Gamma_1), \ldots, (C_n, \Gamma_n) = (A, \Gamma)$$

is called a *defining sequence* for (A, Γ).

A defining sequence $(C, \Gamma_0) = (C_0, \Gamma_0), (C_1, \Gamma_1), \ldots, (C_n, \Gamma_n) = (A, \Gamma)$ is called *reduced* if, for any i $(1 \le i \le n)$, whenever (C_i, Γ_i) is obtained from (C_{i-1}, Γ_{i-1}) by an operation of type (ad 2), (ad 2*), (ad 3) or (ad 3*), then there is no defining sequence $(C, \Gamma_0) = (B_0, \Delta_0), (B_1, \Delta_1), \ldots, (B_t, \Delta_t) = (C_i, \Gamma_i)$ such that (C_i, Γ_i) is obtained from (B_{t-1}, Δ_{t-1}) by an operation of type (ad 1) or (ad 1*). In this case, the corresponding sequence of admissible operations is also called *reduced*.

In other words, a sequence of admissible operations transforming (C, Γ_0) into (A, Γ) is reduced if the operations that appear before each (ad 2), (ad 3), (ad 2*) or (ad 3*) operation in the sequence are only those that do not commute with such an operation (see [7, (3.1)]).

We shall keep the notation introduced above throughout the rest of the paper.

2.5. In this section we will be concerned with a more general situation. The results obtained here will be quite useful in the rest of our work.

Let C be an algebra and Γ_0 be a standard coil in Γ_C. Assume that (A, Γ) is obtained from (C, Γ_0) by a sequence of n (ad 1) operations. Then we have the following definitions.

The projectives inserted in Γ_0 by the above operations are called *consecutive* if:

(i) each operation has parameter zero,

(ii) for $1 \leq i < n$, the pivot of the $(i+1)$th operation is the unique projective inserted by the ith operation.

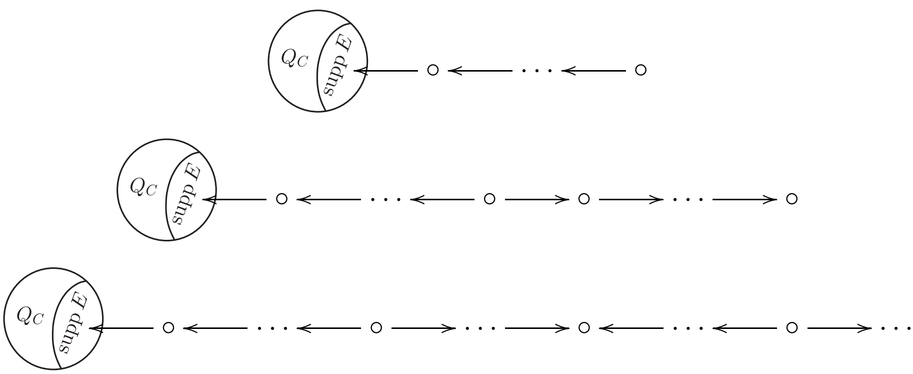
The projectives inserted in Γ_0 by the above operations are called *line-coline consecutive* if:

(i) each operation but the last has parameter zero,

(ii) for $1 \leq i < n$, the pivot of the $(i+1)$th operation is the unique projective inserted by the ith operation.

The projectives inserted in Γ_0 by the above operations are said to be *in line* or *aligned* if, for $1 \leq i < n$, the pivot of the $(i+1)$th operation is either the simple projective inserted by the ith operation, or an $(\mathrm{ad}\,1)$-pivot on the ray starting at the unique projective inserted by the ith operation.

Note that if the unique projective P inserted by the ith operation is not injective, then P is the only $(\mathrm{ad}\,1)$-pivot on the ray starting at P, and thus the pivot of the $(i+1)$th operation. If P is injective, then there are at least two $(\mathrm{ad}\,1)$-pivots on the ray starting at P, and thus the pivot of the $(i+1)$th operation may not be P. If we assume that the pivot of the sequence of $(\mathrm{ad}\,1)$ operations that transforms (C, Γ_0) into (A, Γ) is the indecomposable C-module E, then the bound quiver of A has, in each of the three situations described above, the following form:



all bound by the relations of $C[E]$ and, in the last case, possibly by additional zero relations from points on the line to points in $\operatorname{supp} E$.

Lemma. *Let C be an algebra and Γ_0 be a standard coil in Γ_C. Assume that (A, Γ) is obtained from (C, Γ_0) by a sequence of n $(\mathrm{ad}\,1)$ operations with pivot E, and that there is a sincere A-module U in Γ. Then the projectives inserted by these operations in Γ_0 are in line, and $U|_C$ is a sincere module in Γ_0 lying on the ray that starts at E.*

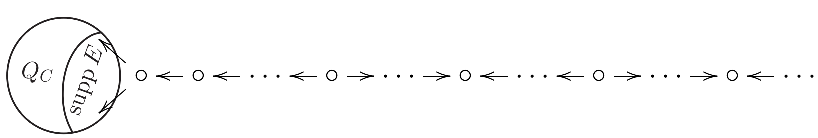
Proof. By induction on n. Let $(C, \Gamma_0) = (A_0, \Gamma_0), (A_1, \Gamma_1), \ldots, (A_n, \Gamma_n) = (A, \Gamma)$ be the corresponding sequence of modified algebras and components. Since U is sincere and Γ is standard, U lies on the ray containing all the projectives inserted in Γ_{n-1} by the nth operation, and $U' = U|_{A_{n-1}}$ is a sincere module in Γ_{n-1} lying on the ray that starts at the pivot X of the nth operation. Again, since U' is sincere and Γ_{n-1} is standard, U' lies on the ray starting at P, where P is either the simple projective or the unique projective inserted in Γ_{n-2} by the $(n-1)$th operation. Hence the ray $[X, \infty)$ is contained in the maximal ray $[P, \infty)$. If P is the simple projective inserted by the $(n-1)$th operation, then the unique $(\mathrm{ad}\,1)$-pivot on the ray $[P, \infty)$ is P, and thus $X \cong P$. The same is true if P is the unique projective inserted by the $(n-1)$th operation and P is not injective. If P is injective, then X is any $(\mathrm{ad}\,1)$-pivot on the ray $[P, \infty)$. The proof is completed by applying the induction hypothesis to (A_{n-1}, Γ_{n-1}). Note that if P is injective, then also $\mathrm{rad}\,P$ is injective. ∎

2.6. Corollary. *Let C be an algebra and Γ_0 be a standard coil in Γ_C. Let (A, Γ) be obtained from (C, Γ_0) by a sequence of* $(\mathrm{ad}\,1)$ *operations with pivot E such that the projectives inserted in Γ_0 by those operations are in line. Then an A-module U is a sincere indecomposable in Γ if and only if $U|_C$ is a sincere indecomposable in Γ_0 lying on the ray that starts at E and $U(x) = K$ for every vertex $x \in Q_A \setminus Q_C$.*

Proof. Let U be a sincere A-module in Γ. From (2.5) it follows that $U|_C$ is a sincere indecomposable in Γ_0 lying on the ray that starts at E, and $\dim_K \mathrm{Hom}_A(P(x), U) = 1$ for every vertex $x \in Q_A \setminus Q_C$.

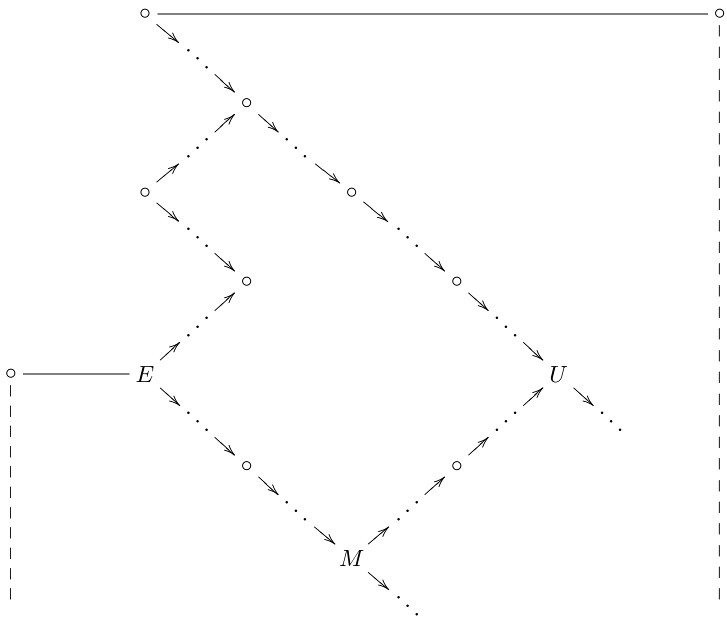
Conversely, if U is as stated above, then it follows from the description of the modified component after applying an $(\mathrm{ad}\,1)$ operation that U is a sincere module in Γ lying on the ray that contains all the projectives inserted by the last operation. ∎

2.7. Returning to our original problem, we first assume that Γ contains projectives or injectives, but not both. By duality, we may assume that Γ contains only projectives. Then Γ is obtained from Γ_0 by a sequence of $(\mathrm{ad}\,1)$'s, for the remaining operations give rise to injectives in Γ. Thus Γ is a standard non-stable tube and A is a branch extension of C. From (2.5) we obtain a better description of A and Γ. Let E in Γ_0 be the pivot of C-module which is the pivot of the sequence of $(\mathrm{ad}\,1)$'s that transforms (C, Γ_0) into (A, Γ). Then the bound quiver of A has the following form:



bound by the relations of $C[E]$.

The sincere non-stable tube Γ has the following shape:



The following result is a direct consequence of (2.6).

THEOREM. *Let* (A, Γ) *be obtained from* (C, Γ_0) *by a sequence of* $(\mathrm{ad}\,1)$ *operations with pivot* E *such that the projectives inserted in* Γ_0 *by those operations are in line. Then an* A*-module* U *is a weakly sincere indecomposable module in* Γ *if and only if there exists an indecomposable* C*-module* M *in* Γ_0 *lying on the ray starting at* E *such that* U *is isomorphic to*

$$M \rightleftarrows K \leftarrow K \leftarrow \cdots \leftarrow K \rightarrow \cdots \rightarrow K \leftarrow \cdots \leftarrow K \rightarrow \cdots \rightarrow K \leftarrow \cdots$$

Moreover, U *is sincere if and only if* M *is sincere.*

2.8. We now assume that Γ contains projectives and injectives. We first consider the case where (A, Γ) is obtained from (C, Γ_0) by a sequence of $(\mathrm{ad}\,1)$ and $(\mathrm{ad}\,1^*)$ operations. By [7, (3.1), (3.2)], we can replace this sequence by another one in which all the $(\mathrm{ad}\,1)$'s appear first, leading to (A^+, Γ^+) (see [7, (3.5)]), and then all the $(\mathrm{ad}\,1^*)$'s follow.

LEMMA. *Assume that* (A, Γ) *is obtained from* (C, Γ_0) *by a sequence of admissible operations consisting of a block of* $(\mathrm{ad}\,1)$*'s followed by a block of* $(\mathrm{ad}\,1^*)$*'s, and that there is a sincere* A*-module* U *in* Γ. *Let* (A^+, Γ^+) *denote the term of the defining sequence of* (A, Γ) *obtained by the block of* $(\mathrm{ad}\,1)$*'s. Then*:

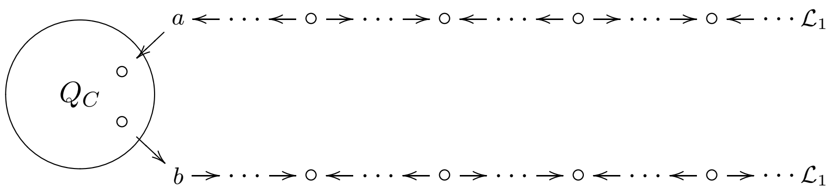
(i) *The projectives inserted in* Γ_0 *by the block of* $(\mathrm{ad}\,1)$*'s are in line.*
(ii) *The injectives inserted in* Γ^+ *by the block of* $(\mathrm{ad}\,1^*)$*'s are in line.*

Proof. Since Γ^+ is a standard coil in Γ_{A^+}, (ii) follows from the dual of (2.5). By the same result, $U|_{A^+}$ is a sincere indecomposable in Γ^+. Hence (i) follows from (2.5). ∎

Let E in Γ_0 be the pivot of the block of (ad 1)'s, and E' in Γ^+ be the pivot of the block of (ad 1*)'s. There are three possible situations:

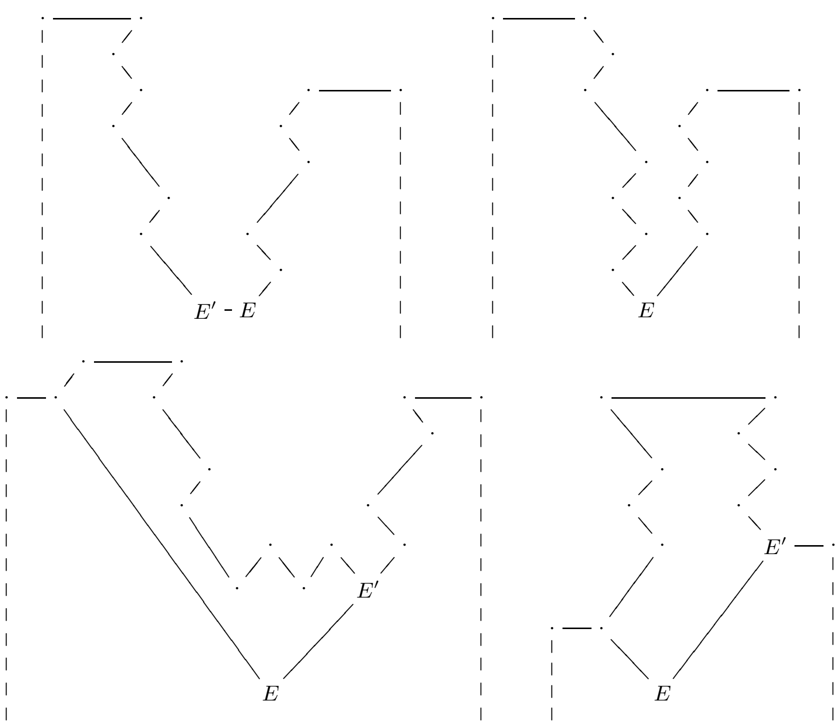
1. E and E' are both C-modules and $E \not\cong E'$.
2. E and E' are both C-modules and $E \cong E'$.
3. E' is any (ad 1*)-pivot (not isomorphic to E) on the sectional path from E to the mouth of Γ^+.

The bound quiver of A then has the following form:



bound by the relations of $[E'](C[E])$, an additional zero relation from a to b in case 2, and possibly additional zero relations from points on $\mathcal{L}_1$ to points on $\mathcal{L}_2$ in case 3.

The sincere tube Γ in Γ_A has one of the following shapes:



THEOREM. *Let (A,Γ) be obtained from (C,Γ_0) by a sequence of admissible operations consisting of a block of* $(\mathrm{ad}\,1)$ *operations with pivot E, followed by a block of* $(\mathrm{ad}\,1^*)$ *operations with pivot E', such that the projectives* (*respectively*, *injectives*) *inserted by the first* (*respectively*, *the last*) *block are in line. Then an A-module U is a weakly sincere indecomposable in Γ if and only if there exists an indecomposable C-module M in Γ_0 lying on the intersection of the ray that starts at E and the coray that ends at $E'|_C$ such that U is isomorphic to*

$$
\begin{tikzcd}[row sep=small, column sep=small]
 & K & K \arrow[l] & \cdots \arrow[l] & K \arrow[l] \arrow[r] & \cdots \arrow[r] & K & \cdots \arrow[l] & K \arrow[l] \arrow[r] & \cdots \arrow[r] & K & \cdots \arrow[l] \\
M \arrow[ur, leftarrow] \arrow[dr] & & & & & & & & & & & \\
 & K \arrow[r] & K \arrow[r] & \cdots \arrow[r] & K & \cdots \arrow[l] & K \arrow[l] \arrow[r] & \cdots \arrow[r] & K & \cdots \arrow[l] & K \arrow[l] \arrow[r] & \cdots
\end{tikzcd}
$$

Moreover, *U is sincere if and only if M is sincere.*

Proof. Let (A^+,Γ^+) be as in the preceding lemma. Since the coray that ends at $E'|_C$ in Γ_0 is the coray through E' in Γ^+, the proof follows from (2.6) and its dual. ■

3. Coils with weakly sincere indecomposable modules

3.1. The aim of this section is to describe all coil enlargements B of the algebra C, using modules from the standard stable tube $\mathcal{T}$, which admit a (weakly) sincere indecomposable module lying in the standard coil $\mathcal{C}$ obtained from $\mathcal{T}$ by the corresponding coil enlargement. We shall show that any such algebra B can be obtained from C by adding a suitable braid of two linear quivers.

In order to formulate our main result, we define some families of bound quiver algebras KQ/I, and then introduce the notion of a braid algebra.

In the bound quivers (Q,I) listed below we use the following notation:

(i) The unoriented edge $\circ \!\!-\!\!-\!\!-\!\! \circ$ means $\circ \longrightarrow \circ$ or $\circ \longleftarrow \circ$.

(ii)
$$
\begin{tikzcd}[row sep=small, column sep=small]
 & \circ \arrow[r, "\alpha_2"] & \circ \arrow[r] & \cdots \arrow[r] & \circ \arrow[r, "\alpha_{r-1}"] & \circ \arrow[dr, "\alpha_r"] & \\
\circ \arrow[ur, "\alpha_1"] \arrow[dr, "\beta_1"'] \arrow[rrrrrr, dash, dashed] & & & & & & \circ \\
 & \circ \arrow[r, "\beta_2"'] & \circ \arrow[r] & \cdots \arrow[r] & \circ \arrow[r, "\beta_{s-1}"'] & \circ \arrow[ur, "\beta_s"'] &
\end{tikzcd}
\qquad r,s \geq 2
$$

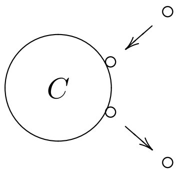
means that $\alpha_r \ldots \alpha_2\alpha_1 - \beta_s \ldots \beta_2\beta_1 \in I$ but $\alpha_r \ldots \alpha_2\alpha_1 \notin I$ and $\beta_s \ldots \beta_2\beta_1 \notin I$.

(iii)
$$
\begin{tikzcd}
\circ \arrow[r, "\alpha_1"] \arrow[rrrrr, dash, dashed, bend right] & \circ \arrow[r, "\alpha_2"] & \circ \arrow[r] & \cdots \arrow[r] & \circ \arrow[r, "\alpha_r"] & \circ
\end{tikzcd}
\qquad r \geq 2
$$

means that $\alpha_r \ldots \alpha_2\alpha_1 \in I$ but $\alpha_r \ldots \alpha_3\alpha_2 \notin I$, $\alpha_{r-1} \ldots \alpha_2\alpha_1 \notin I$.

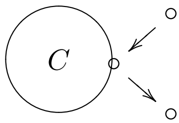
(iv) $\star \xrightarrow{\alpha_1} \circ \xrightarrow{\alpha_2} \circ \longrightarrow \ldots \longrightarrow \circ \xrightarrow{\alpha_r} \bullet \xrightarrow{\beta_1} \circ \xrightarrow{\beta_2} \circ \longrightarrow \ldots \longrightarrow \circ \xrightarrow{\beta_s} \star$, $r,s \geq 1$, means that possibly there are sequences $1 \le i_1 < \ldots < i_t \le r$ and $1 \le j_1 < \ldots < j_t \le s$ such that the paths $\beta_{j_l}\beta_{j_l-1}\ldots\beta_1\alpha_r\ldots\alpha_{i_l+1}\alpha_{i_l}$, for $1 \le l \le t$, belong to I.

(v)



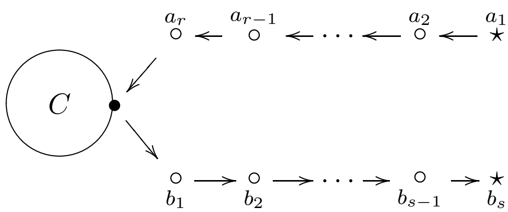
means the extension-coextension algebra $[N](C[M]) = ([N]C)[M]$ given by two different modules M and N lying on the mouth of $\mathcal{T}$.

(vi)



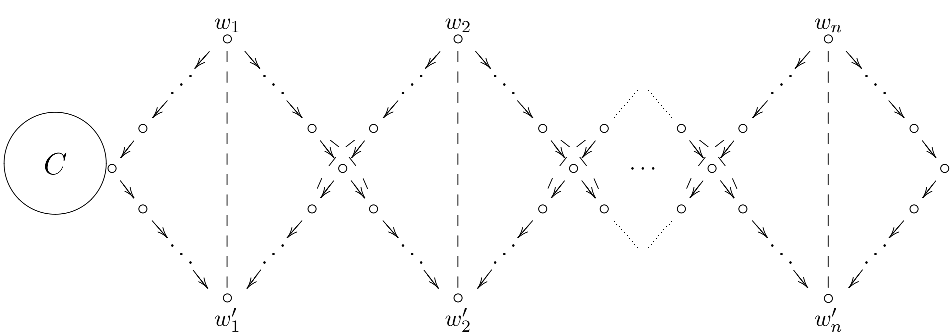
means the extension-coextension algebra $[M'](C[M])$ where M is a module lying on the mouth of $\mathcal{T}$ and $M' = (K, M, \mathrm{id})$ is the new indecomposable projective module of the one-point extension $C[M]$ of C by M.

(vii)



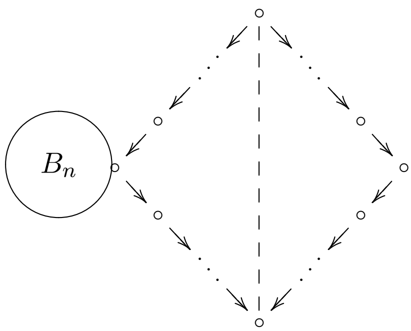
means that the restrictions of $P(a_1), \ldots, P(a_r)$ (respectively, $I(b_1), \ldots, I(b_s)$) to C are equal to a module M lying on the mouth of $\mathcal{T}$, and possibly there are sequences $1 \le i_1 < \ldots < i_t \le r$ and $1 \le j_1 < \ldots < j_t \le s$ such that the paths from a_{i_l} to b_{j_l}, for all $1 \le l \le t$, belong to I.

