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PSEUDO SHIFT OPERATORS WITH LARGE IMAGES

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Abstract. We give suitable conditions for the existence of many holomorphic functions f on a disc such that the image of any nonempty open subset under the action of pseudo shift operators on f is arbitrarily large. This generalizes an earlier result about images of derivatives and completes another one on infinite order differential operators.

1. Introduction. In 1987 R. M. Gethner and J. H. Shapiro [4, Theorem 2.4] proved that for any continuous linear operator T on a separable complete linear metric space X whose iterates T^n converge to zero on a dense subset of X and any sequence (x_n) in X converging to zero $(n \to \infty)$, the set of vectors x in X for which $\liminf_{n\to\infty} \|T^n x - T^n x_n\| = 0$ is a dense G_δ subset of X. Here and in what follows, we write $\|x\| = d(x, 0)$ for $x \in X$, where d is a translation invariant metric in X.

For instance, given a domain G of the complex plane $\mathbb{C}$ and an exhaustive nondecreasing sequence (K_n) of compact subsets of G, the translation invariant metric

$$d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\sup_{K_n} |f-g|}{1+\sup_{K_n} |f-g|}$$

generates the topology of uniform convergence on compact subsets on the space $H(G)$ of holomorphic functions on G.

As an application of the above result to function theory, Gethner and Shapiro [4, Theorem 3.7] provide entire functions for which many derivatives have "large images" on given arbitrarily small open subsets (see Theorem 1.1 below). In what follows we denote by $B(a,r)$ the open disc with center at $a \in \mathbb{C}$ of radius $r > 0$.

THEOREM 1.1. *Suppose (ϱ_n) is an unbounded increasing sequence of positive numbers for which $\lim_{n\to\infty} \varrho_n^{1/n}/n = 0$. Then there exists a dense G_δ*

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subset M of $H(\mathbb{C})$ such that, for every $f \in M$ and for every nonempty open set $V \subset \mathbb{C}$, there are infinitely many $n \in \mathbb{N}$ such that $f^{(n)}(V) \supset B(0, \varrho_n)$.

For future reference, it is convenient to introduce the following definition.

DEFINITION 1.1. Let $G \subset \mathbb{C}$ be a domain, $T_n : H(G) \to H(G)$ a sequence of continuous operators and (ϱ_n) an increasing unbounded sequence of positive numbers. We say that the sequence (T_n) has *(ϱ_n)-large image* if the set

$$M := \{f \in H(G) : \text{for each nonempty open subset } V \subset G, \quad (T_n f)(V) \supset B(0, \varrho_n) \text{ for infinitely many } n \in \mathbb{N}\}$$

is residual. We say that a single operator T on $H(G)$ has (ϱ_n)-large image whenever the sequence of iterates $T_n = T \circ \ldots \circ T$ (n times) has (ϱ_n)-large image.

With this terminology, Gethner and Shapiro proved that the derivative operator D has (ϱ_n)-large image whenever $\lim_{n\to\infty} \varrho_n^{1/n}/n = 0$. In 1999 L. Bernal [3] generalized this result to infinite order differential operators in the following way: Let $G \subset \mathbb{C}$ be a simply connected domain and $\phi(z) = \sum_{j=0}^{\infty} c_j z^j$ be a nonconstant entire function of subexponential type with a zero of multiplicity m at the origin ($m \geq 0$). Then for any unbounded sequence (ϱ_n) such that either

(1) $c_0 \neq 0$ and $\limsup_{n\to\infty} \varrho_n/|c_0|^n = 0$, or
(2) $c_0 = 0$ and $\limsup_{n\to\infty} \varrho_n^{1/n}/n^m \leq (m/(e \cdot \mathrm{diam}(G)))^m \cdot |c_m|$,

the operator $T = \phi(D) = \sum_{j=0}^{\infty} c_j D^j$ has (ϱ_n)-large image. In fact, Bernal established the property of having large images for sequences $(\phi_n(D))$ of infinite order differential operators under two different sufficient conditions (see [3, Theorems 2–3]).

In this note we study the property of having large images for a kind of shift operators (see the next section) which also constitute a natural generalization of the derivative operators.

2. Preliminaries. In 1996 Bernal [2] introduced the Taylor shift operators on $H(\mathbb{D})$ and $H(\mathbb{C})$. Actually, his proof easily extends to $H(\mathbb{D}_R)$, where $R \in (0, +\infty]$ and $\mathbb{D}_R = B(0, R)$. In fact the continuity is automatic by the closed graph theorem in F-spaces. So, we recall the Taylor shift, or pseudo shift in the more general definition of K.-G. Grosse-Erdmann [5] for F-sequence spaces, as follows. Let $\mathbb{N}$ be the set of positive integers, and set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

DEFINITION 2.1. An operator $T : H(\mathbb{D}_R) \to H(\mathbb{D}_R)$ is said to be a *pseudo shift* if there are an injective mapping $m : \mathbb{N}_0 \to \mathbb{N}_0$ and a sequence (b_j) of nonzero complex numbers such that

(1) There exists $\beta > 0$ such that $m(j) \geq \beta j$ (respectively $m(j) \leq \beta j$) for all $j \in \mathbb{N}$ if $R > 1$ (respectively, if $R \leq 1$).

