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# WHEN EVERY POINT IS EITHER TRANSITIVE OR PERIODIC

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**Abstract.** We study transitive non-minimal $\mathbb{N}$-actions and $\mathbb{Z}$-actions. We show that there are such actions whose non-transitive points are periodic and whose topological entropy is positive. It turns out that such actions can be obtained by perturbing minimal systems under some reasonable assumptions.

**1. Introduction.** It is known [K] that the set of non-transitive points in a transitive non-minimal system ($\mathbb{N}$-action) is dense. In [K-W] a transitive recurrent non-minimal system is constructed (hence with all non-transitive points recurrent). In [D] we find a similar example with additional regularity properties (weakly almost periodic).

Transitive systems with dense minimal (or periodic) points have gained some attention in more recent papers: in [G-W] it is proved that a transitive system with a dense set of minimal points must be sensitive; we also mention [A-K-L-S] where some relations between density of periodic orbits and topological entropy in low dimensional dynamics are studied.

When one reads these papers, a natural question comes to mind: do there exist transitive non-minimal systems all of whose non-transitive points are minimal or even periodic? How restrictive is such a combination of constraints (it seems to be the strongest of the discussed type)? In this note we will provide answers to these questions.

Let $\mathbb{S}$ denote either the semigroup $\mathbb{N}$ of all positive integers or the group $\mathbb{Z}$ of all integers. Let $X$ be a compact metric space without isolated points (hence uncountable). We will consider the action of $\mathbb{S}$ on $X$, i.e., a dynamical system $(X,T) = (X, T^n : n \in \mathbb{S})$, where $T$ is a continuous map from $X$ to $X$ (in case $\mathbb{S} = \mathbb{Z}$, $T$ must be a homeomorphism).

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A point $x \in X$ is called *recurrent* if it is contained in the closure of the set $\{T^n x : n \in \mathbb{S}, n \neq 0\}$. A system $(X,T)$ is called *recurrent* if every $x \in X$ is recurrent. A point $x \in X$ is called *transitive* if its orbit $\{T^n x : n \in \mathbb{S}\}$ is dense in $X$. The system $(X,T)$ is called *transitive* if it has at least one transitive point. In transitive systems all the transitive points form a dense $G_\delta$ set. Since the space has no isolated points, every transitive point is recurrent. A non-empty subset $F \subset X$ is called *invariant* if $T^n F \subset F$ for each $n \in \mathbb{S}$; $F$ is *minimal* if it is closed, invariant, and contains no proper closed invariant subsets. A point $x$ is called *minimal* if it is contained in a minimal subset of $X$ or, equivalently, if its orbit closure is minimal. In particular, each periodic point is minimal. It is well known that the whole space $X$ is minimal if and only if each point is transitive. We then say that $(X,T)$ is a *minimal system.* It is clear that a minimal system is recurrent.

We are now in a position to introduce the notions which are the main subject of investigation in this note:

DEFINITION. A dynamical system $(X,T)$ will be called a *ToM-system* if it is transitive, not minimal, and every point $x \in X$ is either transitive or minimal. A special case is a *ToP-system*, where each point is either transitive or periodic.

Clearly, in both cases of $\mathbb{S}$, ToM implies that the system is recurrent. In particular, this forces that in the case of an $\mathbb{N}$-action, $T$ is surjective. A nice and obvious fact about the ToM property is that it is inherited by any non-minimal factor of $(X,T)$. Moreover, the ToP property is invariant under orbit equivalence (by surjectivity finite orbits are periodic).

We emphasize that some properties of ToM $\mathbb{N}$-actions and $\mathbb{Z}$-actions are quite different (see Theorem A below and the following remark).

FACT 1 (see e.g. [K-W]). *If $(X,T)$ is a transitive, recurrent, non-minimal $\mathbb{N}$-action then the space is not totally disconnected; every transitive point is contained in a connected set intersecting all minimal subsets.*

*Proof.* Suppose the assertion is not true. Then there exists a transitive point $x$, a minimal subset $F$, and an open and closed set $U$ so that $F \subset U$ and $x \notin U$. Then $T^n x \in U$ along arbitrarily long intervals of time, and, on the other hand, $T^n x \notin U$ for arbitrarily large values of $n$. Thus for each $k \in \mathbb{N}$ we can find an integer $n_k \in \mathbb{N}$ such that $T^{n_k} x \notin U$ and $T^{n_k+i} x \in U$ for every $i = 1, \ldots, k$. Then any accumulation point of the sequence $(T^{n_k} x)$ is not recurrent, a contradiction. ∎

FACT 2 (see [K] for the original proof). *Let $(X,T)$ be a transitive non-minimal $\mathbb{N}$-action. Then the set of non-transitive points is dense in $X$.*