By a *regular braid algebra* of degree $n \ge 0$ we mean an algebra B_n of the form



where $B_0 = C$.

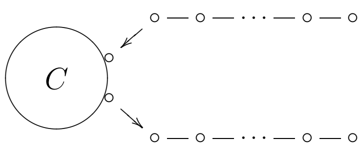
(viii) For a regular braid algebra B_n, $n \geq 0$,



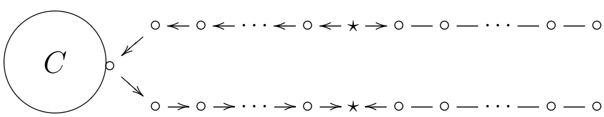
means a regular braid algebra of degree $n+1$.

3.2. Consider now the following families of bound quiver algebras KQ/I:

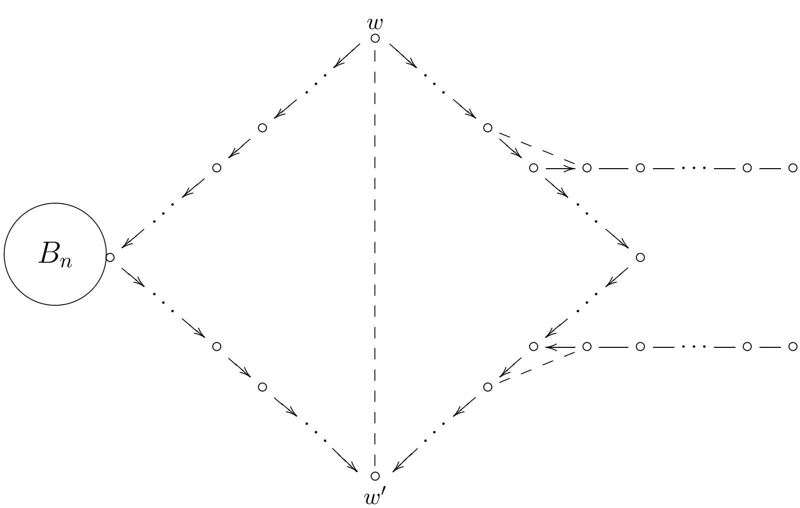
(1)



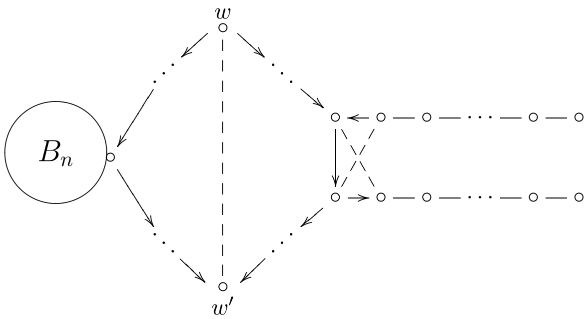
(2)



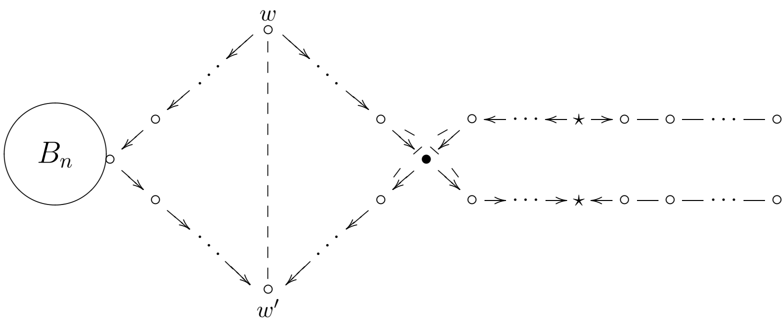
(3)



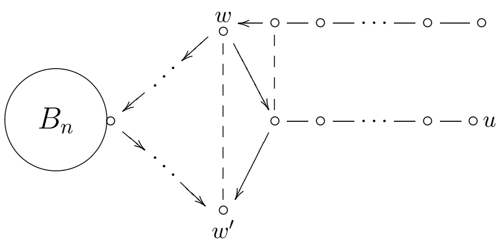
(4)



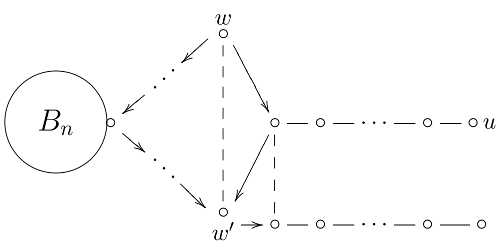
(5)



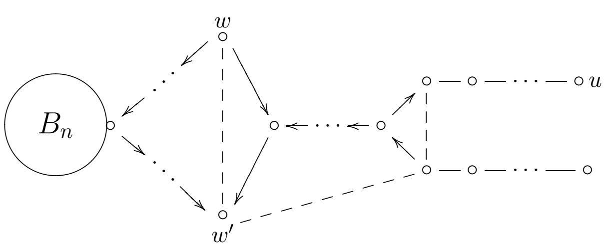
(6)



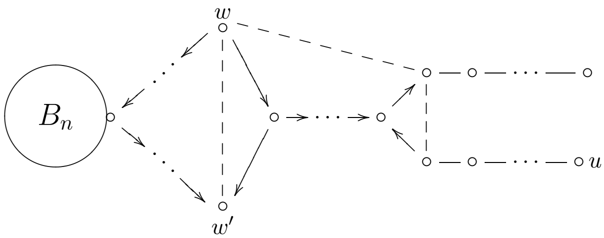
(7)



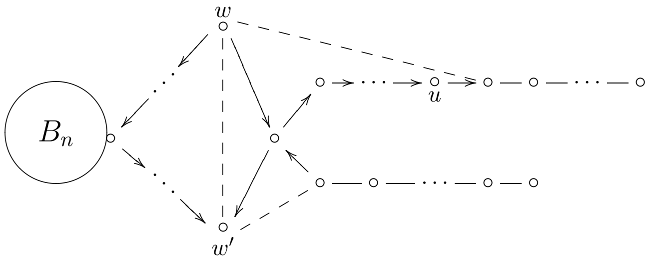
(8)



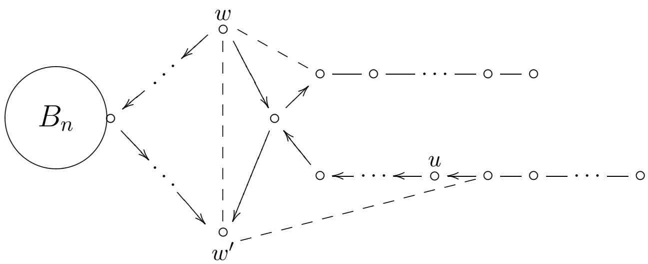
(9)



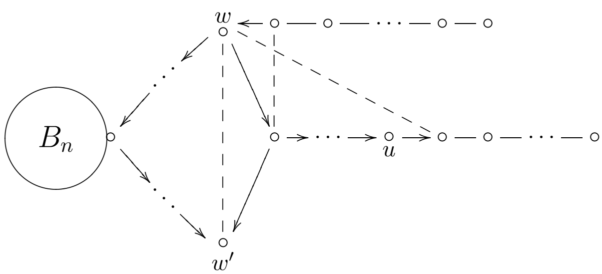
(10)



(11)



(12)



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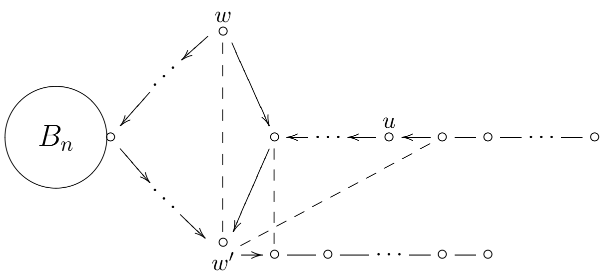

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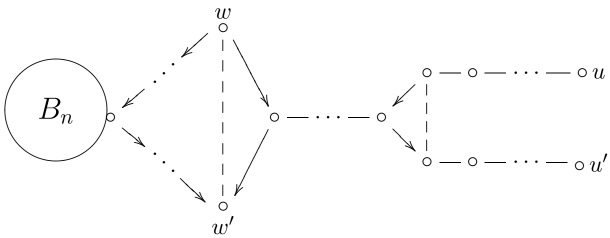

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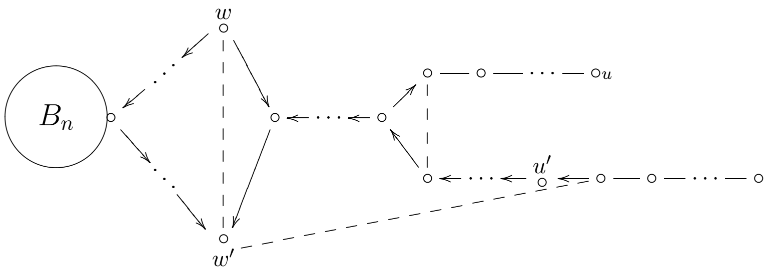

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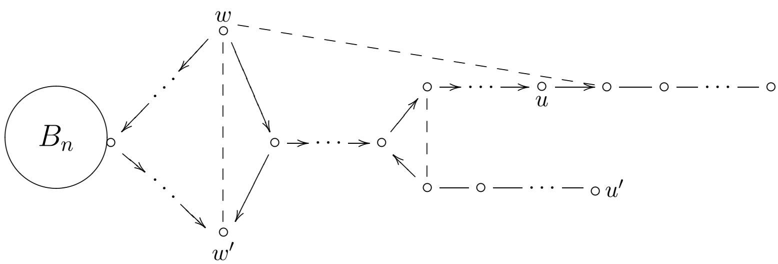

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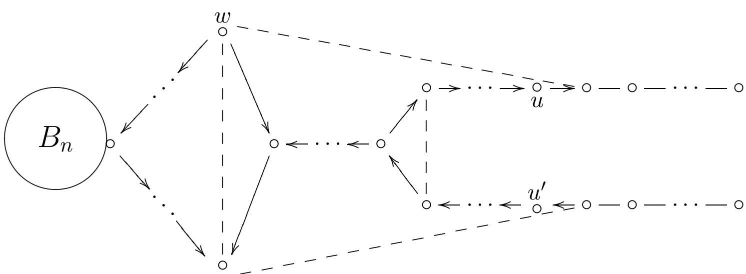

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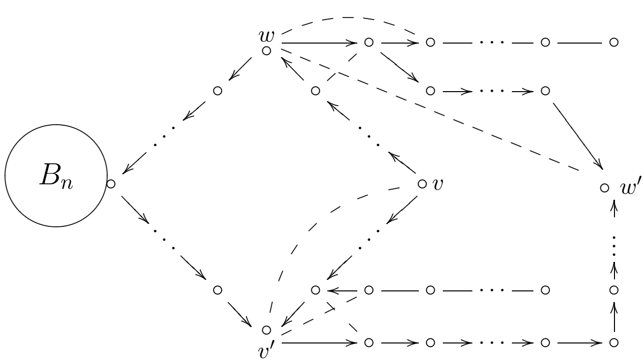

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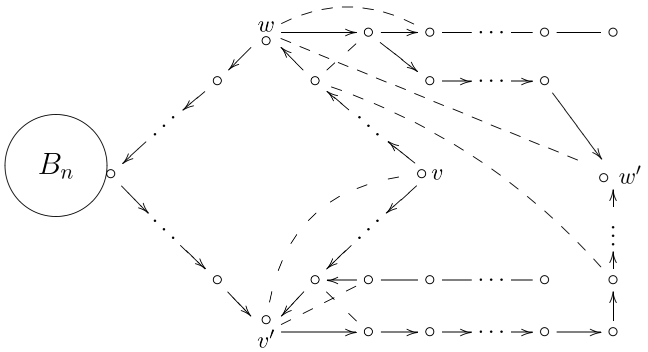
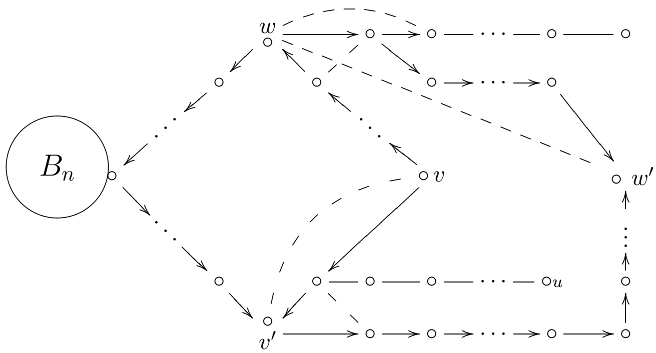


(20)



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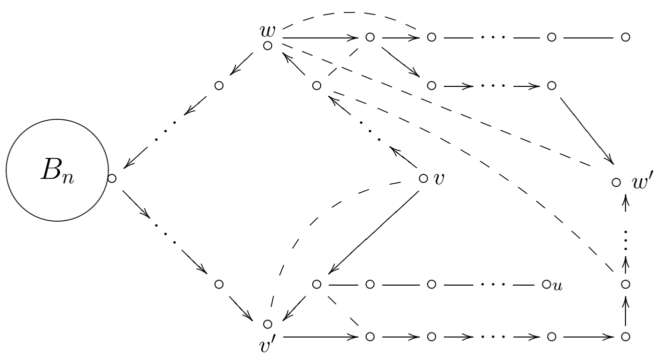

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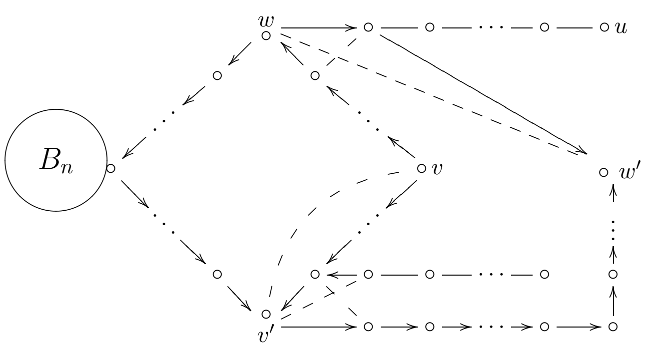
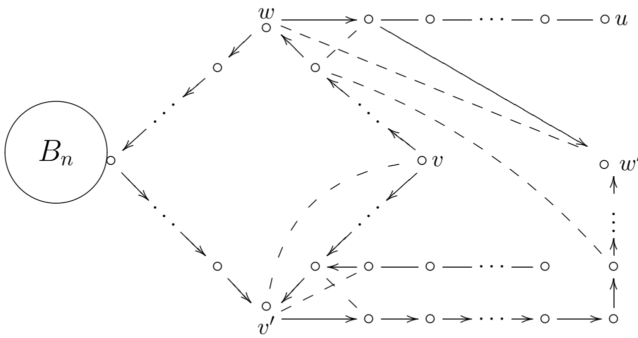


(23)



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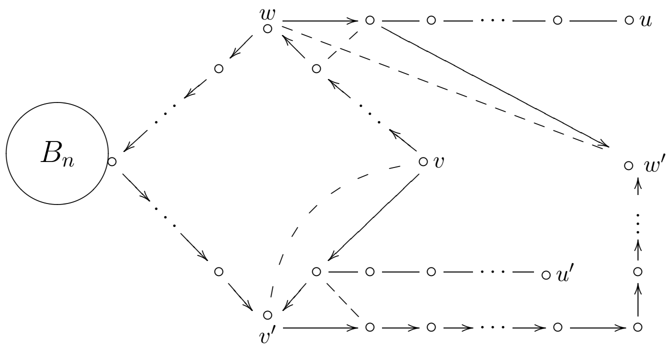

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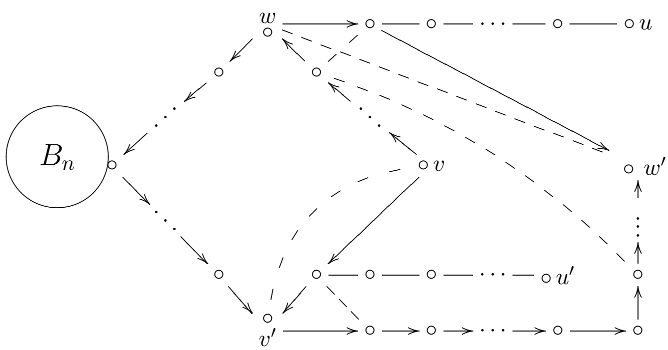


By a *braid algebra on C* we mean a connected convex subcategory of one of the bound quiver algebras (1)–(25) containing the algebra C and the vertices w, w', u, u', v and v' (whenever they exist).

Now we are able to state our main result.

3.3. THEOREM. *Let C be an algebra, $\mathcal{T}$ a sincere standard stable tube of Γ_C, B a coil enlargement of C using modules from $\mathcal{T}$, and $\mathcal{C}$ the standard coil of Γ_B obtained from $\mathcal{T}$ by the corresponding sequence of admissible operations. Then the following conditions are equivalent*:

(i) *B is a braid algebra on C.*

(ii) *$\mathcal{C}$ admits a weakly sincere indecomposable B-module.*

(iii) *$\mathcal{C}$ admits a sincere indecomposable B-module.*

3.4. In this section we will consider the case where (A,Γ) is obtained from (C,Γ_0) by a sequence of $(\mathrm{ad}\,1)$, $(\mathrm{ad}\,1^*)$, $(\mathrm{ad}\,2)$ and $(\mathrm{ad}\,2^*)$ operations. By [7, (3.3)], we may assume that the sequence consists only of $(\mathrm{ad}\,1)$, $(\mathrm{ad}\,1^*)$ and $(\mathrm{ad}\,2)$ operations. In fact, we assume that there are $m\geq 2$ $(\mathrm{ad}\,2)$'s in the sequence, and, for $1\leq i\leq m$, we denote by (B_i,Γ_i) the term in the defining sequence of (A,Γ) corresponding to the ith $(\mathrm{ad}\,2)$, by (B_i',Γ_i') the term preceding the latter, and by P_i the projective-injective inserted in Γ_i' by the ith $(\mathrm{ad}\,2)$. We will also assume that the defining sequence of (A,Γ) is reduced.

Two situations are possible:

(a) There is no $(\mathrm{ad}\,1^*)$ between the $(m-1)$th and mth $(\mathrm{ad}\,2)$'s, that is, (B_m,Γ_m) is obtained from (B_{m-1},Γ_{m-1}) by a sequence of $(\mathrm{ad}\,1)$'s followed by an $(\mathrm{ad}\,2)$.

(b) There is at least one $(\mathrm{ad}\,1^*)$ between the $(m-1)$th and mth $(\mathrm{ad}\,2)$'s, that is, (B_m,Γ_m) is obtained from (B_{m-1},Γ_{m-1}) by a sequence of $(\mathrm{ad}\,1)$ and $(\mathrm{ad}\,1^*)$ operations followed by an $(\mathrm{ad}\,2)$.

We use the following notation throughout our analysis of case (a).

Let $X=\mathrm{rad}\,P_m$ and let $\mathrm{supp}\,\mathrm{Hom}_{\Gamma_m'}(X,-)$ be

$$Y_t\leftarrow\cdots\leftarrow Y_2\leftarrow Y_1\leftarrow X=X_0\to X_1\to X_2\to\cdots$$

where $t\geq 1$ with $X,Y_1,\dots,Y_t$ injectives.

Let $X'=\mathrm{rad}\,P_{m-1}$ and let $\mathrm{supp}\,\mathrm{Hom}_{\Gamma_{m-1}'}(X',-)$ be

$$Y_r'\leftarrow\cdots\leftarrow Y_2'\leftarrow Y_1'\leftarrow X'=X_0'\to X_1'\to X_2'\to\cdots$$

where $r\geq 1$ with $X',Y_1',\dots,Y_r'$ injectives.

Since the defining sequence of (A,Γ) is reduced, and each $(\mathrm{ad}\,1)$ operation consists in inserting one or several rays, it follows that the rectangle $\mathrm{supp}\,\mathrm{Hom}_{\Gamma_{m-1}}(P_{m-1},-)$ determined by P_{m-1} in Γ_{m-1} is enlarged to the rectangle $\mathrm{supp}\,\mathrm{Hom}_{\Gamma_m'}(P_{m-1},-)$, and X lies on the ray forming the upper border of the latter. In other words, there is a sectional path from P_{m-1} to the mouth of Γ_m', which ends in an $(\mathrm{ad}\,1)$-pivot R and X belongs to the ray starting at R.

We will assume moreover that Γ contains a weakly sincere indecomposable A-module.

LEMMA. *With the above setting, let $(A,\Gamma)=(B_m,\Gamma_m)$. Then:*

(i) *The projectives inserted by those* $(\mathrm{ad}\,1)$ *operations that transform $(B_{m-1},\,\Gamma_{m-1})$ into (B_m',Γ_m') are consecutive.*

(ii) *The pivot of the first of these operations is the indecomposable B_{m-1}-module $\overline{X_1'}=(K,X_1',1)$.*

Proof. Let W be a weakly sincere indecomposable in Γ. Then W belongs to the rectangle $\operatorname{supp}\operatorname{Hom}_\Gamma(P_m,-)$ and receives non-zero morphisms from all the projectives inserted by the (ad 1)'s mentioned above. It follows from the standardness of Γ and the description of the modified component after applying an (ad 1) that these projectives must lie on the coray ending in R. This proves (i) and (ii). ■

3.5. LEMMA. *With the above setting, let* $(A,\Gamma)=(B_m,\Gamma_m)$ *and let* k *be the least positive integer such that for* $k\le i<m$, *there is no* (ad 1) *operation between the* i*th and* $(i+1)$*th* (ad 2) *operations. Then* $k=m-1$.