(2) The sequence $\{|b_j|^{1/j}\}$ is bounded if $R = +\infty$, and satisfies $\limsup_{j\to\infty} |b_j|^{1/j} \leq 1/R$ if R is finite.

(3) If $f(z) := \sum_{j=0}^{\infty} a_j z^j \in H(\mathbb{D}_R)$, then

$$Tf(z) = \sum_{j=0}^{\infty} b_j a_{m(j)} z^j.$$

We will denote by $T_{m,b}$ the pseudo shift determined by the mapping m and the sequence $b = (b_j)$. Finally, we recall the following auxiliary lemma (see [3, Theorem 1]).

LEMMA 2.1. *Assume that X and Y are linear metric spaces, and X is a Baire space. Let $\sigma = (x_n)$ be a sequence in X and $\tau = (T_n)$ a sequence of continuous linear operators from X to Y. Suppose that*

(a) *σ is relatively compact.*

(b) *For every limit point α of σ, $\lim_{n\to\infty} T_n\alpha = 0$.*

(c) *There exists a dense subset $\mathcal{D} \subset X$ such that $\liminf_{n\to\infty} \|T_n x\| = 0$ for all $x \in \mathcal{D}$.*

Then

$$M(\sigma,\tau) := \{x \in X : \liminf_{n\to\infty} d(T_n x, T_n x_n) = 0\}$$

is residual in X.

3. Pseudo shifts with large images. We are now ready to state the large image property for pseudo shifts. Note that the condition involving the radii ϱ_n only concerns the first two terms of each sequence $b^{(n)}$ and of each mapping m_n.

THEOREM 3.1. *Let $\{T_n : H(\mathbb{D}_R) \to H(\mathbb{D}_R)\}$ be a sequence of pseudo shift operators, $T_n = T_{m_n, b^{(n)}}$, and (ϱ_n) an increasing unbounded sequence of positive numbers. Suppose that*

(1) $\lim_{n\to\infty} (\inf\{m_n(j) : j \in \mathbb{N}_0\}) = +\infty$.

(2) $\liminf_{n\to\infty} \left(\max\left\{ \dfrac{(\varrho_n)^{1/(1+m_n(j))}}{|b_j^{(n)}|^{1/(1+m_n(j))}} : j = 0,1 \right\}\right) = 0.$

Then the sequence (T_n) has (ϱ_n)-large image.

Proof. We have to prove, in the notation of Definition 1.1, that the set M is residual in $H(G)$, where $G = \mathbb{D}_R$. Let $\{B(w_k, \varepsilon_k)\}_{k\in\mathbb{N}}$ be any enumeration of all balls contained in $\mathbb{D}_R$ with centers having rational real and imaginary parts, and with rational radii. For any ball $B(w,\varepsilon) \subset \mathbb{D}_R$ we set

$$H(w,\varepsilon) := \{f \in H(\mathbb{D}_R) : B(0,\varrho_n) \subset (T_n f)(B(w,\varepsilon)) \quad \text{for infinitely many } n \in \mathbb{N}\}.$$

It is an easy exercise to check the equality

$$M = \bigcap_{k\in\mathbb{N}} H(w_k,\varepsilon_k).$$

Therefore we only have to prove the residuality of each $H(w,\varepsilon)$. Fix a ball $B(w,\varepsilon) \subset \mathbb{D}_R$ and, for each $n \in \mathbb{N}$, consider the monomial of degree one

$$f_n(z) = \frac{2\varrho_n}{\varepsilon}(z-w).$$

By (2), there exists a sequence of positive integers $n_1 < n_2 < \dots$ satisfying

$$\lim_{p\to\infty} \frac{(\varrho_{n_p})^{1/(1+m_{n_p}(j))}}{|b_j^{(n_p)}|^{1/(1+m_{n_p}(j))}} = 0 \qquad (j = 0,1).$$

Then, for every $r \in (0,R)$ and every $\delta \in (0,1)$ there exists $p_0 \in \mathbb{N}$ such that

$$\frac{(\varrho_{n_p})^{1/(1+m_{n_p}(j))}}{|b_j^{(n_p)}|^{1/(1+m_{n_p}(j))}} < \frac{\delta}{r} \qquad (j = 0,1;\ p > p_0).$$

So, we have $\varrho_{n_p} r^{m_{n_p}(j)}/|b_j^{(n_p)}| < \delta/r$, and

$$\text{(A)} \qquad \lim_{p\to\infty} \frac{\varrho_{n_p} r^{m_{n_p}(j)}}{|b_j^{(n_p)}|} = 0 \qquad (j = 0,1).$$

For each $p \in \mathbb{N}$ we define the polynomial

$$g_p(z) = \frac{2\varrho_{n_p}}{\varepsilon}\left(\frac{z^{m_{n_p}(1)}}{b_1^{(n_p)}} - w\,\frac{z^{m_{n_p}(0)}}{b_0^{(n_p)}}\right).$$