*Proof* (based on the proof of Lemma K in [K-W]). Let $A$ denote the (non-empty) set of all non-transitive points. Clearly, the image of a non-transitive point is non-transitive. Thus $T(\overline{A}) \subset \overline{A}$. Since proper closed invariant sets contain no transitive points, it follows that either $\overline{A} = X$ (which we claim) or $\overline{A}$ contains no transitive points, i.e., $A$ is closed. Since the preimage of a transitive point consists of transitive points, we can find an open neighborhood $U$ of $A$ with $\overline{U}$ disjoint from one such preimage. Then $T(\overline{U}) \neq X$. Thus $\bigcap_{n=1}^{\infty} T^n(\overline{U})$ is a proper, closed and (forward) invariant subset, hence contains no transitive points, hence is contained in $A$. This implies that for some $n_0$, $\bigcap_{n=1}^{n_0} T^n(\overline{U}) \subset U$. Setting $K := \bigcap_{n=0}^{n_0-1} T^n(\overline{U})$ we have $T(K) \subset \bigcap_{n=1}^{n_0} T^n(\overline{U}) \subset U$, hence $T(K) \subset \overline{U} \cap T(K) \subset K$. Thus, by the same argument a third time, $K$ contains no transitive points. But $K$ contains the open set $\bigcap_{n=0}^{n_0-1} T^n(U)$, which is non-empty (contains $\bigcap_{n=0}^{n_0-1} T^n(A) = T^n(A)$). Thus $K$ does contain transitive points, a contradiction. ■

Applying the above two facts to ToM $\mathbb{N}$-actions we obtain the following

THEOREM A. *Let* $(X, T)$ *be a ToM* $\mathbb{N}$*-action. Then the space* $X$ *is not totally disconnected* (*there exists a connected set intersecting all minimal sets*), *and the union of all minimal sets is dense.* ■

As we shall see in the next construction, both statements fail for $\mathbb{Z}$-actions: there exist totally disconnected (even symbolic) ToP $\mathbb{Z}$-actions where the union of minimal subsets reduces to a single fixpoint.

It should be noted that, by Theorem A and the above-mentioned result of [G-W], every ToP $\mathbb{N}$-action is chaotic in the sense of Devaney. According to a recent result in [H-Y], it is then chaotic in the sense of Li–Yorke. Thus, our construction of Section 3 automatically adds to the variety of examples of Li–Yorke chaotic systems.

In the following sections we shall prove two theorems (for $\mathbb{S} = \mathbb{Z}$ and $\mathbb{S} = \mathbb{N}$), which show that even the seemingly strong ToP property does not impose much restriction on the dynamics. For $\mathbb{S} = \mathbb{Z}$, the system may behave (up to $\varepsilon$) just like any minimal system $(X, T)$. For $\mathbb{S} = \mathbb{N}$ we prove a slightly weaker measure-theoretic statement. If $\mathbb{S} = \mathbb{Z}$ then among transitive points, those which are transitive only backwards are admitted (i.e., $\{T^{-n} : n \in \mathbb{N}\}$ is dense in $X$ while $\{T^n : n \in \mathbb{N}\}$ is not), which makes the case considerably easier. For simplicity, we will state the theorem only for $X$ totally disconnected. It follows from the results of E. Lindenstrauss and B. Weiss (Theorem 4.2 and the preceding statement in [L-W] and Theorem 6.2 in [L]) that every minimal system $(X, T)$ with finite entropy has a base of topology consisting of sets whose boundaries are null sets with respect to all invariant measures. This easily implies the existence of a totally disconnected extension of $(X, T)$ with "the same" dynamics, i.e., of an almost one-to-one extension with an isomorphic simplex of invariant measures and which, for every invariant measure, is measure-theoretically isomorphic to $(X,T)$. This explains that the totally disconnected case is in a sense the most important one.

**2. $\mathbb{Z}$-actions.** This section is devoted to proving the existence theorem for ToP $\mathbb{Z}$-actions and finding out how general such systems can be.

THEOREM 1. *Let $(X,T)$ be an arbitrary non-periodic totally disconnected minimal $\mathbb{Z}$-action and let $\varepsilon > 0$ be given. Then there exists a totally disconnected ToP $\mathbb{Z}$-action $(Y,S)$ with one fixpoint as a unique minimal subset, and a closed set $F \subset Y$ with $\nu(F) > 1-\varepsilon$ for every ergodic measure $\nu$ on $Y$ not supported by the fixpoint, such that $(F,S_F)$ is topologically conjugate to $(X,T)$, where $S_F$ is the first return map induced on $F$ (which includes that $S_F$ is continuous).*