Proof. Let $m>2$, for otherwise there is nothing to prove. Assume that $k<m-1$, and let W be a weakly sincere indecomposable in Γ. Then W belongs to the rectangle $\operatorname{supp}\operatorname{Hom}_\Gamma(P_m,-)$. If $W=\overline{X_i'}=(K,X_i',1)$ for some $i\ge 0$, then X_i is a weakly sincere indecomposable in Γ_m'. But there is no weakly sincere module in Γ_m' since such a module should belong to $\bigcap_{i=k}^{m-1}\operatorname{supp}\operatorname{Hom}_{\Gamma_m'}(-,P_i)=\operatorname{supp}\operatorname{Hom}_{\Gamma_m'}(-,P_k)$ and to the coray through $X,Y_1,\ldots,Y_t$, and the intersection of these sets is empty. Indeed, otherwise, it is cofinite. But on the coray through X there are an infinite number of indecomposables which are modules over B_k' and do not send non-zero morphisms to $\operatorname{rad}P_k$. Therefore $W=Z_{ij}=(K,X_i\oplus Y_j,(\begin{smallmatrix}1\\1\end{smallmatrix}))$ for some $i\ge 1$ and $1\le j\le t$. Then $X_i\oplus Y_j$ is a weakly sincere B_m'-module. By the previous lemma and the description of the modified component after applying an (ad 2), $X_i|_{B_{m-1}'}=X_l'$ for some $l>i$, and $X_l'\oplus Y_j$ is a weakly sincere B_{m-1}'-module. But $Y_j(x)\ne 0$ only at the vertices $x\in Q_{B_{m-1}'}$ corresponding to the injectives $Y_j,Y_{j+1},\ldots,Y_t$. Hence $X_l'(y)\ne 0$ at the remaining vertices $y\in Q_{B_{m-1}'}$, in particular, at the vertices corresponding to the injectives $X',Y_1',\ldots,Y_r'$. Since Γ_{m-1}' is standard, X_l' must be on the coray containing $X',Y_1',\ldots,Y_r'$ and in $\bigcap_{i=k}^{m-2}\operatorname{supp}\operatorname{Hom}_{\Gamma_{m-1}'}(-,P_i)=\operatorname{supp}\operatorname{Hom}_{\Gamma_{m-1}'}(-,P_k)$, and for $k\le m-2$, the intersection of the two sets is empty. This contradiction proves the lemma. ■

3.6. LEMMA. *With the above setting, let* $(A,\Gamma)=(B_m,\Gamma_m)$. *Then*:

(i) *There is no weakly sincere module in* Γ_{m-1}.

(ii) *The weakly sincere modules in* Γ *are of the form* $Z_{i1}=(K,X_i\oplus Y_1,(\begin{smallmatrix}1\\1\end{smallmatrix}))$, *where* X_i *is the extension in* $\operatorname{mod}B_m'$ *of an indecomposable* B_{m-1}'-*module that lies on the intersection of the ray and the coray through* $X'=\operatorname{rad}P_{m-1}$ *in* Γ_{m-1}', *and sends non-zero morphisms to all the indecomposable injective* B_{m-1}'-*modules not isomorphic to* $Y_1,\ldots,Y_t$.

Proof. (i) As in the proof of the foregoing lemma, Γ_{m-1} does not contain weakly sincere modules, for these should belong to both the corectangle $\operatorname{supp}\operatorname{Hom}_{\Gamma_{m-1}}(-,P_{m-1})$ determined by P_{m-1} in Γ_{m-1} and the coray through $Y_1,,\dots,Y_t$, and their intersection is empty.

(ii) Let W be a weakly sincere indecomposable in Γ. By (i) and (2.6), there is no weakly sincere module in Γ'_m. Then, as in the proof of the preceding lemma, $W=Z_{ij}=(K,X_i\oplus Y_j,(\begin{smallmatrix}1\\1\end{smallmatrix}))$ for some $i\ge 1$ and $1\le j\le t$, and X_i is the extension in $\operatorname{mod} B'_m$ of an indecomposable B'_{m-1}-module X'_l that lies on the intersection of the ray and the coray through X' in Γ'_{m-1} and sends non-zero morphisms to all the indecomposable injective B'_{m-1}-modules not isomorphic to $Y_1,\dots,Y_t$. Since $X'_l\oplus Y_j$ is a weakly sincere B'_{m-1}-module, $j=1$. Hence $W=Z_{i1}$ is as in the above statement. The converse is clear. ∎

3.7. We now assume that $m>2$. By (3.4), $X|_{B'_{m-1}}=X'_k$ for some $k>2$. Due to (3.5) and the reducibility of the defining sequence of (A,Γ), there must be at least two $(\mathrm{ad}\,1^*)$'s with non-trivial parameter between the $(m-2)$th and $(m-1)$th $(\mathrm{ad}\,2)$'s, namely, those giving rise to $X'_k,Y_1,\dots,Y_t$, and to $X'_1,Y'_1,\dots,Y'_r$. By [7, (3.1), (3.2)], we may assume that (B_{m-1},Γ_{m-1}) is obtained from (B_{m-2},Γ_{m-2}) by a sequence of $s\ge 0$ $(\mathrm{ad}\,1)$'s followed by a sequence of $n\ge 2$ $(\mathrm{ad}\,1^*)$'s and the corresponding $(\mathrm{ad}\,2)$. We assume that of the n $(\mathrm{ad}\,1^*)$'s, it is the n_1th $(1\le n_1<n)$ that inserts $X'_k,Y_1,\dots,Y_t$ and, due to reducibility, the last that inserts $X'_1,Y'_1,\dots,Y'_r$. We denote by (A',Γ') the term of the defining sequence of (A,Γ) that is obtained from (B_{m-2},Γ_{m-2}) by the s $(\mathrm{ad}\,1)$'s, and by (A'',Γ'') the term obtained from (A',Γ') by the first n_1 $(\mathrm{ad}\,1^*)$'s. Then (B'_{m-1},Γ'_{m-1}) is the term obtained from (A'',Γ'') by the remaining $n_2=n-n_1$ $(\mathrm{ad}\,1^*)$'s.

LEMMA. *With the above setting, let $(A,\Gamma)=(B_m,\Gamma_m)$. Then:*

(i) *The injectives inserted by the first n_1 $(\mathrm{ad}\,1^*)$ operations are line-coline consecutive.*

(ii) *The injectives inserted by the remaining n_2 $(\mathrm{ad}\,1^*)$'s are line-coline consecutive.*

(iii) *The pivot of the (n_1+1)th $(\mathrm{ad}\,1^*)$ is the indecomposable A''-module $\underline{E}=(E,K,1)$, where $E=X'_{k+1}$ is the pivot of the n_1th $(\mathrm{ad}\,1^*)$.*

(iv) *If $s\ne 0$, the projectives inserted by the s $(\mathrm{ad}\,1)$ operations are consecutive, and the pivot of the first $(\mathrm{ad}\,1^*)$ is the unique projective inserted by the last $(\mathrm{ad}\,1)$.*

Proof. Let W be a weakly sincere indecomposable in Γ. By (3.6), there is an indecomposable $M=X'_l$ $(l>k)$ in Γ'_{m-1} that lies on the ray and the coray through X' and sends non-zero morphisms to all the indecomposable injective B'_{m-1}-modules not isomorphic to $Y_1,\dots,Y_t$. It follows from the fact that X' is an $(\mathrm{ad}\,2)$-pivot and the standardness of Γ'_{m-1} that all the injectives inserted by the n $(\mathrm{ad}\,1^*)$'s, except $Y'_1,\dots,Y'_r$ and $Y_1,\dots,Y_t$, must lie on the ray $[X',\infty)$. This proves (i)–(iii).

For (iv), let $N = M|_{A'}$. Then N is a weakly sincere indecomposable in Γ' lying on the coray that ends at the pivot F of the first (ad 1*), and on the ray that starts at F. Again the standardness of Γ' and the fact that F is an (ad 1*)-pivot imply that the projectives inserted by the s (ad 1)'s lie on the coray $(\infty, F]$, where F is the unique projective inserted by the last (ad 1). ∎

3.8. We now turn to case (b).

Again let $X = \operatorname{rad} P_m$ and let $\operatorname{supp}\operatorname{Hom}_{\Gamma'_m}(X, -)$ be

$$Y_t \leftarrow \cdots \leftarrow Y_2 \leftarrow Y_1 \leftarrow X = X_0 \to X_1 \to X_2 \to \cdots$$

where $t \geq 1$ with $X, Y_1, \ldots, Y_t$ injectives. By [7, (3.1), (3.2)], we may assume that (B_m, Γ_m) is obtained from (B_{m-1}, Γ_{m-1}) by a sequence of $s \geq 0$ (ad 1)'s followed by a sequence of $r \geq 1$ (ad 1*)'s and an (ad 2). By the reducibility of the defining sequence of (A, Γ), the last (ad 1*) inserts $X, Y_1, \ldots, Y_t$. We denote by (A', Γ') the term of the defining sequence of (A, Γ) that is obtained from (B_{m-1}, Γ_{m-1}) by the s (ad 1)'s. Then (B'_m, Γ'_m) is obtained from (A', Γ') by the r (ad 1*)'s.

LEMMA. *With the above setting, let $(A, \Gamma) = (B_m, \Gamma_m)$. Then:*

(i) *There is a weakly sincere module X_i $(i > 0)$ in Γ'_m lying on the intersection of the ray and the coray through X.*

(ii) *The weakly sincere modules in Γ are of the form $\overline{X}_i = (K, X_i, 1)$, $Z_{ij} = (K, X_i \oplus Y_j, \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right))$ with $1 \leq j \leq t$ and $Z_{i-j,k} = (K, X_{i-j} \oplus Y_k, \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right))$ with $1 \leq j \leq t$ and $1 \leq k \leq t+1-j$, where X_i $(i > 0)$ is a weakly sincere module in Γ'_m lying on the intersection of the ray and the coray through X.*

Proof. (i) Let W be a weakly sincere indecomposable in Γ. Then W belongs to the rectangle $\operatorname{supp}\operatorname{Hom}_{\Gamma}(P_m, -)$ determined by P_m in Γ. If $W = \overline{X}_i = (K, X_i, 1)$ for some $i \geq 0$ and $i = 0$, then $\overline{X}_0 = P_m$ is weakly sincere. As $m \geq 2$, we obtain an oriented cycle between projective-injectives in Γ, contradicting [4, (4.5)]. Hence $i > 0$ and X_i is a weakly sincere indecomposable in Γ'_m lying on the ray through X. Since Γ'_m is standard, X_i lies also on the coray containing $X, Y_1, \ldots, Y_t$. If $W = Z_{lj} = (K, X_l \oplus Y_j, \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right))$ for some $l \geq 1$ and $1 \leq j \leq t$, then $X_l \oplus Y_j$ is a weakly sincere B'_m-module. Since $Y_j(x) \neq 0$ only at the vertices $x \in Q_{B'_m}$ corresponding to the injectives $Y_j, Y_{j+1}, \ldots, Y_t$, $X_l(y) \neq 0$ at the remaining vertices $y \in Q_{B'_m}$, in particular, at the vertex corresponding to X. Therefore X_l belongs to the corectangle $\operatorname{supp}\operatorname{Hom}_{\Gamma'_m}(-, X)$ determined by X in Γ'_m. Let X_i with $i \geq l$ be the indecomposable B'_m-module corresponding to the first point of intersection in Γ'_m of the ray $[X_l, \infty)$ and the coray containing $X, Y_1, \ldots, Y_t$. It follows from the description of the modified component after applying an (ad 1*) that X_i is a weakly sincere indecomposable in Γ'_m lying on the ray and the coray through X. Note that in both cases $i > t + 1$.

(ii) Let X_i with $i > t+1$ be a weakly sincere indecomposable in Γ'_m lying on the intersection of the ray and the coray through X. Along the ray through X, the modules $X_{i-t}, X_{i-t+1}, \ldots, X_i$ are all in the corectangle $\operatorname{supp}\operatorname{Hom}_{\Gamma'_m}(-, X)$ determined by X in Γ'_m. Since X_i is weakly sincere, it is clear that the modules $\overline{X}_i$ and Z_{ij} with $1 \le j \le t$ are all weakly sincere in Γ. For X_{i-j}, with $1 \le j \le t$, we have $X_{i-j}(z) = X_i(z)$ for every vertex $z \in Q_{B'_m}$ not corresponding to the injectives $Y_1, \ldots, Y_t$, $\operatorname{Hom}_{B'_m}(X_{i-j}, Y_k) \neq 0$ for $1 \le k \le t-j$ and $\operatorname{Hom}_{B'_m}(X_{i-j}, Y_k) = 0$ for $t-j < k \le t$. Therefore $Z_{i-j,k}$ is weakly sincere only for $1 \le k \le t-(j-1) = t+1-j$.

The converse follows from the proof of (i) and the discussion of the last paragraph. ∎

3.9. Lemma. *With the above setting, let* $(A, \Gamma) = (B_m, \Gamma_m)$. *Then*:

(i) *The injectives inserted by the* r $(\operatorname{ad} 1^*)$ *operations that transform* (A', Γ') *into* (B'_m, Γ'_m) *are line-coline consecutive.*

(ii) *If* $s > 0$, *then the projectives inserted by the* s $(\operatorname{ad} 1)$ *operations that transform* (B_{m-1}, Γ_{m-1}) *into* (A', Γ') *are consecutive and the pivot of the first* $(\operatorname{ad} 1^*)$ *is the unique projective inserted by the last* $(\operatorname{ad} 1)$.

Proof. (i) Let W be a weakly sincere indecomposable in Γ. By (3.8) there is a weakly sincere indecomposable $M = X_i$ $(i > t+1)$ in Γ'_m lying on the ray and the coray through X. The assertion follows from the weak sincerity of M, the standardness of Γ'_m and the fact that X is an $(\operatorname{ad} 2)$-pivot.

(ii) Let $N = M|_{A'}$. Then N is a weakly sincere indecomposable in Γ' lying on the coray that ends at the pivot F of the first $(\operatorname{ad} 1^*)$ and on the ray that starts at F. Again the assertion follows from the weak sincerity of N, the standardness of Γ' and the fact that F is an $(\operatorname{ad} 1^*)$-pivot. ∎

3.10. Lemma. *With the above setting, let* $(A, \Gamma) = (B_m, \Gamma_m)$. *Then, for* $1 \le i \le m-1$, *there is at least one* $(\operatorname{ad} 1^*)$ *operation between the* i*th and* $(i+1)$*th* $(\operatorname{ad} 2)$*'s.*

Proof. By induction on m. If $m = 2$, then there is nothing to prove. Let $m > 2$ and W be a weakly sincere indecomposable in Γ. By (3.8), there is a weakly sincere indecomposable $M = X_i$ $(i > t+1)$ in Γ'_m. Then $N = M|_{B_{m-1}}$ is a weakly sincere indecomposable in Γ_{m-1}. By (3.6), there is at least one $(\operatorname{ad} 1^*)$ between the $(m-2)$th and $(m-1)$th $(\operatorname{ad} 2)$'s. The proof is completed by applying the induction hypothesis to (B_{m-1}, Γ_{m-1}). ∎

3.11. In view of the results we have just proved, we shall see that we may assume that in case (a), the sequence of admissible operations that transforms (C, Γ_0) into (A, Γ) consists of:

(i) an ith block of $(\operatorname{ad} 1)$'s, an ith block of $(\operatorname{ad} 1^*)$'s, the ith $(\operatorname{ad} 2)$, where $1 \le i \le m-2$,

(ii) an $(m-1)$th block of $(\mathrm{ad}\,1)$'s, an $(m-1)$th block of $(\mathrm{ad}\,1^*)$'s (which gives rise to the restriction of the pivot of the mth $(\mathrm{ad}\,2)$), an mth block of $(\mathrm{ad}\,1^*)$'s (which gives rise to the pivot of the $(m-1)$th $(\mathrm{ad}\,2)$), the $(m-1)$th $(\mathrm{ad}\,2)$,

(iii) an mth block of $(\mathrm{ad}\,1)$'s, the mth $(\mathrm{ad}\,2)$,

(iv) a last block of $(\mathrm{ad}\,1)$'s and a last block of $(\mathrm{ad}\,1^*)$'s;

and, in case (b), it consists of:

(i) an ith block of $(\mathrm{ad}\,1)$'s, an ith block of $(\mathrm{ad}\,1^*)$'s, the ith $(\mathrm{ad}\,2)$, where $1 \leq i \leq m$,

(ii) a last block of $(\mathrm{ad}\,1)$'s and a last block of $(\mathrm{ad}\,1^*)$'s.

The following proposition provides a detailed description of such a sequence.

PROPOSITION. *Assume that* (A, Γ) *is obtained from* (C, Γ_0) *by a reduced sequence of* $(\mathrm{ad}\,1)$, $(\mathrm{ad}\,1^*)$ *and* $(\mathrm{ad}\,2)$ *operations having* $m \geq 2$ $(\mathrm{ad}\,2)$*'s, and that there is a weakly sincere* A*-module in* Γ. *Then the sequence is as one of the two described above. Moreover*,

(i) *If there is no* $(\mathrm{ad}\,1^*)$ *between the* $(m-1)$*th and* m*th* $(\mathrm{ad}\,2)$, *then*:

(1) *The projectives inserted by each block of* $(\mathrm{ad}\,1)$*'s, except the last one, are consecutive. The projectives inserted by the last block are in line.*

(2) *The injectives inserted by each block of* $(\mathrm{ad}\,1^*)$*'s, except the last one, are line-coline consecutive. The injectives inserted by the last block are in line.*

(3) *If for* $1 \leq i \leq m-1$, *the* i*th block of* $(\mathrm{ad}\,1)$*'s is non-empty, then the pivot of the first* $(\mathrm{ad}\,1^*)$ *in the* i*th block is the unique projective inserted by the last* $(\mathrm{ad}\,1)$ *in it. If it is empty, then for* $2 \leq i \leq m-1$, *the pivot of the first* $(\mathrm{ad}\,1^*)$ *in the* i*th block is one of the simples inserted by the last* $(\mathrm{ad}\,1^*)$ *in the preceding block.*

(4) *If for* $2 \leq i \leq m-1$, *the* i*th block of* $(\mathrm{ad}\,1)$*'s is non-empty, then the pivot of the first operation in it is one of the simples inserted by the last* $(\mathrm{ad}\,1^*)$ *in the preceding block.*

(5) *The pivot of the first* $(\mathrm{ad}\,1^*)$ *in the* m*th block is* $\underline{E} = (E, K, 1)$, *where* E *is the pivot of the last* $(\mathrm{ad}\,1^*)$ *in the preceding block.*

(6) *If the* m*th block of* $(\mathrm{ad}\,1)$*'s is non-empty, then the pivot of the first operation in it is* $\overline{F} = (K, F, 1)$, *where* F *is the pivot of the last* $(\mathrm{ad}\,1^*)$ *in the preceding block.*

(7) *If the last block of* $(\mathrm{ad}\,1)$*'s is non-empty, then the pivot of the first operation in it is the simple* $\operatorname{soc} Y_1$, *where* $X'_k, Y_1, \ldots, Y_t$ *are the injectives inserted by the last* $(\mathrm{ad}\,1^*)$ *in the* $(m-1)$*th block.*

(8) *If the last block of* $(\mathrm{ad}\,1^*)$*'s is non-empty, then the pivot of the first operation in it is the simple injective* Y_r' *inserted by the last* $(\mathrm{ad}\,1^*)$ *in the* m*th block.*

(ii) *If there is an* $(\mathrm{ad}\,1^*)$ *between the* $(m-1)$*th and* m*th* $(\mathrm{ad}\,2)$*'s, then*:

(1) *The projectives inserted by each block of* $(\mathrm{ad}\,1)$*'s, except the last one, are consecutive. The projectives inserted by the last block are in line.*

(2) *The injectives inserted by each block of* $(\mathrm{ad}\,1^*)$*'s, except the last one, are line-coline consecutive. The injectives inserted by the last block are in line.*

(3) *If for* $1 \le i \le m$*, the* i*th block of* $(\mathrm{ad}\,1)$*'s is non-empty, then the pivot of the* i*th block of* $(\mathrm{ad}\,1^*)$*'s is the unique projective inserted by the last operation in it. If it is empty, then for* $2 \le i \le m$*, the pivot of the* i*th block of* $(\mathrm{ad}\,1^*)$*'s is one of the simples inserted by the last* $(\mathrm{ad}\,1^*)$ *in the preceding block.*

(4) *If for* $2 \le i \le m$*, the* i*th block of* $(\mathrm{ad}\,1)$*'s is non-empty, then its pivot is one of the simples inserted by the last* $(\mathrm{ad}\,1^*)$ *in the preceding block.*

(5) *If the last block of* $(\mathrm{ad}\,1)$*'s is non-empty, then its pivot* F *is either one of the simples inserted by the last* $(\mathrm{ad}\,1^*)$ *in the preceding block or* $\overline{E} = (K, E, 1)$*, where* E *is the pivot of this last* $(\mathrm{ad}\,1^*)$*.*

(6) *If the last block of* $(\mathrm{ad}\,1)$*'s is empty, then the pivot* F' *of the last block of* $(\mathrm{ad}\,1^*)$*'s is either one of the simples inserted by the last* $(\mathrm{ad}\,1^*)$ *in the preceding block or* $\underline{E} = (E, K, 1)$*, where* E *is the pivot of this last operation. Otherwise,* F' *depends on the choice of* F*, namely*: (α) *if* $F = \overline{E}$*, then* $F' = Y_t$*,* (β) *if* $F = \operatorname{soc} Y_j$*,* $1 < j \le t$*, then* F' *is either* $\operatorname{soc} Y_{j-1}$*, or* $\operatorname{soc} Y_i$ *with* $i > j$*, or any* $(\mathrm{ad}\,1^*)$*-pivot on the coray through* $\operatorname{soc} Y_j$*,* (γ) *if* $F = \operatorname{soc} Y_1$*, then* F' *is either* $\underline{E}$*, or* $\operatorname{soc} Y_i$ *with* $i > 1$*, or any* $(\mathrm{ad}\,1^*)$*-pivot on the coray through* $\operatorname{soc} Y_1$*.*

Proof. By induction on m. Let U be a weakly sincere A-module in Γ, and assume that there are some (ad 1) and (ad 1*) operations after the mth (ad 2) in the above sequence. By [7, (3.1), (3.2)], we may assume that (A, Γ) is obtained from (B_m, Γ_m) by a block of (ad 1)'s followed by a block of (ad 1*)'s. Then (2.5) and its dual show that the projectives and injectives inserted respectively by these two blocks are each in line, and that $V = U|_{B_m}$ is a weakly sincere indecomposable in Γ_m. It follows from (3.5), (3.10) and [7, (3.1), (3.2)] that the sequence has one of the two forms described before the proposition.