Then $g_p \in H(\mathbb{D}_R)$ and, for each $r \in (0,R)$,

$$|g_p(z)| \le \frac{2\varrho_{n_p}}{\varepsilon}\left(\frac{r^{m_{n_p}(1)}}{|b_1^{(n_p)}|} + |w|\,\frac{r^{m_{n_p}(0)}}{|b_0^{(n_p)}|}\right) \qquad (|z| \le r).$$

By (A),

$$g_p \to 0 \quad \text{in } H(\mathbb{D}_R) \qquad (p\to\infty).$$

On the other hand, given any polynomial P, from (1) we have the existence of $n_0 \in \mathbb{N}$ such that $m_n(j) > \deg(P)$ for all $j \in \mathbb{N}_0$ and $n > n_0$. Then

$$T_n(P) = 0 \qquad \text{for all } n > n_0.$$

We now apply Lemma 2.1, taking $X = Y = H(\mathbb{D}_R)$, $T_p = T_{n_p}$, $(x_p) = (g_p)$ and $\mathcal{D} = \{\text{polynomials}\}$. Therefore, the set

$$M((g_p),(T_p)) = \{f \in H(\mathbb{D}_R) : \liminf_{p\to\infty} d(T_p f, T_p g_p) = 0\}$$

is residual in $H(\mathbb{D}_R)$. But

$$T_p g_p = T_{n_p}\left(\frac{2\varrho_{n_p}}{\varepsilon}\left[\frac{z^{m_{n_p}(1)}}{b_1^{(n_p)}} - w\,\frac{z^{m_{n_p}(0)}}{b_0^{(n_p)}}\right]\right) = \frac{2\varrho_{n_p}}{\varepsilon}(z-w) = f_{n_p}.$$

Thus,

$$M((g_p),(T_{n_p})) = \{f \in H(\mathbb{D}_R) : \text{there exists a subsequence } (n_k) \text{ of } (n_p) \text{ such that } T_{n_k} f - f_{n_k} \to 0 \text{ on } H(\mathbb{D}_R)\ (k\to\infty)\}.$$

Let $f \in M((g_p),(T_{n_p}))$. Consider the compact subset $K = \{z : |z-w| = \varepsilon\} \subset \mathbb{D}_R$. There exists a new subsequence of (n_k), call it (n_k) again, such that for every $k \in \mathbb{N}$,

$$|T_{n_k} f(z) - f_{n_k}(z)| < 1 \quad \text{for all } z \in K$$

and $\varrho_{n_k} > 1$. Fix $k \in \mathbb{N}$ and let $a \in B(0, \varrho_{n_k})$. If $z \in K$ then

$$|(T_{n_k} f)(z) - a - (f_{n_k}(z) - a)| < 1 < \varrho_{n_k} = 2\varrho_{n_k} - \varrho_{n_k} < \frac{2\varrho_{n_k}}{\varepsilon}\cdot\varepsilon - |a| = |f_{n_k}(z)| - |a| \le |f_{n_k}(z) - a|.$$

But $|a| < \varrho_{n_k}$, so there is a solution of the equation $f_{n_k}(z) = a$ in $B(w,\varepsilon)$, and, by Rouché's theorem [1, p. 153], $T_{n_k} f$ takes the value a in $B(w,\varepsilon)$. Hence,

$$B(0,\varrho_{n_k}) \subset (T_{n_k} f)(B(w,\varepsilon)) \quad (k \in \mathbb{N}).$$

In particular, $M((g_p),(T_{n_p})) \subset H(w,\varepsilon)$ and we obtain the residuality of $H(w,\varepsilon)$ in $H(\mathbb{D}_R)$. ■

REMARK 3.2. For each $n \in \mathbb{N}$, the nth derivative operator $D^n f = f^{(n)}$ is a pseudo shift by just taking $b_j^{(n)} = (j+1)\ldots(j+n)$ and $m_n(j) = j+n$. Then it is clear that $\liminf_{n\to\infty}(\inf\{m_n(j) : j \in \mathbb{N}_0\}) = +\infty$, and that condition (2) of Theorem 3.1 can be written as $\liminf_{n\to\infty} \varrho_n^{1/n}/n = 0$. Hence we obtain Theorem 1.1 as a special case.

REFERENCES

[1] L. V. Ahlfors, *Complex Analysis*, McGraw-Hill, London, 1979.

[2] L. Bernal-González, *Universal functions for Taylor shifts*, Complex Variables 31 (1996), 121–129.

[3] —, *Holomorphic functions having large images under the action of differential operators*, J. Math. Anal. Appl. 230 (1999), 390–399.

[4] R. M. Gethner and J. H. Shapiro, *Universal vectors for operators on spaces of holomorphic functions*, Proc. Amer. Math. Soc. 100 (1987), 281–288.
[5] K.-G. Grosse-Erdmann, *Hypercyclic and chaotic weighted shifts*, Studia Math. 139 (2000), 47–68.

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