Assume now that we are in case (i). From (3.6) we know that $U = Z_{i1} = (K, X_i \oplus Y_1, \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right))$ lies on the ray starting at $\operatorname{soc} Y_i$ and on the coray ending at Y_r'. The only way to preserve weak sincerity after applying the (ad 1) and (ad 1*) operations is that the pivot of the first (ad 1) be $\operatorname{soc} Y_1$ and the pivot of the first (ad 1*) be Y_r', thus proving (7) and (8). If the mth block of (ad 1)'s is non-empty, (3.4) shows that the projectives inserted by those operations are consecutive and the pivot of the first is $\overline{F}$, where $F = X'_1$ is the pivot of the last $(\mathrm{ad}\,1^*)$ in the mth block, thus proving (6). From (3.7) we know that the injectives inserted by the $(m-1)$th and mth blocks of $(\mathrm{ad}\,1^*)$'s are line-coline consecutive, and the pivot of the mth block is $\underline{E}$, where $E = X'_{k+1}$ is the pivot of the last operation in the $(m-1)$th block, thus proving (5). From (3.7) we also know that if the $(m-1)$th block of $(\mathrm{ad}\,1)$'s is non-empty, then the projectives inserted by those operations are consecutive, and the pivot G of the $(m-1)$th block of $(\mathrm{ad}\,1^*)$'s is the unique projective inserted by the last of those operations. By (3.6), $W = V|_{B_{m-2}} = X_i|_{B_{m-2}}$ is a weakly sincere indecomposable in Γ_{m-2}, and by (3.5), B_{m-2} is one of the algebras considered in case (ii). If the $(m-1)$th block of $(\mathrm{ad}\,1)$'s is non-empty, then W lies on the ray starting at the pivot D of the first operation in it. From (3.8) and the fact that G is an $(\mathrm{ad}\,1^*)$-pivot, we see that D is one of the simples inserted by the last $(\mathrm{ad}\,1^*)$ in the preceding block. If the $(m-1)$th block of $(\mathrm{ad}\,1)$'s is empty, then a similar argument shows that the pivot of the $(m-1)$th block of $(\mathrm{ad}\,1^*)$'s is one of the simples mentioned above. The rest of the proof is obtained by applying the induction hypothesis to B_{m-2}.

Finally, assume that we are in case (ii), and let $V = U|_{B'_m}$ as above. By (2.5) and its dual, V is a weakly sincere indecomposable in Γ_m lying on the ray that starts at the pivot F of the last block of $(\mathrm{ad}\,1)$'s if it is non-empty, and on the coray that ends at the pivot F' of the last block of $(\mathrm{ad}\,1^*)$'s otherwise. From (3.8) it follows that, in the former case, F is one of the simples inserted by the last $(\mathrm{ad}\,1^*)$ in the preceding block or $\overline{E}$, where $E = X_1$ is the pivot of such an operation, and, in the latter case, F' is one of those simples or $\underline{E}$. If the last block of $(\mathrm{ad}\,1)$'s is non-empty, let (A', Γ') be obtained from (B_m, Γ_m) by those operations and let $U' = U|_{A'}$. Then U' is a weakly sincere indecomposable in Γ' lying on the coray that ends at F'. In this case the choice of F' depends on that of F due to the distribution of the weakly sincere indecomposable B_m-modules in the squares $\operatorname{supp}\operatorname{Hom}_{\Gamma_m}(-, P_m) \cap \operatorname{supp}\operatorname{Hom}_{\Gamma_m}(P_m, -)$ given in (3.8). Therefore, the rest of (6) follows from (3.8). By (3.9), the injectives inserted by the mth block of $(\mathrm{ad}\,1^*)$'s are line-coline consecutive and, if the mth block of $(\mathrm{ad}\,1)$'s is non-empty, then the projectives inserted by those operations are consecutive and the pivot G of the mth block of $(\mathrm{ad}\,1^*)$'s is the unique projective inserted by the last of those operations. By (3.8), there is a weakly sincere indecomposable X_i in Γ'_m such that $W = V|_{B_{m-1}} = X_i|_{B_{m-1}}$ is a weakly sincere indecomposable in Γ_{m-1}, and by (3.10), B_{m-1} falls into case (ii). As in case (i), if the mth block of $(\mathrm{ad}\,1)$'s is non-empty, then its pivot is one of the simples inserted by the last $(\mathrm{ad}\,1^*)$ in the preceding block. If it is empty, then the pivot of the mth block of $(\mathrm{ad}\,1^*)$'s is one of the above simples. The rest of the proof is obtained by applying the induction hypothesis to B_{m-1}. ∎

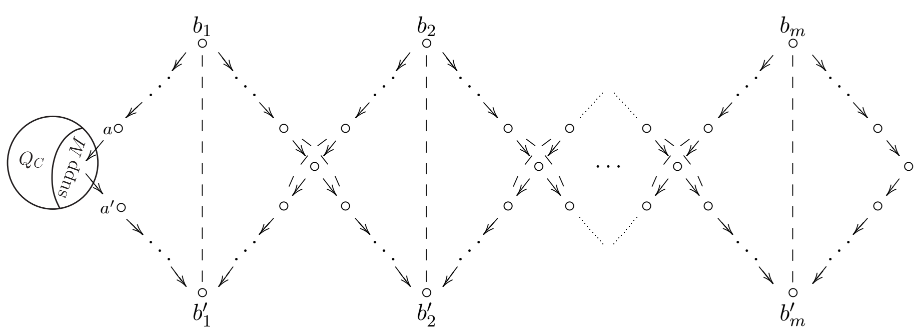
Note that the case where there is only one (ad 2) in the reduced sequence of (ad 1), (ad 1*) and (ad 2) operations that transforms (C, Γ_0) into (A, Γ) can be regarded within case (b) above. In this case, and only then, the indecomposable projective-injective P_1 may be weakly sincere. In fact, if E in Γ_0 is the pivot of such a sequence, then P_1 is weakly sincere if and only if E is weakly sincere. As we shall see in the next section, this is also the only case where the weakly sincere indecomposable is faithful.

4. Weakly sincere indecomposable modules in coils

4.1. In this section we show that the braid algebras defined in the previous section are weakly sincere coil enlargements of the algebra C. Moreover, we describe completely all weakly sincere indecomposable representations of algebras of types (1)–(25), and hence of all braid algebras. This will give the proof of the implication (i)⇒(ii) of Theorem 3.3. As a direct consequence of the considerations below we find that a braid algebra has in fact infinitely many pairwise non-isomorphic weakly sincere indecomposable modules.

Let (A, Γ) be obtained from (C, Γ_0) by a reduced sequence of (ad 1), (ad 1*) and (ad 2) operations, and assume that Γ contains a weakly sincere A-module. In this section we give the description of the bound quiver of A, of the weakly sincere coil Γ, and of all the weakly sincere indecomposable A-modules in Γ.

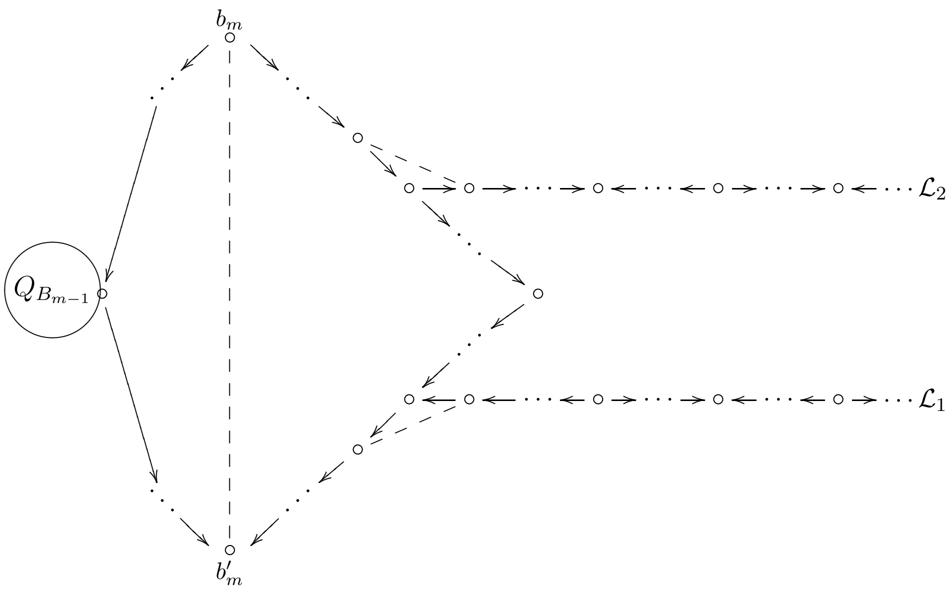
From the preceding proposition we immediately obtain the description of the bound quiver of A and of the weakly sincere coil Γ. We start by describing the bound quiver of A when A is one of the algebras considered in case (b). Assume first that $(A, \Gamma) = (B_m, \Gamma_m)$. Let M be the simple regular C-module that is the pivot of the reduced sequence that transforms (C, Γ_0) into (A, Γ). Let a be the extension vertex of $C[M]$, and a' be the coextension vertex of $[\overline{M}](C[M])$, where $\overline{M} = P(a)$. For $1 \leq i \leq m$, let b_i and b'_i be the extension and coextension vertices, respectively, of B_i. Then $P_i = P(b_i) = I(b'_i)$. From (3.11)(ii)(1)–(4), we obtain as the bound quiver of A the following quiver:

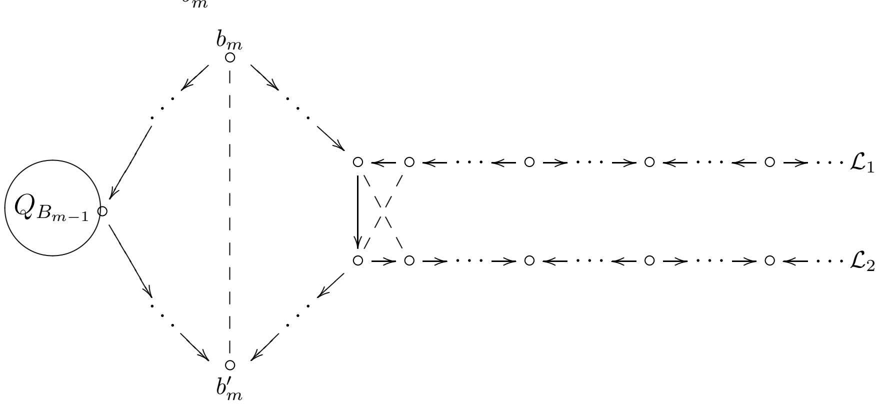


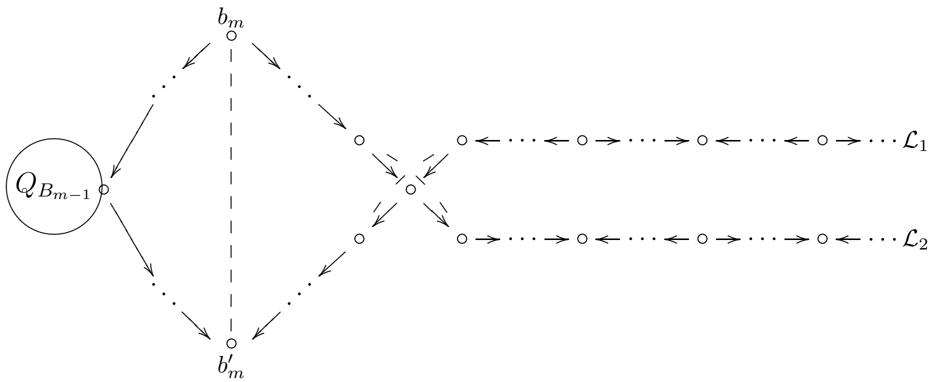
bound by the relations of $[\overline{M}](C[M])$, the commutativity relations and the zero relations indicated by the dashed lines.

DEFINITION. A bound quiver as above, where C is an algebra and M is an indecomposable C-module lying on the mouth of a standard stable tube of Γ_C, will be called a *simple braid* or a *braid of type 1*, and the algebra given by such a bound quiver will be called a *simple braid algebra* or a *braid algebra of type 1*.

The general case is obtained from the one above in the following way. From (3.11)(ii)(5)–(6), we deduce that the bound quiver of A consists of a simple braid (the bound quiver of B_m) together with two lines $\mathcal{L}_1$ and $\mathcal{L}_2$ formed, respectively, by the projectives and the injectives inserted by the last two blocks in the sequence, and has one of the following forms (with $B_{m-1} = C$ if $m = 1$):



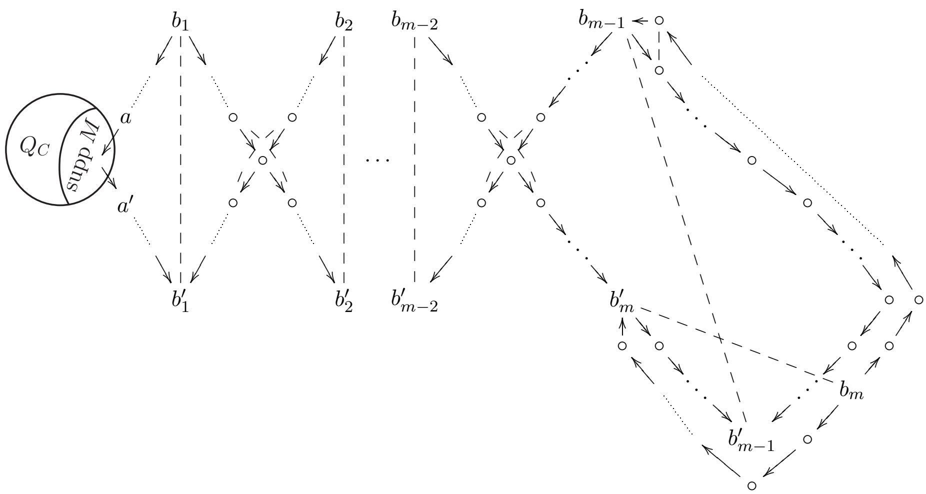




all bound by the relations of B_m, the zero relations indicated by the dashed lines and, in the last case, possibly by zero relations from points on $\mathcal{L}_1$ to points on $\mathcal{L}_2$.

Definition. A bound quiver as above, consisting of a simple braid along with two lines, will be called a *braid of type 2*, and an algebra given by such a bound quiver will be called a *braid algebra of type 2*.

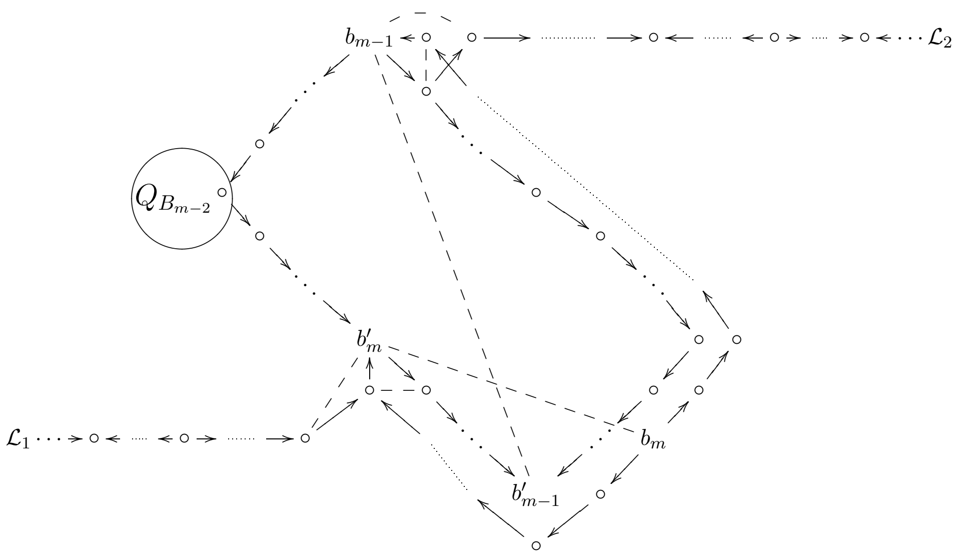
We now describe the bound quiver of A when A is one of the algebras considered in case (a). Again, assume first that $(A, \Gamma) = (B_m, \Gamma_m)$. From (3.11)(i)(1)–(6), we deduce that the bound quiver of A is a braid with a handle, namely, the quiver



bound by the relations of $[\overline{M}](C[M])$, the commutativity relations and the zero relations indicated by the dashed lines. Note that the vertices corresponding to the projectives inserted by the $(m-1)$th block of $(\mathrm{ad}\,1)$'s, the injectives inserted by the $(m-1)$th and mth blocks of $(\mathrm{ad}\,1^*)$'s, and b_{m-1} form the last loop of the above braid and part of the handle. The rest of the handle is formed by the vertices corresponding to the projectives inserted by the mth block of (ad 1)'s, and b_m.

Definition. A bound quiver as above, consisting of a simple braid and a handle, will be called a *braid of type 3*, and an algebra given by such a bound quiver will be called a *braid algebra of type 3*.

The general case is obtained from the one above as follows. From (3.11)(i)(7)–(8), we find that the bound quiver of A is a braid of type 3 together with two lines $\mathcal{L}_1$ and $\mathcal{L}_2$ formed, respectively, by the projectives and injectives inserted by the last block of (ad 1) and (ad 1*) operations, namely, the quiver



(with $B_{m-2} = C$ if $m = 2$) bound by the relations of B_m and the zero relations indicated by the dashed lines.

Definition. A bound quiver as above, consisting of a braid of type 3 along with two lines, will be called a *braid of type 4*, and an algebra given by such a bound quiver will be called a *braid algebra of type 4*.

We have thus proved that the algebras we are interested in are, in case (a), braid algebras of type 3 or 4 and, in case (b), braid algebras of type 1 or 2.

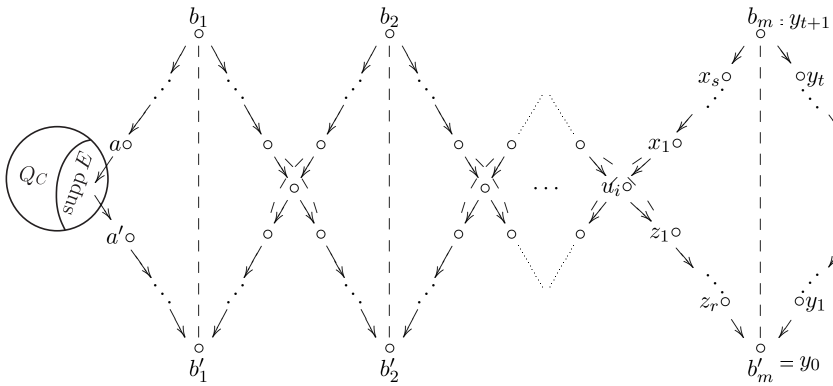
4.2. Conversely, if A is a braid algebra of type 2 (respectively, 4) with bound quiver as shown above, then the following two statements are clear:

(i) A is obtained from the algebra C by means of a sequence of (ad 1), (ad 1*) and (ad 2) operations as described in (3.11)(ii) (respectively, (i)).

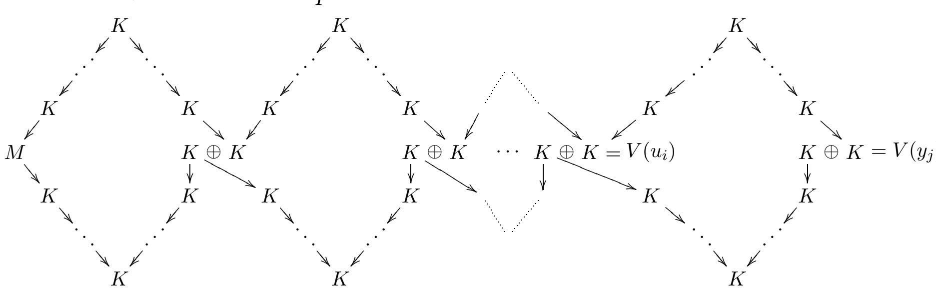
(ii) There exists a coil Γ in Γ_A that is obtained from the tube Γ_0 in Γ_C containing the indecomposable C-module M by means of the same sequence of operations.

Now, from the fact that A is a coil enlargement of C, and that Γ contains all the indecomposable projective-injective A-modules, it follows that in case there are indecomposable weakly sincere A-modules, these must belong to Γ. We shall prove that a braid algebra of any type has indeed indecomposable weakly sincere modules by exhibiting all of them. We start with the following result.

LEMMA. *Let $A = B_m$ be a simple braid algebra with bound quiver*



and let Γ be the coil in Γ_A containing the indecomposable projective-injective A-modules. Then, for $1 \le j \le t$, there is a weakly sincere indecomposable A-module V in Γ isomorphic to



where M is a weakly sincere indecomposable C-module lying on the intersection of the ray starting at E and the coray ending at E in Γ_0. Moreover, V lies on the intersection of the ray starting at $S(y_j)$ and the coray ending at $S(y_j)$.

Proof. By induction on m. Let $m > 1$, and assume that a similar indecomposable W has been constructed over B_{m-1}. Note that $(A, \Gamma) = (B_m, \Gamma_m)$ is obtained from (B_{m-1}, Γ_{m-1}) by the following sequence of admissible operations with pivot $S(u_i)$:

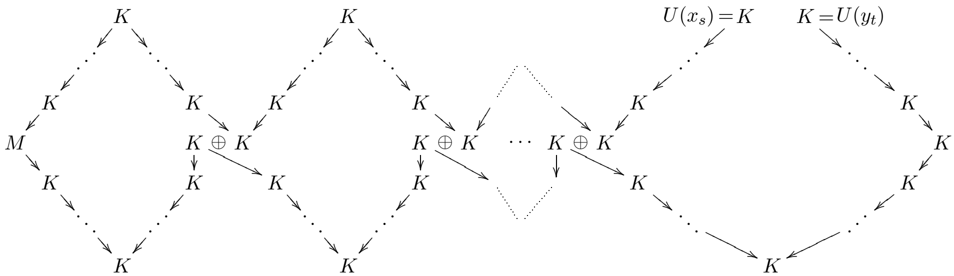
(i) consecutive one-point extensions with extension vertices $x_1, \ldots, x_s$,

(ii) consecutive one-point coextensions with coextension vertices $z_1, \ldots$ $\ldots, z_r$ and pivot $P(x_s)$ over the corresponding algebra in the sequence,

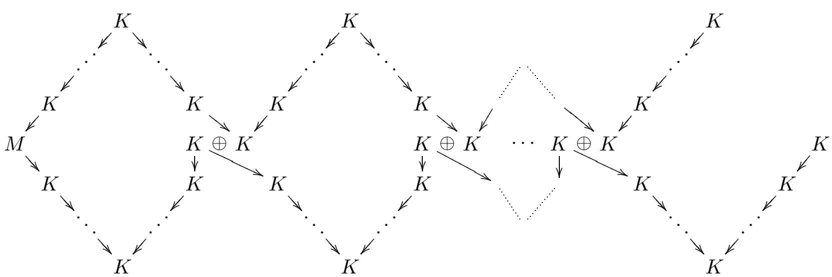
(iii) a branch coextension with root vertex b'_m and pivot $I(z_r)$ over the corresponding algebra in the sequence,

(iv) a one-point extension with extension vertex b_m and pivot $I(b'_m)$ over B'_m (in the notation of (3.4)).

Since W lies on the intersection of the ray starting at $S(u_i)$ and the coray ending at $S(u_i)$ in Γ_{m-1}, after applying (i), (ii) and (iii), we obtain the following indecomposable weakly sincere B'_m-module U:



which lies on the intersection of the ray and the coray through $I(b'_m) = \operatorname{rad} P(b_m) = X_0$ in Γ'_m. From the description of the modified component after applying an (ad 1^*) operation, it follows that the indecomposable B'_m-module

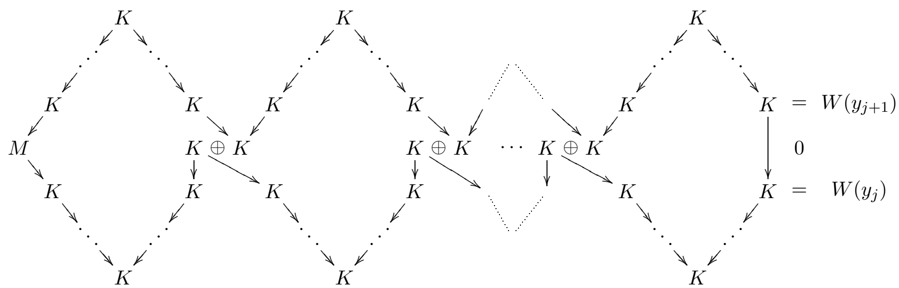


lies on the intersection of the ray through X_0 and the coray ending at $S(y_j)$. Therefore it is a module of the form X_i in $\operatorname{supp} \operatorname{Hom}_{\Gamma'_m}(X_0, -)$. Let $Y_j = I(y_j)$ over B'_m. Then $V = Z_{ij} = (K, X_i \oplus Y_j, \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right))$ is a weakly sincere indecomposable module in Γ lying on the intersection of the ray starting at $S(y_j)$ and the coray ending at $S(y_j)$. (Note that $S(y_j)$ is indeed in the socle and the top of V.) ∎

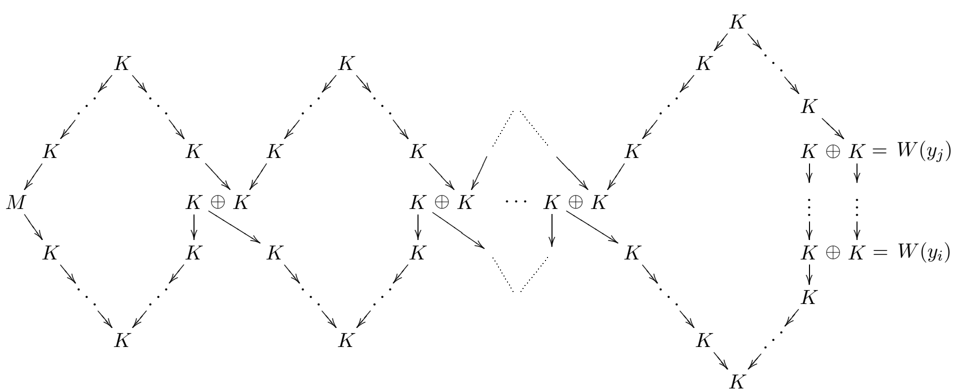
4.3. Let A be again a simple braid algebra with bound quiver as in the previous lemma, and let Γ be the weakly sincere coil in Γ_A. We now describe all the weakly sincere indecomposable A-modules in Γ. Let E in Γ_0 be the pivot of the sequence of admissible operations transforming (C, Γ_0)

into (A, Γ), and let $\mathcal{E}$ be the set of all indecomposable C-modules in Γ_0 lying on the intersection of the ray starting at E and the coray ending at E. Then we have the following result.

THEOREM. *Let A be a simple braid algebra with bound quiver as in the preceding lemma, and let Γ be the weakly sincere coil in Γ_A. Then an A-module W is a weakly sincere indecomposable module in Γ if and only if there exists $M \in \mathcal{E}$ such that W is isomorphic to one of the following modules*:



with $0 \le j \le t$,

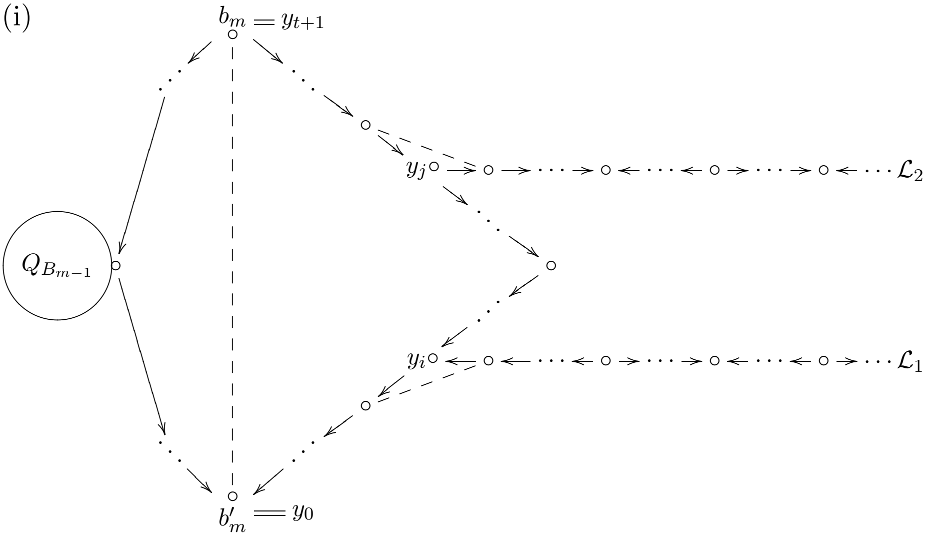


with $1 \le i \le j \le t$.

Proof. Let $X = X_0 = \operatorname{rad} P(b_m)$. By (3.8) the weakly sincere indecomposable A-modules in Γ are of the form either $\overline{X}_i = (K, X_i, 1)$, or $Z_{ij} = (K, X_i \oplus Y_j, \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right))$ with $1 \le j \le t$, or $Z_{i-j,k} = (K, X_{i-j} \oplus Y_k, \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right))$ with $1 \le j \le t$ and $1 \le k \le t+1-j$, where X_i $(i > t+1)$ is a weakly sincere indecomposable B'_m-module (with the notation of (3.4)) lying on the intersection of the ray and the coray through X in Γ'_m. From the description of the sequence of admissible operations with pivot $S(u_i)$ that transforms (B_{m-1}, Γ_{m-1}) into $(A, \Gamma) = (B_m, \Gamma_m)$, it follows that $V = X_i|_{B_{m-1}}$ is a weakly sincere indecomposable in Γ_{m-1} lying on the intersection of the ray starting at $S(u_i)$ and the coray ending at $S(u_i)$. Hence V is one of the modules described in (4.2), so that, in particular, $M = V|_C$ belongs to $\mathcal{E}$. The rest of the proof is clear. ∎

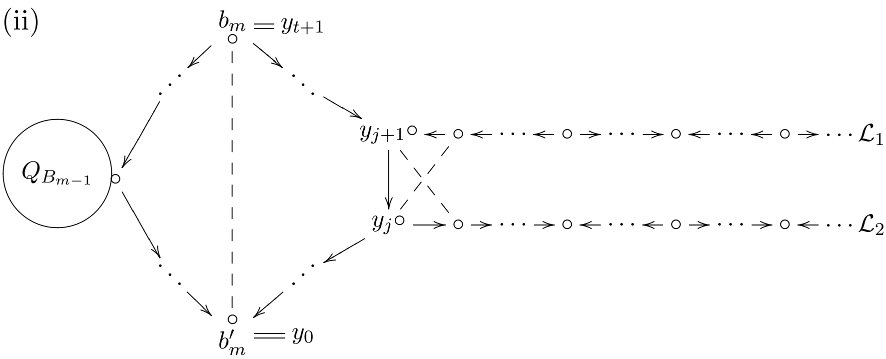
4.4. Let A be a braid algebra of type 2 with bound quiver one of the following three:

(i)



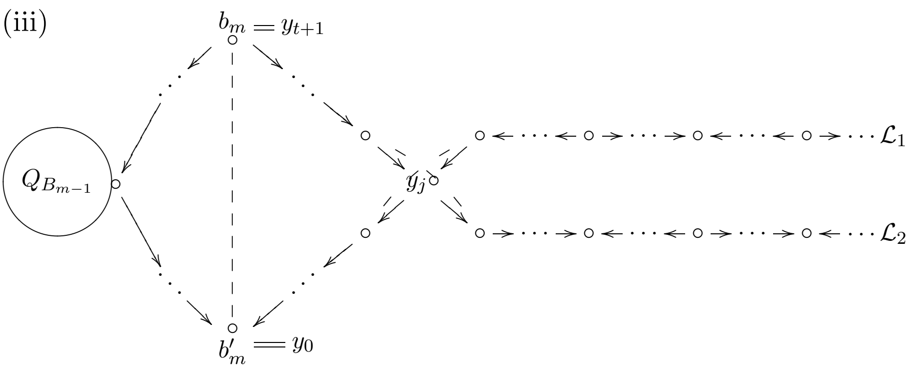
with $1 \leq i < j \leq t$,

(ii)



with $0 \leq j \leq t$,

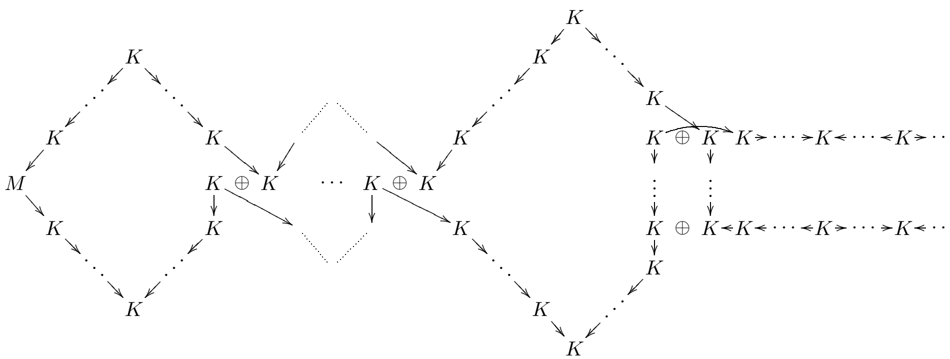
(iii)



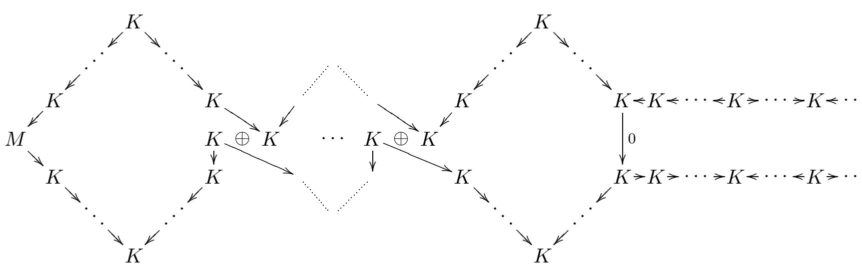
with $1 \le j \le t$, which are all bound by the relations of B_m, the zero relations indicated by the dashed lines and, in the last case, possibly by zero relations from points on $\mathcal{L}_1$ to points on $\mathcal{L}_2$.

In the three cases, A is obtained from the simple braid algebra B_m by applying a sequence of (ad 1) and (ad 1^*) operations in the weakly sincere coil Γ_m of Γ_{B_m}, and the projectives (respectively, injectives) inserted by those operations are in line. Let Γ be the coil in Γ_A obtained in this way from Γ_m. By (2.6) and its dual, Γ is weakly sincere. The following result is then a consequence of (4.3).

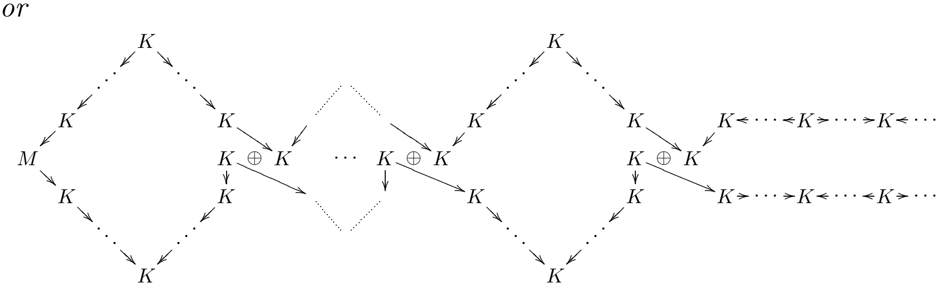
Corollary. *Let A be a braid algebra of type 2 with bound quiver as in* (i) (*respectively*, (ii), (iii)) *above*, *and let Γ be the weakly sincere coil in Γ_A. Then an A-module U is a weakly sincere indecomposable in Γ if and only if there exists $M \in \mathcal{E}$ such that U is isomorphic respectively to*



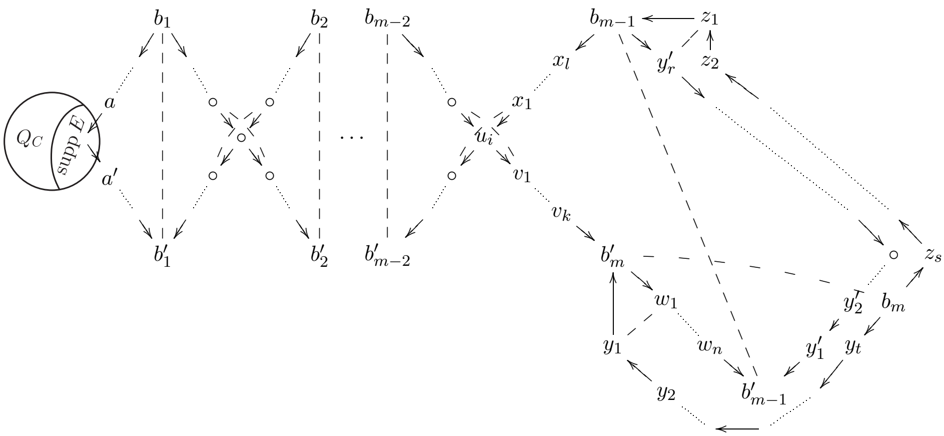
or



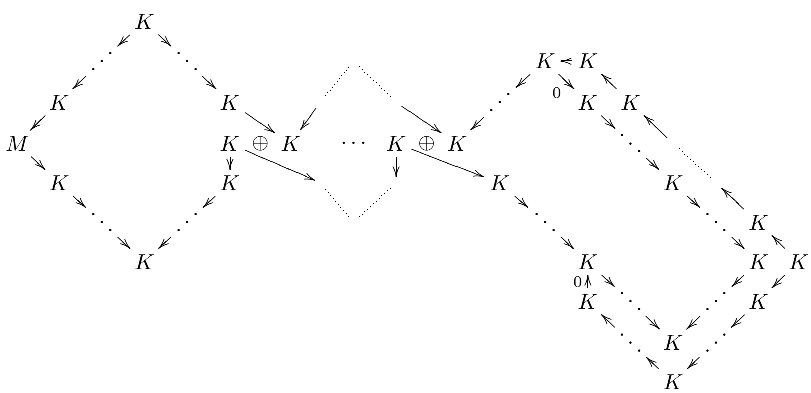
or



4.5. LEMMA. *Let $A = B_m$ be a braid algebra of type 3 with bound quiver*

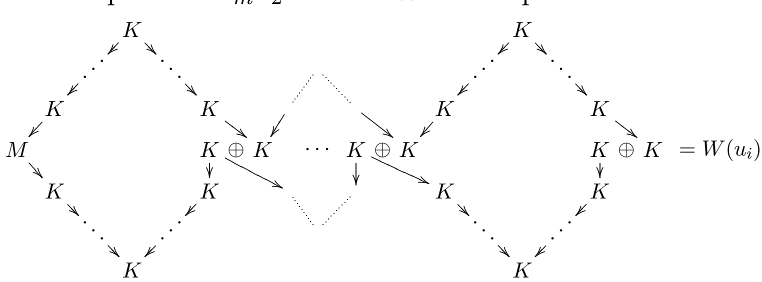


and let Γ be the coil in Γ_A containing the indecomposable projective-injective A-modules. Then there is a weakly sincere indecomposable A-module V in Γ isomorphic to



where M is an indecomposable C-module lying on the intersection of the ray starting at E and the coray ending at E in Γ_0. Moreover, V lies on the intersection of the ray starting at $S(y_1)$ and the coray ending at $S(y'_r)$.

Proof. Since B_{m-2} is a simple braid algebra, by (4.2), there is a weakly sincere indecomposable B_{m-2}-module W isomorphic to



where M is a weakly sincere indecomposable C-module lying on the intersection of the ray starting at E and the coray ending at E in Γ_0. Moreover, W lies on the intersection of the ray starting at $S(u_i)$ and the coray ending at $S(u_i)$ in Γ_{m-2}.

Note that $(A, \Gamma) = (B_m, \Gamma_m)$ is obtained from (B_{m-2}, Γ_{m-2}) by the following sequence of admissible operations with pivot $S(u_i)$:

(i) consecutive one-point extensions with extension vertices $x_1, \ldots, x_l$,

(ii) consecutive one-point coextensions with coextension vertices $v_1, \ldots, v_k$ and pivot $P(x_l)$ over the corresponding algebra,

(iii) a branch coextension with root vertex b'_m and pivot $I(v_k) = E$ over the corresponding algebra,

(iv) consecutive one-point coextensions with coextension vertices $w_1, \ldots, w_n$ and pivot $\underline{E} = (E, K, 1)$,

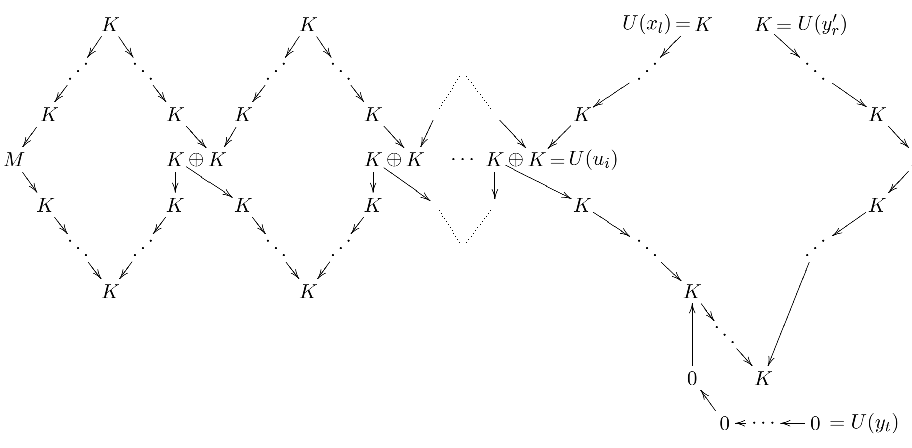
(v) a branch coextension with root vertex b'_{m-1} and pivot $I(w_n) = F$ over the corresponding algebra,

(vi) a one-point extension with extension vertex b_{m-1} and pivot $I(b'_{m-1})$ over B'_{m-1} (in the notation of (3.4)),

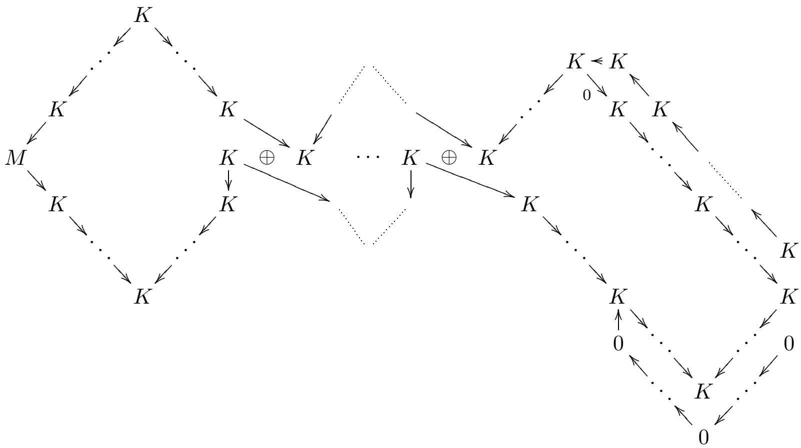
(vii) consecutive one-point extensions with extension vertices $z_1, \ldots, z_s$ and pivot $\overline{F} = (K, F, 1)$,

(viii) a one-point extension with extension vertex b_m and pivot $I(b'_m)$ over B'_m.

After applying (i) to (v) we obtain the following indecomposable B'_{m-1}-module U:



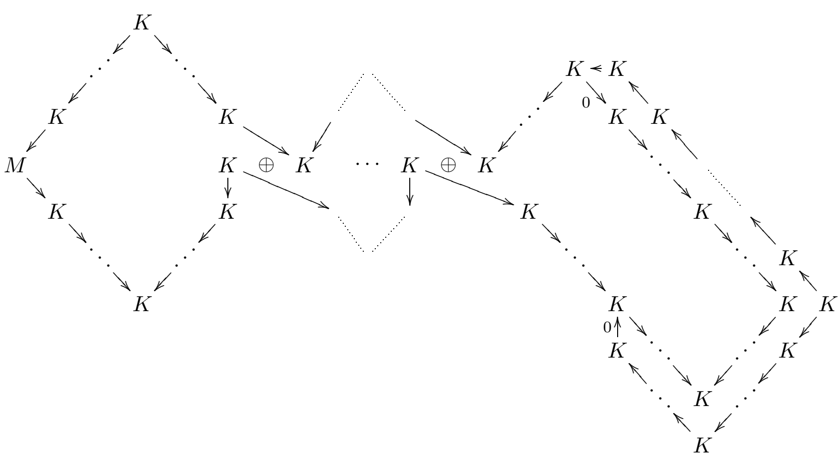
which lies on the intersection of the ray and the coray through $\operatorname{rad} P(b_{m-1}) = I(b'_{m-1})$ in Γ_{m-1}. Then, after applying (vi) and (vii), we obtain the following indecomposable B'_m-module:



which lies on the intersection of the ray through $X_0 = \operatorname{rad} P(b_m) = I(b'_m)$ and the coray ending at $S(y'_r)$. Hence it is a module of the form X_i in $\operatorname{supp} \operatorname{Hom}_{\Gamma'_m}(X_0, -)$. Let $Y_1 = I(y_1)$ over B'_m. Then $V = Z_{i1} = (K, X_i \oplus Y_1, \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right))$ is a weakly sincere indecomposable A-module in Γ lying on the intersection of the ray starting at $S(y_1)$ and the coray ending at $S(y'_r)$. (Note that $S(y_1)$ is indeed in the socle of V and $S(y'_r)$ is in its top.) ∎

4.6. We now describe all weakly sincere indecomposable modules over a braid algebra A of type 3 with bound quiver as in the previous lemma. Let E, Γ_0 and $\mathcal{E}$ be as in (4.3). From the preceding lemma and (3.6) we immediately obtain the following result.

THEOREM. *Let A be a braid algebra of type 3 with bound quiver as in the preceding lemma, and let Γ be the weakly sincere coil in Γ_A. Then an A-module U is a weakly sincere indecomposable in Γ if and only if there exists $M \in \mathcal{E}$ such that U is isomorphic to*

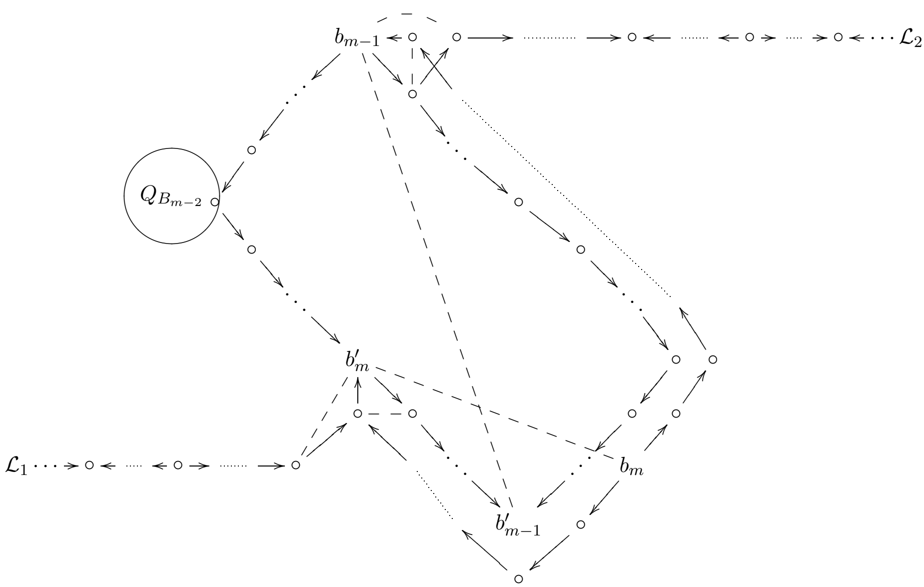


Proof. Let $X = X_0 = \operatorname{rad} P(b_m)$ and $Y_1 = I(y_1)$ over B'_m. By (3.6) the weakly sincere indecomposable A-modules in Γ are of the form

$$Z_{i1} = (K, X_i \oplus Y_1, (\begin{smallmatrix}1\\1\end{smallmatrix})),$$

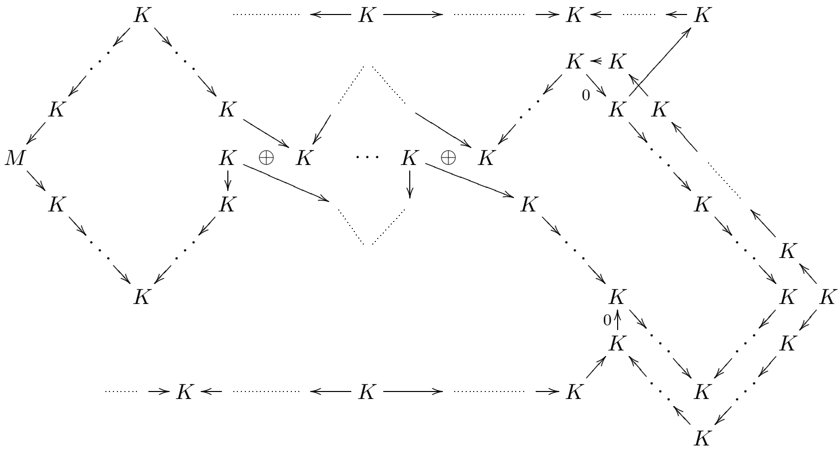
where X_i is the extension in $\operatorname{mod} B'_m$ of an indecomposable B'_{m-1}-module lying on the intersection of the ray and the coray through $X' = \operatorname{rad} P(b_{m-1})$ in Γ'_{m-1}, and sends non-zero morphisms to all the indecomposable injective B'_{m-1}-modules not isomorphic to $I(y_1), \ldots, I(y_t)$. Let $V = X_i|_{B_{m-2}}$. Then V is a weakly sincere indecomposable in Γ_{m-2} lying on the intersection of the ray starting at $S(u_i)$ and the coray ending at $S(u_i)$. Therefore V is one of the modules described in (4.1). The rest of the proof is clear. ∎

4.7. Finally, let A be a braid algebra of type 4 with bound quiver



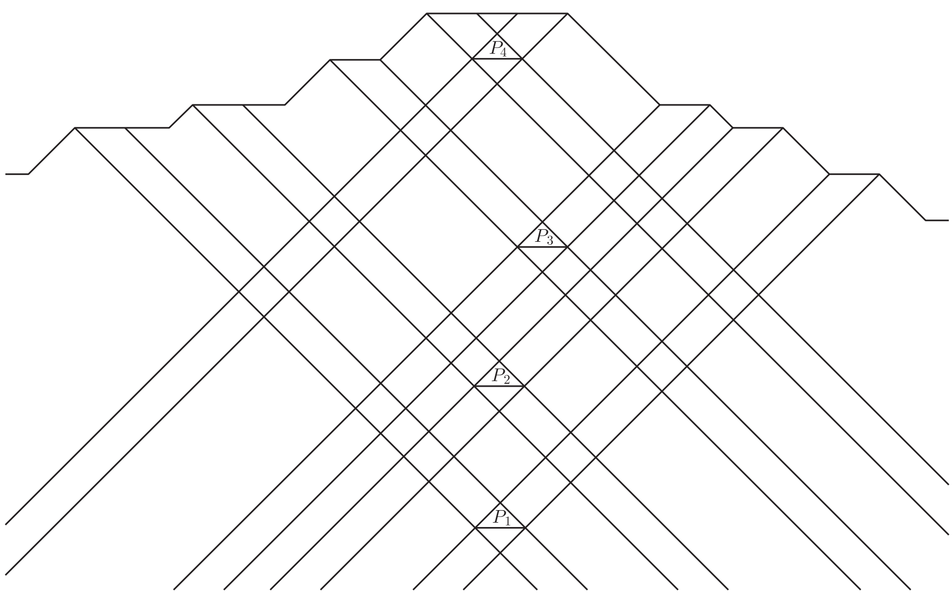
bound by the relations of B_m and the zero relations indicated by the dashed lines. Then A is obtained from the braid algebra B_m of type 3 by applying a sequence of (ad 1) and (ad 1*) operations in the weakly sincere coil Γ_m of Γ_{B_m}, and the projectives (respectively, injectives) inserted by those operations are in line. Let Γ be the coil in Γ_A obtained in this way from Γ_m. Then Γ is weakly sincere, and we have the following result.

Corollary. *Let A be a braid algebra of type 4 with bound quiver as above, and let Γ be the weakly sincere coil in Γ_A. Then an A-module U is a weakly sincere indecomposable in Γ if and only if there exists $M \in \mathcal{E}$ such that U is isomorphic to*



REMARK. A simple braid algebra $A = B_m$ with bound quiver as in (4.2) has a weakly sincere indecomposable projective-injective if and only if the indecomposable C-module E which is the pivot of the sequence of admissible operations that transforms (C, Γ_0) into (A, Γ) is weakly sincere and $m = 1$. This is the only case where the weakly sincere indecomposable is faithful.

4.8. Let A be a simple braid algebra and Γ be the weakly sincere coil in Γ_A. Then Γ has the following shape:



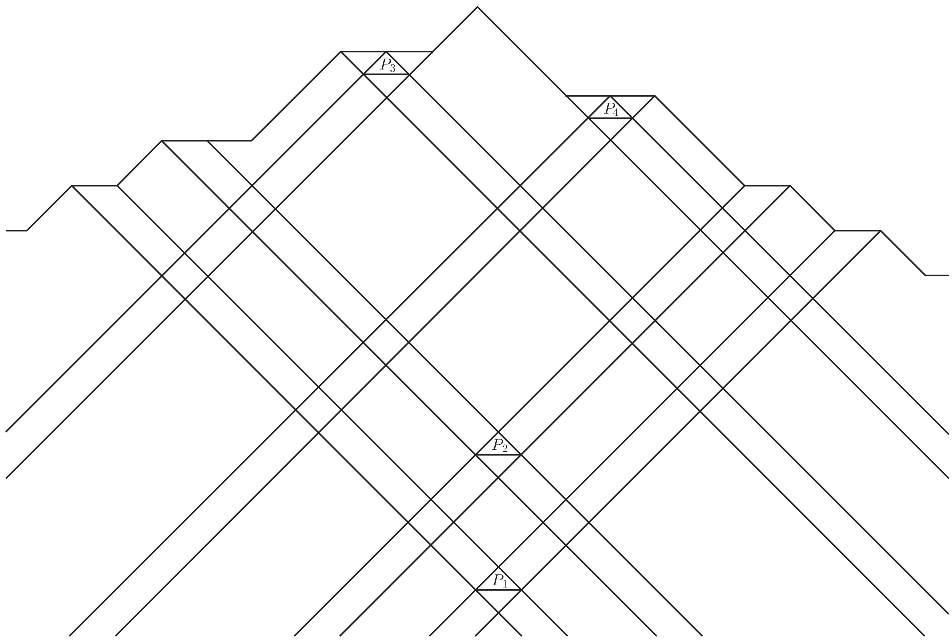
Note that Γ is a proper quasi-tube (see [2, (1.3)] or [5, (2.1)]) that satisfies:

(i) the projective-injective modules $P_1, \dots, P_m$ in Γ are ordered in such a way that the rectangle determined by P_i in Γ is cofinite in the rectangle determined by P_{i-1}, for $1 < i \leq m$,

(ii) the rays corresponding to projective non-injective modules do not intersect the rectangle determined by P_m,

(iii) the corays corresponding to injective non-projective modules do not intersect the corectangle determined by P_m.

4.9. Let A be a braid algebra of type 3 and Γ be the weakly sincere coil in Γ_A. Then Γ has the following shape:



Note that Γ is a proper quasi-tube that satisfies:

(i) the projective-injective modules $P_1, \dots, P_m$ in Γ are ordered in such a way that the rectangle determined by P_i in Γ is cofinite in the rectangle determined by P_{i-1}, for $1 < i \leq m-1$, and the rectangle determined by P_m is contained in the rectangle determined by P_{m-1},

(ii) the rays corresponding to projective non-injective modules do not intersect the rectangle determined by P_m,

(iii) the corays corresponding to injective non-projective modules do not intersect the corectangle determined by P_m.

5. Braid algebras of forms (1) and (2)

5.1. The next three sections are devoted to the proof of the implication (ii)$\Rightarrow$(i) of the main theorem. We also show that the weakly sincere indecomposable representations of the braid algebras of forms (1) to (25) are precisely those given in Section 3. We start with some definitions.

Let B be an algebra and $\mathcal{C}$ a (standard) coil of Γ_B. Moreover, let $(A, \mathcal{C}')$ be one of the terms in a defining sequence for $(B, \mathcal{C})$, and assume that $(B, \mathcal{C})$ is obtained from $(A, \mathcal{C}')$ by a sequence of n admissible operations of type (ad 1). Then we make the following definitions.

The projectives inserted in $\mathcal{C}'$ by such a sequence appear in a *single stairs configuration* if, for $1 < i \le n$, the pivot of the ith operation is either the unique projective inserted by the $(i-1)$th operation, in case it has parameter zero, or the simple projective inserted by the $(i-1)$th operation, in case it has non-zero parameter.

The projectives inserted in $\mathcal{C}'$ by such a sequence appear in a *double stairs configuration* if there is an integer $1 < k < n$ such that:

(i) the projectives inserted by the first k operations appear in a single stairs configuration,

(ii) the parameter of the kth operation is $t > 1$, and the pivot of the $(k+1)$th operation is one of the simple non-projective modules inserted by the kth operation,

(iii) the projectives inserted by the remaining $n-k-1$ operations appear in a single stairs configuration.

The projectives inserted in $\mathcal{C}'$ by such a sequence appear in a (*broken line*, *single stairs*) *configuration* if some of them are injective and for $1 < i \le n$, the pivot of the ith operation is either a ray module on the ray starting at the unique projective inserted by the $(i-1)$th operation, in case it has parameter zero, or the simple projective inserted by the $(i-1)$th operation, in case it has non-zero parameter.

These names are suggested by the way the projectives appear in the coil $\mathcal{C}$. There are also the corresponding dual definitions.

5.2. Let C be an algebra and $\mathcal{T}$ be a standard stable tube of Γ_C. In this section we consider the case where the pair $(B, \mathcal{C})$ is obtained from the pair $(C, \mathcal{T})$ by a sequence of admissible operations of types (ad 1) and (ad 1*). To this end, and also for future use, we prove the following more general results.

LEMMA. *Let A be an algebra and $\mathcal{C}$ be a standard coil in Γ_A. Let $(B, \mathcal{C}')$ be obtained from $(A, \mathcal{C})$ by a sequence of admissible operations of type* (ad 1) *with pivot M. If $\mathcal{C}'$ contains a weakly sincere indecomposable B-module, then the projectives inserted in $\mathcal{C}$ by this sequence appear in a* (*broken line*, *single stairs*) *configuration.*

Proof. Let N be a weakly sincere indecomposable B-module in $\mathcal{C}'$. Since $\mathcal{C}'$ is standard, N must lie on the ray containing all the projectives inserted by the last operation, and there must be a sectional path pointing to the mouth of $\mathcal{C}'$ from each of the projectives inserted by the previous operations to the ray containing N. It follows from the description of the modified component after applying (ad 1) that the projectives inserted in $\mathcal{C}$ by the sequence must appear in a (broken line, single stairs) configuration. ∎

5.3. COROLLARY. *With the hypothesis of the above lemma, N is a weakly sincere indecomposable B-module in $\mathcal{C}'$ if and only if $N|_A$ is a weakly sincere indecomposable in $\mathcal{C}$ lying on the ray that starts at M, and $N(x) = K$ for every vertex $x \in Q_B \setminus Q_A$.*

Proof. This follows from (5.2) and the description of the modified component after applying (ad 1). ∎

We now turn to the case where $(B, \mathcal{C})$ is obtained from $(C, \mathcal{T})$ by a sequence of admissible operations of types (ad 1) and (ad 1*). By [7, Section 3], we may replace this sequence by another one consisting of a block of operations of type (ad 1) followed by a block of operations of type (ad 1*). Applying (5.2) and its dual, we find that B is a braid algebra of one of the forms (1) or (2), and applying (5.3) and its dual, we see that the list of its weakly sincere indecomposable representations is complete.

6. Braid algebras of forms (18) to (25)

6.1. In the next two sections we consider the case where $(B, \mathcal{C})$ is obtained from $(C, \mathcal{T})$ by a sequence of admissible operations containing operations of types (ad 2), (ad 3) and their duals. By [7, Section 3], we may assume that the sequence consists only of operations of types (ad 1), (ad 1*), (ad 2) and (ad 3), that it is reduced, and that those parts of the sequence consisting only of operations of types (ad 1) and (ad 1*) may be replaced by a block of (ad 1)'s followed by a block of (ad 1*)'s.

Recall that the projectives inserted by operations of types (ad 2) and (ad 3) are called *exceptional*. We assume that there are $m \geq 2$ exceptional projectives $P_1, \ldots, P_m$ in $\mathcal{C}$ and, for $1 \leq i \leq m$, we denote by $(A_i', \mathcal{C}_i')$ the term in the defining sequence of $(B, \mathcal{C})$ corresponding to the insertion of P_i, and by $(A_i, \mathcal{C}_i)$ the term preceding the latter. Hence, $(B, \mathcal{C})$ is obtained from $(A_m', \mathcal{C}_m')$ by a block of (ad 1)'s followed by a block of (ad 1*)'s. By (5.2), (5.3) and their duals, the projectives and injectives inserted respectively by these two blocks appear in a (broken line, single stairs) configuration, and $\mathcal{C}_m'$ contains a weakly sincere indecomposable A_m'-module N. Two situations are possible:

(a) $(A_m, \mathcal{C}_m)$ is obtained from $(A_{m-1}', \mathcal{C}_{m-1}')$ by a block of (ad 1)'s,

(b) $(A_m, \mathcal{C}_m)$ is obtained from $(A_{m-1}', \mathcal{C}_{m-1}')$ by a block of (ad 1)'s followed by a block of (ad 1*)'s.

The aim of this section is the analysis of case (a). Throughout, we use the following notation. Let $X = \operatorname{rad} P_m$ and let the support $\mathcal{S}(X)$ of $\operatorname{Hom}_{A_m}(X,-)|_{\mathcal{C}_m}$ be

$$Y_t \leftarrow \cdots \leftarrow Y_2 \leftarrow Y_1 \leftarrow X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots$$

where $t \geq 1$ and $X, Y_1, \ldots, Y_t$ are injectives if X is an (ad 2)-pivot, and

$$\begin{array}{ccccccccc} Y_1 & \rightarrow & Y_2 & \rightarrow & \cdots & \rightarrow & Y_t & & \\ \uparrow & & \uparrow & & & & \uparrow & & \\ X = X_0 & \rightarrow & X_1 & \rightarrow & \cdots & \rightarrow & X_{t-1} & \rightarrow & X_t & \rightarrow \cdots \end{array}$$

where $t \geq 2$ and X_{t-1}, Y_t are injectives if X is an (ad 3)-pivot. Let $U = \operatorname{rad} P_{m-1}$ and let the support $\mathcal{S}(U)$ of $\operatorname{Hom}_{A_{m-1}}(U,-)|_{\mathcal{C}_{m-1}}$ be

$$V_s \leftarrow \cdots \leftarrow V_2 \leftarrow V_1 \leftarrow U = U_0 \rightarrow U_1 \rightarrow U_2 \rightarrow \cdots$$

where $s \geq 1$ and $U, V_1, \ldots, V_s$ are injectives if U is an (ad 2)-pivot, and

$$\begin{array}{ccccccccc} V_1 & \rightarrow & V_2 & \rightarrow & \cdots & \rightarrow & V_s & & \\ \uparrow & & \uparrow & & & & \uparrow & & \\ U = U_0 & \rightarrow & U_1 & \rightarrow & \cdots & \rightarrow & U_{s-1} & \rightarrow & U_s & \rightarrow \cdots \end{array}$$

where $s \geq 2$ and U_{s-1}, V_s are injectives if U is an (ad 3)-pivot.

6.2. Lemma. *$(A_m, \mathcal{C}_m)$ is obtained from $(A'_{m-1}, \mathcal{C}'_{m-1})$ by a sequence of consecutive one-point extensions having as pivot any ray module on the ray starting at $U'_1 = (K, U_1, 1)$ (respectively, $U'_s = (K, U_s, 1)$) if U is an (ad 2)-pivot (respectively, (ad 3)-pivot).*

Proof. Since the defining sequence of $(B, \mathcal{C})$ is reduced, and the operations of types (ad 1), (ad 2) and (ad 3) consist of inserting one or several rays, the rectangle determined by P_{m-1} in $\mathcal{C}'_{m-1}$ (i.e., the support of $\operatorname{Hom}_{A'_{m-1}}(P_{m-1},-)|_{\mathcal{C}'_{m-1}}$) must be enlarged, by applying the block of (ad 1) operations with pivot a ray module on the ray forming its upper border, to the rectangle determined by P_{m-1} in $\mathcal{C}_m$ (i.e., the support of $\operatorname{Hom}_{A_m}(P_{m-1},-)|_{\mathcal{C}_m}$), and X must lie on the ray forming the upper border of the latter. In other words, if U is an (ad 2)-pivot (respectively, (ad 3)-pivot), then the pivot of the block of (ad 1)'s is any ray module on the ray starting at U'_1 (respectively, U'_s), there is a sectional path from U'_1 (respectively, U'_s) to a module R on the mouth of $\mathcal{C}_m$, and X lies on the ray starting at R. On the other hand, the weakly sincere indecomposable N in $\mathcal{C}'_m$ receives non-zero morphisms from P_m and from all the projectives inserted in $\mathcal{C}'_{m-1}$ by the block of (ad 1)'s. It follows from the standardness of $\mathcal{C}'_m$ and the description of the modified component after applying (ad 1) that N belongs to the rectangle determined by P_m in $\mathcal{C}'_m$, and so the projectives inserted by the block of (ad 1)'s lie on the coray ending in R, where R is the last of them. ■

6.3. Note that each exceptional projective P_i determines a unique exceptional injective Q_i lying on the rectangle determined by P_i in $\mathcal{C}_i'$. The reducibility of the defining sequence of $(B,\mathcal{C})$ implies that the number of corays in the corectangle determined by Q_i in $\mathcal{C}_i'$ is the same as the number of rays in the rectangle determined by P_i in $\mathcal{C}_i'$.

LEMMA. *Let r be the least positive integer such that for $r < i \le m$, $(A_i,\mathcal{C}_i)$ is obtained from $(A_{i-1}',\mathcal{C}_{i-1}')$ by a block of operations of type* (ad 1). *Then $r = m$.*

Proof. Let $m > 2$, for otherwise there is nothing to prove. Assume that $r < m$, and consider the weakly sincere indecomposable N in $\mathcal{C}_m'$ which belongs to the rectangle determined by P_m in $\mathcal{C}_m'$. As above, the reducibility of the defining sequence of $(B,\mathcal{C})$ and the standardness of $\mathcal{C}_m'$ imply that the projectives inserted by the block of operations of type (ad 1) that transforms $(A_{m-2}',\mathcal{C}_{m-2}')$ into $(A_{m-1},\mathcal{C}_{m-1})$ lie on the coray ending at the mouth module at which the ray that contains U starts, and the pivot of such block is any ray module on the ray forming the upper border of the rectangle determined by P_{m-2} in $\mathcal{C}_{m-2}'$. If $N = X_i' = (K,X_i,1)$ for some $i \ge 0$, then X_i is a weakly sincere indecomposable in $\mathcal{C}_m$, and as such it belongs to the corectangle determined by Q_{m-1} in $\mathcal{C}_m$. But no module in this corectangle sends non-zero morphisms to the indecomposable injective A_m-module Y_t. Therefore $N = Z_{ij} = (K, X_i \oplus Y_j, \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right))$ for some $i \ge 1$ and some $1 \le j \le t$. Then $X_i \oplus Y_j$ is a weakly sincere A_m-module. By (6.2) and the description of the modified component after applying (ad 2), $X_i|_{A_{m-1}} = U_k$ for some $k > i$. Thus $U_k \oplus Y_j$ is a weakly sincere A_{m-1}-module, and as such sends non-zero morphisms to the indecomposable injective A_{m-1}-modules Q_{m-2} and V_s. Since Y_j does not do that, U_k should. But again, if U_k belongs to the corectangle determined by Q_{m-2} in $\mathcal{C}_{m-1}$, then $\mathrm{Hom}_{A_{m-1}}(U_k,V_s) = 0$. This contradiction proves the lemma. ∎

6.4. The modules $Y_1,\dots,Y_t$ in $\mathcal{S}(X)$ arise from applying one or several operations of type (ad 1*) which form a branch in the sense of [24, (4.4)]. We denote by H the branch obtained from the latter by deleting its branching point. Then $Y_1,\dots,Y_t$ are indecomposable H-modules. Similarly, the modules $V_1,\dots,V_s$ in $\mathcal{S}(U)$ are indecomposable modules over a certain branch H'. Note that the reducibility of the defining sequence of $(B,\mathcal{C})$ implies that the branches H and H' have respectively t and s vertices.

LEMMA. (i) *There are unique integers $1 \le j \le t$ and $1 \le l \le s$ such that Y_j is a weakly sincere H-module and V_l is a weakly sincere H'-module.*

(ii) *The weakly sincere modules in $\mathcal{C}_m'$ are of the form $Z_{ij} = (K, X_i \oplus Y_j, \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right))$, where X_i is an indecomposable in $\mathcal{C}_m$ that lies on the same coray* *as V_l and sends non-zero morphisms to all the indecomposable injective A_m-modules but those that are H-modules.*

Proof. We prove (i) and (ii) jointly. Let M be a weakly sincere indecomposable in $\mathcal{C}'_m$. Since there are no weakly sincere modules in $\mathcal{C}_m$, $M = Z_{ij} = (K, X_i \oplus Y_j, (\begin{smallmatrix}1\\1\end{smallmatrix}))$ for some $i \geq 1$ and $1 \leq j \leq t$, and so $X_i \oplus Y_j$ is a weakly sincere A_m-module. Therefore, X_i sends non-zero morphisms to all the indecomposable injective A_m modules which are not H-modules, in particular, to Q_{m-1}. Thus, X_i belongs to the corectangle determined by Q_{m-1} in $\mathcal{C}_m$, and $\mathrm{Hom}_{A_m}(X, I) = 0$ for every indecomposable injective H-module I. Hence Y_j is a weakly sincere H-module, the indecomposable injective H-modules appear in a single stairs configuration, and j is the unique index with this property. Let $U_k = X_i|_{A_{m-1}}$. Then U_k sends non-zero morphisms to all the indecomposable injective H'-modules. Consequently, these appear in a single stairs configuration, and there is a unique index l such that V_l is a weakly sincere H'-module. Clearly, X_i, U_k and V_l all lie on the same coray, thus proving that M is as stated above. The converse is clear. Note that $j = 1$ (respectively, $l = 1$) if X (respectively, U) is an (ad 2)-pivot. ∎

6.5. COROLLARY. *$(B, \mathcal{C})$ is obtained from $(A'_m, \mathcal{C}'_m)$ by a block of operations of type* (ad 1) *with pivot* $\operatorname{soc} Y_1$ (*respectively*, Y_t) *if X is an* (ad 2)*-pivot* (*respectively*, (ad 3)*-pivot*), *followed by a block of operations of type* (ad 1*) *with pivot V_s* (*respectively*, V_1) *if U is an* (ad 2)*-pivot* (*respectively*, (ad 3)*-pivot*). *Moreover, the projectives and injectives inserted respectively by those two blocks appear in a single stairs configuration.*

Proof. This follows from (6.4) and the description of the modified component after applying (ad 2) or (ad 3). ∎

6.6. Let $m > 2$. By (6.3), $(A_{m-1}, \mathcal{C}_{m-1})$ is obtained from $(A'_{m-2}, \mathcal{C}'_{m-2})$ by a block of operations of type (ad 1) followed by a block of operations of type (ad 1*). The next lemma describes precisely this sequence.

LEMMA. *$(A_{m-1}, \mathcal{C}_{m-1})$ is obtained from $(A'_{m-2}, \mathcal{C}'_{m-2})$ by a sequence consisting of*:

(i) *a block of consecutive one-point extensions*,

(ii) *a block of consecutive one-point coextensions with pivot the last projective inserted by the preceding block*,

(iii) *the block of* (ad 1*) *'s that insert $Y_1, \ldots, Y_t$, with pivot the last injective inserted by the preceding block*,

(iv) *a block of consecutive one-point coextensions with pivot* $\tau(\operatorname{soc} Y_1)$, *and*

(v) *the block of* (ad 1*) *'s that insert $V_1, \ldots, V_s$, with pivot the last injective inserted by the preceding block.*

Moreover, if X *(respectively,* U*) is an* (ad 3)*-pivot, the injectives inserted by the block of* (ad 1^*)*'s that insert* $Y_1, \ldots, Y_t$ *(respectively,* $V_1, \ldots, V_s$*) appear in a single stairs configuration.*

Proof. Let N be a weakly sincere indecomposable in $\mathcal{C}'_m$. Then $N = Z_{ij} = (K, X_i \oplus Y_j, \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right))$ is as stated in (6.4). Let $U_k = X_i|_{A_{m-1}}$. Then U_k sends non-zero morphisms to all the indecomposable injective A_{m-1}-modules except those which are H-modules. By (6.4), U_k lies on the intersection of the coray through V_l and the ray through U, which starts at $\tau(\operatorname{soc} V_1)$ due to the reducibility of the defining sequence of $(B, \mathcal{C})$. Hence, the injectives inserted in $\mathcal{C}'_{m-2}$ by the sequence, except those which are H- or H'-modules, lie on the ray through U, and the projectives inserted in $\mathcal{C}'_{m-2}$ by the sequence lie on the coray ending at $\tau(\operatorname{soc} V_1)$, this module being the last of them. This proves (i)–(v). The last assertion follows from the proof of (6.4). ∎

6.7. Corollary. *If* $m > 2$*, then* P_{m-2} *is projective-injective, the pivot of the sequence that transforms* $(A'_{m-2}, \mathcal{C}'_{m-2})$ *into* $(A_{m-1}, \mathcal{C}_{m-1})$ *is a simple module on one of the rays that cross the rectangle determined by* P_{m-2} *in* $\mathcal{C}'_{m-2}$*, and* $(A'_{m-2}, \mathcal{C}'_{m-2})$ *falls into case* (b).

Proof. Let $N = Z_{ij} = (K, X_i \oplus Y_j, \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right))$ and $U_k = X_i|_{A_{m-1}}$ be as in the proof of (6.6). Let $M = U_k|_{A'_{m-2}}$ and L be the pivot of the sequence transforming $(A'_{m-2}, \mathcal{C}'_{m-2})$ into $(A_{m-1}, \mathcal{C}_{m-1})$. Then M is a weakly sincere indecomposable in $\mathcal{C}'_{m-2}$ lying on the ray that starts at L and the coray that ends at L, and belonging to the rectangle determined by P_{m-2} in $\mathcal{C}'_{m-2}$. From (6.6), the reducibility of the defining sequence of $(B, \mathcal{C})$ and the description of the modified component after applying (ad 2) or (ad 3), it follows that P_{m-2} is projective-injective, and L is a simple module on one of the rays that cross the rectangle determined by P_{m-2} in $\mathcal{C}'_{m-2}$. In fact, if $R = \operatorname{rad} P_{m-2}$ and the support $\mathcal{S}(R)$ of $\operatorname{Hom}_{A_{m-2}}(R, -)|_{\mathcal{C}_{m-2}}$ is

$$T_r \leftarrow \cdots \leftarrow T_2 \leftarrow T_1 \leftarrow R = R_0 \rightarrow R_1 \rightarrow R_2 \rightarrow \cdots$$

where $r \geq 1$, and $R, T_1, \ldots, T_r$ are injectives, then L is one of the modules $\operatorname{soc} T_1, \ldots, \operatorname{soc} T_{r-1}$, T_r which all lie in the same τ-orbit. By (6.5)(2), $(A'_{m-2}, \mathcal{C}'_{m-2})$ falls into case (b). ∎

7. Braid algebras of forms (3) to (17)

7.1. The aim of this section is the analysis of case (b). We first assume that P_m is projective-injective. Let $X = \operatorname{rad} P_m$ and let the support $\mathcal{S}(X)$ of $\operatorname{Hom}_{A_m}(X, -)|_{\mathcal{C}_m}$ be

$$Y_t \leftarrow \cdots \leftarrow Y_2 \leftarrow Y_1 \leftarrow X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots$$

where $t \geq 1$ and $X, Y_1, \ldots, Y_t$ are injectives. By the reducibility of $(B, \mathcal{C})$, the number of corays and the number of rays of the corectangle and the rectangle determined by P_m in $\mathcal{C}'_m$ are the same.

LEMMA. *If P_m is projective-injective, then the weakly sincere modules in $\mathcal{C}'_m$ are of the form $X'_i = (K, X_i, 1)$, $Z_{ij} = (K, X_i \oplus Y_j, (\begin{smallmatrix}1\\1\end{smallmatrix}))$ with $1 \le j \le t$, and $Z_{i-j,l} = (K, X_{i-j} \oplus Y_l, (\begin{smallmatrix}1\\1\end{smallmatrix}))$ with $1 \le j \le t$ and $1 \le l \le t-j+1$, where X_i with $i > 0$ is a weakly sincere indecomposable in $\mathcal{C}_m$ lying on the coray containing $Y_1, \ldots, Y_t$.*

Proof. Let M be a weakly sincere indecomposable in $\mathcal{C}'_m$. Then M belongs to the rectangle determined by P_m in $\mathcal{C}'_m$. If $M = (K, X_i, 1)$ for some $i \ge 0$, then $i \ne 0$. Indeed, otherwise $P_m = X'_0$ is weakly sincere, and as $m \ge 2$, there is an oriented cycle between projective-injectives in $\mathcal{C}'_m$, contradicting [4, (4.5)]. Hence X_i with $i > 0$ is a weakly sincere indecomposable in $\mathcal{C}_m$, and consequently, it lies on the coray ending in Y_t. Therefore, M is one of the modules described above. If $M = Z_{kl} = (K, X_k \oplus Y_l, (\begin{smallmatrix}1\\1\end{smallmatrix}))$ for some $k \ge 1$ and $1 \le l \le t$, then $X_k \oplus Y_l$ is a weakly sincere A_m-module. Since $\mathrm{Hom}_{A_m}(Y_l, X) = 0$, X_k belongs to the corectangle determined by X in $\mathcal{C}_m$. Let $i \ge k$ with $i - k \le t$ be such that X_i lies on the coray ending in Y_t. It follows from the description of the modified component after applying (ad 1*) and the standardness of $\mathcal{C}_m$ that X_i is a weakly sincere indecomposable in $\mathcal{C}_m$. If $k = i$, then M is one of the modules described above. If $k \ne i$, then $k = i - j$ for some $1 \le j \le t$. Since $\mathrm{Hom}_{A_m}(X_{i-j}, Y_r) = 0$ for $t - j + 1 \le r \le t$, we have $1 \le l \le t - j + 1$, and thus M is one of the modules described above. The converse is clear. ∎

7.2. COROLLARY. *If P_m is projective-injective, then $(B, \mathcal{C})$ is obtained from $(A'_m, \mathcal{C}'_m)$ by a block of operations of type* (ad 1) *with pivot V followed by a block of operations of type* (ad 1*) *with pivot L, where V is any of the modules $\mathrm{soc}\, Y_1, \ldots, \mathrm{soc}\, Y_{t-1}$, Y_t, $X'_1 = (K, X_1, 1)$, and L depends on V as follows*:

(i) *If $V = \mathrm{soc}\, Y_1$, then L is either ${}'X_1 = (X_1, K, 1)$, or $\mathrm{soc}\, Y_i$ with $i > 1$, or any coray module on the coray through $\mathrm{soc}\, Y_1$.*

(ii) *If $V = \mathrm{soc}\, Y_j$ with $1 < j \le t$, then L is either $\mathrm{soc}\, Y_{j-1}$, or $\mathrm{soc}\, Y_i$ with $i > j$, or any coray module on the coray through $\mathrm{soc}\, Y_j$.*

(iii) *If $V = (K, X_1, 1)$, then $L = Y_t$.*

Moreover, the projectives and injectives inserted respectively by these two blocks appear in a (*broken line*, *single stairs*) *configuration.*

7.3. LEMMA. *If P_m is projective-injective, then $(A_m, \mathcal{C}_m)$ is obtained from $(A'_{m-1}, \mathcal{C}'_{m-1})$ by a sequence consisting of*:

(i) *a block of consecutive one-point extensions*,

(ii) *a block of consecutive one-point coextensions with pivot the last projective inserted by the preceding block*, *and*

(iii) *the operation of type* (ad 1^*) *that inserts* $X, Y_1, \ldots, Y_t$, *with pivot the last injective inserted by the preceding block.*

Moreover, P_{m-1} *is also projective-injective*, *the pivot of such a sequence is a simple module on any of the rays that cross the rectangle determined by* P_{m-1} *in* $\mathcal{C}'_{m-1}$, *and* $(A'_{m-1}, \mathcal{C}'_{m-1})$ *falls into case* (b).

Proof. Let N be a weakly sincere indecomposable in $\mathcal{C}'_m$. Since P_m is projective-injective, N belongs both to the rectangle and the corectangle determined by P_m in $\mathcal{C}'_m$. Hence, the projectives inserted in $\mathcal{C}'_{m-1}$ by the sequence lie on the coray ending at ${}'X_1 = (X_1, K, 1)$, this module being the last of them, and the injectives inserted in $\mathcal{C}'_{m-1}$ by the sequence lie on the ray starting at $X'_1 = (K, X_1, 1)$, this module being the last of them. This proves (i)–(iii). The rest of the proof is similar to that of (6.7). ∎

7.4. We now assume that P_m is not injective. Let $X = \operatorname{rad} P_m$ and let the support $\mathcal{S}(X)$ of $\operatorname{Hom}_{A_m}(X, -)|_{\mathcal{C}_m}$ be

$$\begin{array}{ccccccccccc} Y_1 & \to & Y_2 & \to & \cdots & \to & Y_t & & & & \\ \uparrow & & \uparrow & & & & \uparrow & & & & \\ X = X_0 & \to & X_1 & \to & \cdots & \to & X_{t-1} & \to & X_t & \to & \cdots \end{array}$$

where $t \geq 2$ and X_{t-1}, Y_t are injectives. Again the reducibility of the defining sequence of $(B, \mathcal{C})$ implies that the number of rays in the rectangle determined by P_m in $\mathcal{C}'_m$ equals the number of corays in the corectangle determined by Q_m in $\mathcal{C}'_m$.

LEMMA. *If* P_m *is not injective*, *then* $(A_m, \mathcal{C}_m)$ *is obtained from* $(A'_{m-1}, \mathcal{C}'_{m-1})$ *by a sequence consisting of*:

(i) *a block of consecutive one-point extensions*,

(ii) *a block of consecutive one-point coextensions with pivot the last projective inserted by the preceding block*, *and*

(iii) *the block of operations of type* (ad 1^*) *that insert* $Y_1, \ldots, Y_t$, *with pivot the last injective inserted by the preceding block.*

Moreover, *the injectives inserted by the last block of operations of type* (ad 1^*) *appear either in a single stairs configuration or in a double stairs configuration*, P_{m-1} *is projective-injective*, *and the pivot of such a sequence is any of the simple modules on the rays that cross the rectangle determined by* P_{m-1} *in* $\mathcal{C}'_{m-1}$.

Proof. We only prove that the injectives inserted by the block of (ad 1^*)'s that give rise to $Y_1, \ldots, Y_t$ appear either in a single stairs or in a double stairs configuration, since the proof of the remaining statements is similar to the proof of (6.7).

The weakly sincere indecomposable N in $\mathcal{C}'_m$ belongs to the rectangle determined by P_m in $\mathcal{C}'_m$. If $N = X'_i = (K, X_i, 1)$ for some $i \geq 0$, then X_i is a weakly sincere indecomposable in $\mathcal{C}_m$. By the dual of (5.2), the injectives inserted by the block above appear in a single stairs configuration. Assume then that $N = Z_{ij} = (K, X_i \oplus Y_j, (\begin{smallmatrix}1\\1\end{smallmatrix}))$ for some $i \geq 1$ and $1 \leq j \leq t$, and that the injectives inserted by the block above do not appear in a single stairs configuration. Since $X_i \oplus Y_j$ is a weakly sincere A_m-module, we may assume that X_i lies on the coray containing all the injectives inserted by the last operation in the block. Let $r > 1$ be the greatest positive integer such that the pivot of the $(r+1)$th operation in the block is neither the unique injective inserted by the rth operation, in case it has parameter zero, nor the simple injective inserted by the rth operation, in case it has non-zero parameter. Then X_i cannot send non-zero morphisms to all the injectives inserted by the rth operation in the block, and so Y_j must lie on the coray containing them. It follows from the weak sincerity of $X_i \oplus Y_j$ and the standardness of $\mathcal{C}_m$ that the injectives inserted by the block must appear in a double stairs configuration. ∎

7.5. The block of operations of type (ad 1*) that insert $Y_1, \ldots, Y_t$ form a branch. If we denote by H the branch obtained from the latter by deleting its branching point, then $Y_1, \ldots, Y_t$ are indecomposable H-modules.

LEMMA. *If P_m is not injective and the injectives inserted by the block of operations of type* (ad 1*) *that give rise to $Y_1, \ldots, Y_t$ appear in a single stairs configuration, then*:

(i) *There is a unique integer $1 \leq l \leq t$ such that Y_l is a weakly sincere H-module.*

(ii) *The weakly sincere modules in $\mathcal{C}'_m$ are of the form $X'_i = (K, X_i, 1)$, $Z_{ij} = (K, X_i \oplus Y_j, (\begin{smallmatrix}1\\1\end{smallmatrix}))$ with $1 \leq j \leq t$, $Z_{i-k,l} = (K, X_{i-k} \oplus Y_l, (\begin{smallmatrix}1\\1\end{smallmatrix}))$ with $1 \leq k \leq l-1$, and $Z_{i+k,l} = (K, X_{i+k} \oplus Y_l, (\begin{smallmatrix}1\\1\end{smallmatrix}))$ with $1 \leq k \leq t-l$, where X_i with $i > 0$ is a weakly sincere indecomposable in $\mathcal{C}'_m$ lying on the same coray as Y_l.*

Proof. (i) Let $1 \leq l \leq t$ be such that Y_l lies on the coray containing all the injectives inserted by the last operation in the block above.

(ii) Let M be a weakly sincere indecomposable in $\mathcal{C}'_m$. Then M belongs to the rectangle determined by P_m in $\mathcal{C}'_m$. If $M = X'_i = (K, X_i, 1)$ for some $i \geq 0$, then as in the proof of (7.1), M is one of the modules described above. If $M = Z_{r,j} = (K, X_r \oplus Y_j, (\begin{smallmatrix}1\\1\end{smallmatrix}))$ for some $r \geq 1$ and $1 \leq j \leq t$, then $X_r \oplus Y_j$ is a weakly sincere A_m-module. Since $\mathrm{Hom}_{A_m}(Y_j, X_{t-1}) = 0$, X_r belongs to the corectangle determined by X_{t-1} in $\mathcal{C}_m$. Let $i \geq 1$ be such that $|i - r| \leq t$ and X_i lies on the coray through Y_l. Then X_i is a weakly sincere indecomposable in $\mathcal{C}_m$. If $r = i$, then M is one of the modules described above. If $r \neq i$, then $j = l$ since only the modules on the coray through

Y_l send non-zero morphisms to all the indecomposable injective H-modules. Hence either $r = i - k$ with $1 \le k \le l-1$, or $r = i + k$ with $1 \le k \le t - l$, and thus M is one of the modules described above. The converse is clear. ∎

7.6. Corollary. *Under the hypothesis of the above lemma, we have*:

(i) *If* $l \neq 1, t$, *then* $(B, \mathcal{C})$ *is obtained from* $(A'_m, \mathcal{C}'_m)$ *either by a sequence of operations of type* $(\mathrm{ad}\,1)$ *with pivot* Y_t *or* $X'_t = (K, X_t, 1)$, *or by a sequence of operations of type* $(\mathrm{ad}\,1^*)$ *with pivot* $'X_t = (X_t, K, 1)$ *or* Y_1.

(ii) *If* $l = 1$, *then* $(B, \mathcal{C})$ *is obtained from* $(A'_m, \mathcal{C}'_m)$ *either by a sequence of* $(\mathrm{ad}\,1)$*'s with pivot* Y_t *or* $X'_t = (K, X_t, 1)$ *followed by a sequence of* $(\mathrm{ad}\,1^*)$*'s with pivot* Y_1, *or just by a sequence of* $(\mathrm{ad}\,1^*)$*'s with pivot* $'X_t = (X_t, K, 1)$.

(iii) *If* $l = t$, *then* $(B, \mathcal{C})$ *is obtained from* $(A'_m, \mathcal{C}'_m)$ *either by a sequence of* $(\mathrm{ad}\,1)$*'s with pivot* Y_t *followed by a sequence of* $(\mathrm{ad}\,1^*)$*'s with pivot* $'X_t = (X_t, K, 1)$ *or* Y_1, *or just by a sequence of* $(\mathrm{ad}\,1)$*'s with pivot* $X'_t = (K, X_t, 1)$.

In any case, *the projectives* (*respectively*, *injectives*) *inserted by such sequences appear in a single stairs configuration.*

7.7. Lemma. *If* P_m *is not injective and the injectives inserted by the block of* $(\mathrm{ad}\,1^*)$*'s that give rise to* $Y_1, \ldots, Y_t$ *appear in a double stairs configuration*, *then*:

(i) *There are unique integers* $1 \le j < l \le t$ *such that* $Y_j \oplus Y_l$ *is a weakly sincere* H*-module.*

(ii) *The weakly sincere modules in* $\mathcal{C}'_m$ *are of the form* $Z_{ij} = (K, X_i \oplus Y_j, \binom{1}{1})$ *and* $Z_{k,l} = (K, X_k \oplus Y_l, \binom{1}{1})$ *where* X_i *and* X_k *are indecomposables in* $\mathcal{C}_m$ *that lie respectively on the same corays as* Y_l *and* Y_j *and that send non-zero morphisms to all the indecomposable injective* A_m*-modules which are not* H*-modules.*

Proof. (i) Let $r > 1$ be such that the pivot of the $(r+1)$th operation in the block above is neither the unique injective inserted by the rth operation in case it has parameter zero, nor the simple injective inserted by the rth operation in case it has non-zero parameter. Let $1 \le j < l \le t$ be such that Y_j and Y_l lie respectively on the coray containing the injectives inserted by the last and the rth operations in the block.

(ii) Let M be a weakly sincere indecomposable in $\mathcal{C}'_m$. Then M belongs to the rectangle determined by P_m in $\mathcal{C}'_m$. Then $M = Z_{rs} = (K, X_r \oplus Y_s, \binom{1}{1})$ for some $r \ge 1$ and $1 \le s \le t$, for otherwise the injectives inserted by the block above appear in a single stairs configuration. Hence $X_r \oplus Y_s$ is a weakly sincere A_m-module, and the only way in which it can send non-zero morphisms to all the indecomposable injective H-modules is that either $s = j$ and X_r lies on the coray through Y_l, or $s = l$ and X_r lies on the coray through Y_j. Therefore M is one of the modules described above. The converse is clear. ∎

7.8. Corollary. *Under the hypothesis of the lemma*, $(B, \mathcal{C})$ *is obtained from* $(A'_m, \mathcal{C}'_m)$ *by*:

(i) *a sequence of* $(\mathrm{ad}\,1^*)$*'s with pivot* Y_1 *if* $j = 1$ *and* $l \neq t$,

(ii) *a sequence of* $(\mathrm{ad}\,1)$*'s with pivot* Y_t *if* $j \neq 1$ *and* $l = t$,

(iii) *a sequence of* $(\mathrm{ad}\,1)$*'s with pivot* Y_t *followed by a sequence of* $(\mathrm{ad}\,1^*)$*'s with pivot* Y_1 *if* $j = 1$ *and* $l = t$.

In each case, *the projectives* (*respectively*, *injectives*) *inserted by such sequences appear in a single stairs configuration.*

It follows inductively from what we have shown in Sections 6 and 7 that the list of bound quiver algebras given in Section 3 is complete, and so is the list of their weakly sincere indecomposable representations given in Section 4. Those coil enlargements of C belonging to case (a) give braid algebras of forms (18) to (25), and those belonging to case (b) give braid algebras of forms (3) to (17). In particular, the braid algebras of forms (3) to (5) correspond to the case where P_m is projective-injective, and the braid algebras of forms (6) to (17) correspond to the case where P_m is not injective, that is, $X = \mathrm{rad}\, P_m$ is an $(\mathrm{ad}\,3)$-pivot, those of forms (6) to (13) correspond to the case where the injectives inserted by the block of $(\mathrm{ad}\,1^*)$'s that give rise to X appear in a single stairs configuration, and those of forms (14) to (17) correspond to the case where they appear in a double stairs configuration.

8. Representations of polynomial growth strongly simply connected algebras

8.1. Let A be an algebra and $K[x]$ the polynomial algebra in one variable. Following [13], A is said to be *tame* if, for any dimension d, there exist a finite number of $K[x]$-A-bimodules M_i, $1 \leq i \leq n_d$, which are finitely generated and free as left $K[x]$-modules, and all but a finite number of isomorphism classes of indecomposable A-modules of dimension d are of the form $K[x]/(x - \lambda) \otimes_{K[x]} M_i$ for some $\lambda \in K$ and some i. Let $\mu_A(d)$ be the least number of $K[x]$-A-bimodules satisfying the above conditions for d. Then A is said to be of *polynomial growth* [27] if there exists a positive integer m such that $\mu_A(d) \leq d^m$ for any $d \geq 1$. From the validity of the second Brauer–Thrall conjecture we know that A is representation-finite if and only if $\mu_A(d) = 0$ for any $d \geq 1$.

8.2. Following [28] an algebra A is called *strongly simply connected* if its Gabriel quiver has no oriented cycles and the first Hochschild cohomology $H^1(C, C)$ of any full convex subcategory C of A (considered as a finite K-category) vanishes. It is known that a representation-finite algebra A is strongly simply connected if and only if it is simply connected in the sense of [8] (the geometric realization of the quiver Γ_A is simply connected).

8.3. Following [9], by a *critical algebra* we mean an algebra of the form $\mathrm{End}_H(T)$, where H is the path algebra $K\Delta$ of a quiver Δ of Euclidean type $\widetilde{\mathbb{D}}_n$ $(n \geq 4)$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ or $\widetilde{\mathbb{E}}_8$ and T is a postprojective tilting H-module, that is, $\mathrm{Ext}^1_H(T,T) = 0$ and T is a direct sum of $|\Delta_0|$ (number of vertices of Δ) pairwise non-isomorphic indecomposable H-modules lying in τ_H-orbits of indecomposable projective H-modules. We refer to [9] and [15] for a complete classification of critical algebras by quivers and relations. Recall also [9] that a strongly simply connected algebra A is representation-finite if and only if A does not contain a full convex critical subcategory. Moreover, all indecomposable modules over representation-finite strongly simply connected algebras are directing, and their structure is known (see [8], [11], [12], [25]).

8.4. The representation theory of strongly simply connected algebras of polynomial growth has been established in [34]. In contrast to the representation-finite case, besides the indecomposable directing modules, whose supports are tame tilted algebras [24] described in [20], [21], there are many non-directing indecomposable modules. It has been shown in [34, Corollary 4.7] that the one-parameter families of indecomposable modules over polynomial growth strongly simply connected algebras are the one-parameter families of indecomposable modules given by their full convex critical and tubular (in the sense of [24]) subcategories. The remaining non-directing (discrete) indecomposable modules over polynomial growth strongly simply connected algebras are those belonging to coils of convex subcategories which are coil enlargements of critical algebras [34, Corollary 4.8]. Therefore, applying Theorem 3.3 and the results of Section 4, we obtain the following complete classification of all non-directing indecomposable modules over strongly simply connected algebras of polynomial growth.

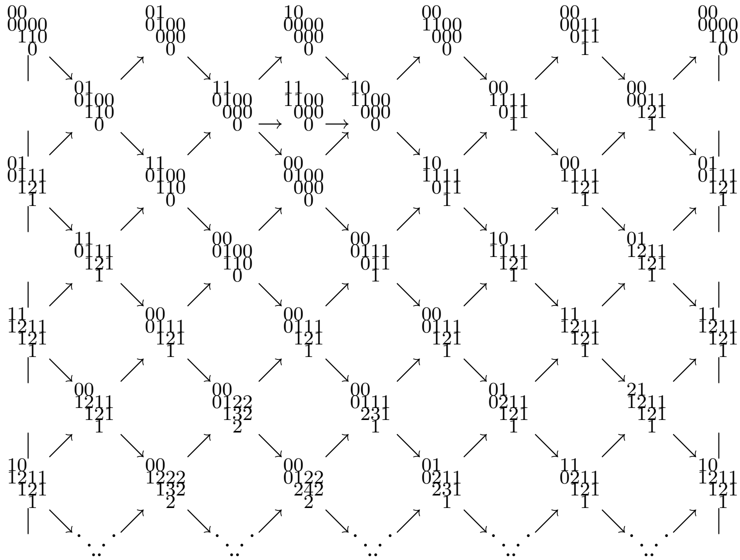
THEOREM. *Let A be a strongly simply connected algebra of polynomial growth and M a non-directing indecomposable A-module. Then there exists a convex subcategory B of A which is tubular or a braid algebra on a critical algebra such that M is a weakly sincere indecomposable B-module lying in a coil of Γ_B.*

9. Examples

9.1. Let A be given by the bound quiver

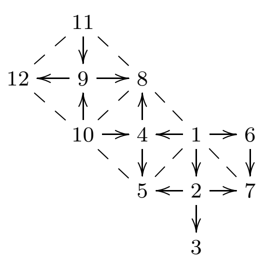
$$\begin{tikzcd}
9 \arrow[r] & 8 & & \\
10 \arrow[u] \arrow[r] \arrow[ur, dash, dashed] \arrow[dr, dash, dashed] & 4 \arrow[u] \arrow[d] & 1 \arrow[l] \arrow[r] \arrow[d] \arrow[ul, dash, dashed] \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & 6 \arrow[d] \\
 & 5 & 2 \arrow[l] \arrow[r] \arrow[d] & 7 \\
 & & 3 &
\end{tikzcd}$$

Then A is a tame simple braid algebra with coil type $c_A = ((5,3,2),(5,3,2))$, in the sense of [7], and Γ_A contains a (weakly) sincere quasi-tube

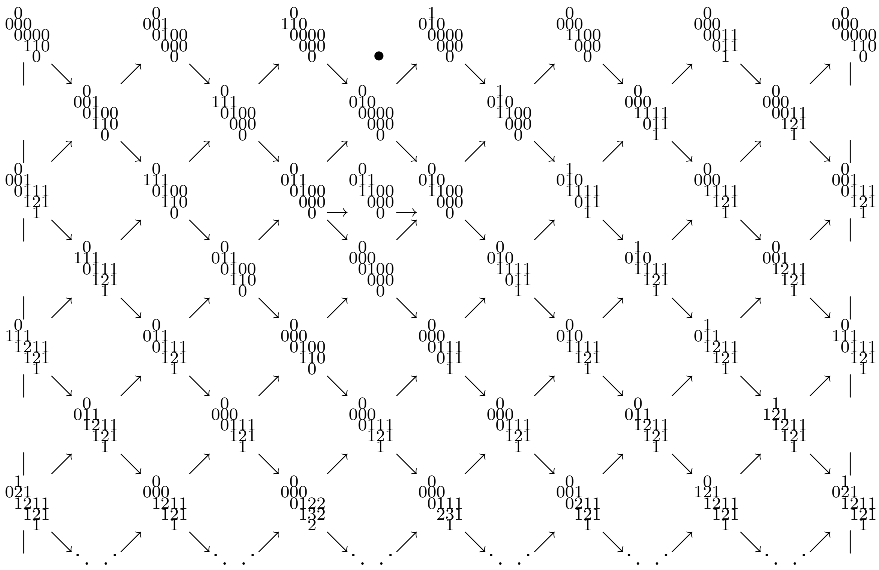


where the indecomposable modules are replaced by their dimension-vectors and the dashed lines have to be identified in order to obtain a (weakly) sincere quasi-tube. We can see three (weakly) sincere indecomposables with dimension-vectors $\begin{smallmatrix}11\\1211\\121\\1\end{smallmatrix}$, $\begin{smallmatrix}21\\1211\\121\\1\end{smallmatrix}$, $\begin{smallmatrix}11\\1211\\121\\1\end{smallmatrix}$ lying on the intersection of the rectangle and the corectangle determined by the unique projective-injective module. If we denote by $X = X_0$ the radical of the unique projective-injective and by Y_1 the simple $S(9)$, then these modules are, respectively, $Z_{41} = (K, X_4 \oplus Y_1, \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right))$, $Z_{51} = (K, X_5 \oplus Y_1, \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right))$ and $\overline{X}_5 = (K, X_5, 1)$ (in the notation of (3.8)). The remaining (weakly) sincere indecomposables are located similarly on the intersections of the rectangle and the corectangle determined by the unique projective-injective.

9.2. Let A be given by the bound quiver

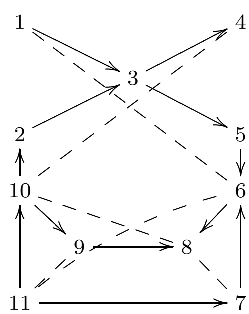


Then A is a tame braid algebra of type (2) having coil type $c_A = ((6,3,2), (6,3,2))$, and Γ_A contains the (weakly) sincere quasi-tube

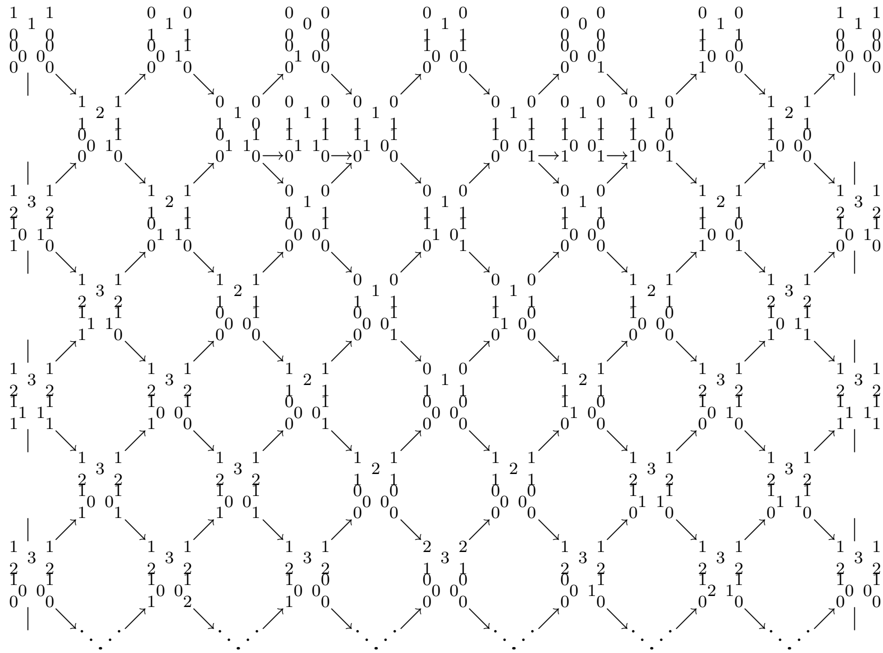


where $X \searrow \bullet \nearrow Y$ denotes that X is injective and Y is projective, the indecomposable modules are replaced by their dimension-vectors and the dashed lines have to be identified in order to obtain a (weakly) sincere quasi-tube. We can see a (weakly) sincere indecomposable with dimension-vector $\begin{smallmatrix} 1 \\ 121 \\ 1211 \\ 121 \\ 1 \end{smallmatrix}$ lying on the intersection of the ray starting at $P(11)$ and the coray ending at $I(12)$. The remaining (weakly) sincere indecomposables correspond to the other points of intersection of the ray and the coray mentioned above.

9.3. Let A be given by the bound quiver



Then A is a tame braid algebra of type (3) with coil type $c_A = ((6,2,2), (6,2,2))$, and Γ_A contains the (weakly) sincere quasi-tube

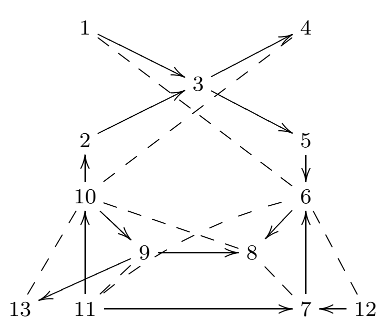


where the indecomposable modules are replaced by their dimension-vectors and the dashed lines have to be identified in order to obtain a (weakly) sincere quasi-tube. We can see a (weakly) sincere indecomposable with dimension-vector

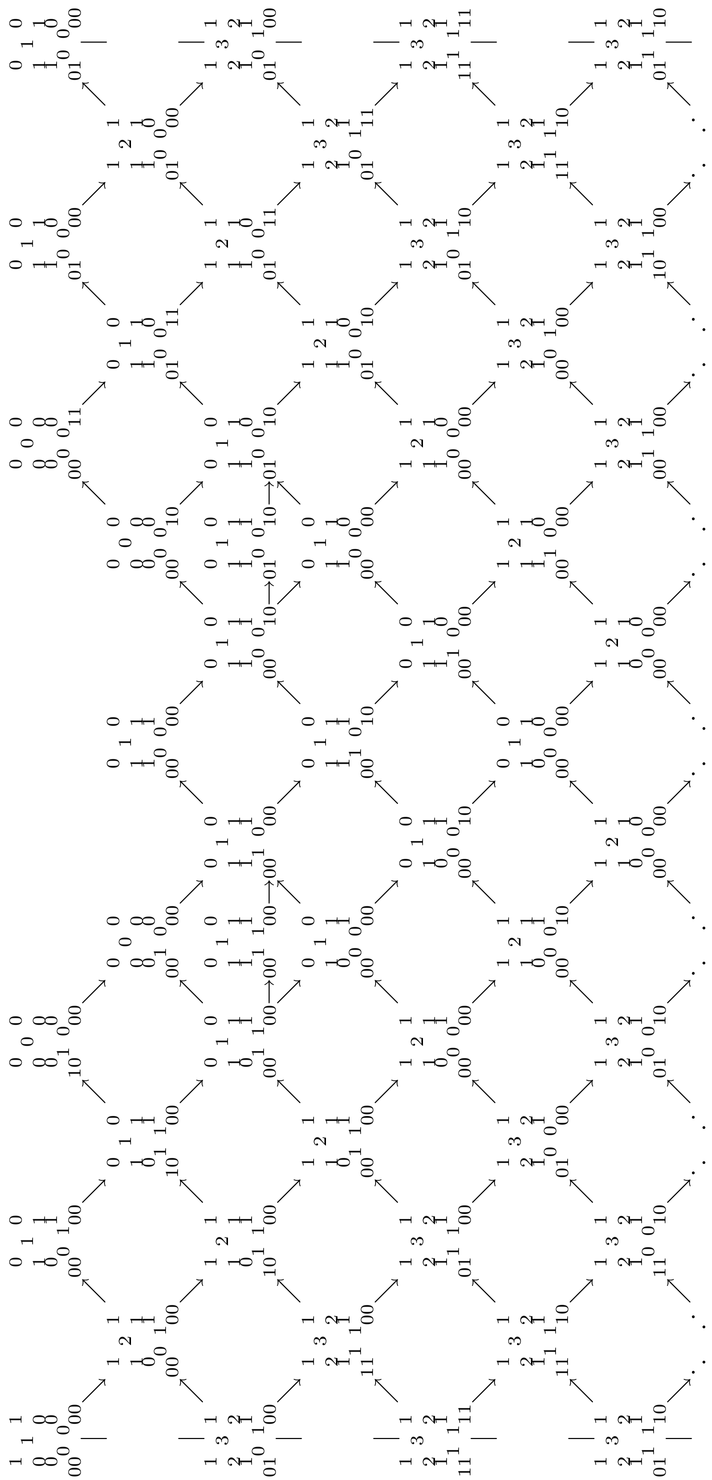
$$\begin{smallmatrix} 1 & & & & 1\\ & & 3 & & \\ 2 & & & & 2\\ 1 & & & & 1\\ & 1 & & 1 & \\ 1 & & & & 1\end{smallmatrix}$$

lying on the intersection of the ray starting at $S(7)$ and the coray ending at $S(9)$. The remaining (weakly) sincere indecomposables correspond to the other points of intersection of the ray and the coray mentioned above.

9.4. Let A be given by the bound quiver



Then A is a tame braid algebra of type (4) having coil type $c_A = ((7,2,2), (7,2,2))$, and Γ_A contains the (weakly) sincere quasi-tube on p. 128, where



the indecomposable modules are replaced by their dimension-vectors and the dashed lines have to be identified in order to obtain a (weakly) sincere quasi-tube. We can see a (weakly) sincere indecomposable with dimension-vector

$$\begin{smallmatrix} & 1 & & & 1 & \\ & & 3 & & & \\ & 2 & & & 2 & \\ & 1 & & & 1 & \\ & & 1 & 1 & & \\ 11 & & & & & 11 \end{smallmatrix}$$

lying on the intersection of the ray starting at $P(12)$ and the coray ending at $I(13)$. The remaining (weakly) sincere indecomposables correspond to the other points of intersection of the ray and the coray mentioned above.

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