*Proof.* Replacing each $x \in X$ by its trajectory $\widetilde{x} := (\widetilde{x}_n)_{n\in\mathbb{Z}} \in X^{\mathbb{Z}}$, where $\widetilde{x}_n := T^n x$, we isomorphically represent $(X,T)$ as a subsystem $\widetilde{X}$ of $X^{\mathbb{Z}}$ with the shift transformation $S$ defined by $S\widetilde{x} := (\widetilde{x}_{n+1})$. By a *block* $\widetilde{x}[m,n)$ we shall mean the finite sequence $\widetilde{x}_m, \widetilde{x}_{m+1}, \widetilde{x}_{m+2}, \ldots, \widetilde{x}_{n-1}$. Let $(m_k)_{k\ge 1}$ be a sequence of positive integers such that

$$\sum_{k=1}^{\infty} \frac{k}{m_k} < \varepsilon.$$

Let $(U_k)_{k\in\mathbb{N}}$ be a sequence of open and closed balls in $X$ with one-point intersection $\{x'\}$ and so small that for each $k$ the sets $U_k, T^{-1}U_k, \ldots, T^{-m_k}U_k$ are pairwise disjoint (this is possible within any minimal non-periodic system). In particular, this implies that $\mu(U_k) < 1/m_k$ for every invariant measure $\mu$ on $X$. Let $c$ be a symbol not contained in $X$. The ToP-system $(Y,S)$ will be built as a subshift $Y \subset (X \cup \{c\})^{\mathbb{Z}}$ ($c$ being an isolated point of $X \cup \{c\}$). A transitive point $y'$ is constructed as follows: Let $(n_i)_{i\in\mathbb{Z}}$ be the sequence of times corresponding to the visits of the trajectory $\widetilde{x}'$ of $x'$ in $U_1$ (so that $n_0 = 0$), and for each integer $i$ let $k_i$ be the maximal $k$ with $T^{n_i}x' \in U_k$. Since $T^{n_i}x' = x'$ only for $i = 0$, each $k_i$ is finite except for $i = 0$. We let

$$y' := \ldots\, c,c,c,c,c,\widetilde{x}'[\underline{n_0},n_1)\, c^{k_1}\, \widetilde{x}'[n_1,n_2)\, c^{k_2}\, \widetilde{x}'[n_2,n_3)\, c^{k_3}\, \ldots,$$

where $c^k$ denotes the block $c,c,c,\ldots,c$ ($k$ times). The zero coordinate is underlined. Roughly speaking, $y'$ is obtained from $\widetilde{x}'$ by inserting before each $n_i$ a block of $k_i$ $c$'s. By definition, $Y$ is the orbit closure of $y'$. Since the sets $U_k$ are both open and closed, it follows by an easy approximation argument that if $y \in Y$ and a symbol $x \in X$ appears in $y$ then $y$ has the form

$$y = \ldots\, c^{k_{-1}}\, \widetilde{x}[n_{-1},n_0)\, c^{k_0}\, \widetilde{x}[n_0,n_1)\, c^{k_1}\, \widetilde{x}[n_1,n_2)\, c^{k_2}\, \widetilde{x}[n_2,n_3)\, c^{k_3}\, \ldots,$$

where the numbers $n_i$ and $k_i$ are defined for $x$ just as they were for $x'$ (via the visits in $U_k$; now, of course, $n_0$ need not be 0). In fact, in $Y$ there appear four types of elements: $y$ as described above with all $k_i$ finite, the points $S^n(y')$ (with infinitely many $c$'s on the left), the points $S^n(y'')$, where

$$y'' = \ldots, c^{k_{-3}}\, \widetilde{x}'[n_{-3}, n_{-2})\, c^{k_{-2}}\, \widetilde{x}'[n_{-2}, n_{-1})\, c^{k_{-1}}\, \widetilde{x}'[n_{-1}, \underline{n_0 - 1}]\, , c, c, c, \ldots,$$

(these have infinitely many $c$'s on the right), and the fixpoint

$$\overline{c} := \ldots, c, c, c, c, c, c, c, \ldots$$

We claim that, except for the fixpoint, every point is transitive (either forward or backward or in both directions). This follows immediately from the fact that every $x \in X$ is both forward and backward transitive in $(X, T)$, and that two sufficiently close points in $X$ generate the same numbers $k_i$ for all $i$ between any preset bounds. We have proved that $(Y, S)$ is a ToP-system with the fixpoint $\overline{c}$ as a unique minimal subset.

Now, $Y$ splits into two closed and open sets: $F := \{y \in Y : y_0 \in X\}$ and $C := Y \setminus F = \{y \in Y : y_0 = c\}$. Every ergodic measure $\nu$ on $Y$ distinct from the point mass at $\overline{c}$ is supported by the set of points $y$ with all blocks $c^{k_i}$ finite. Thus, up to a measure zero set, $C$ splits as the countable union $C = \bigcup C_k$, where

$$C_k := \{y \in Y : \text{the length of the maximal block of } c\text{'s containing } y_0 \text{ is } k\}.$$

Clearly, the trajectory of any $y \in Y$ visits $C_k$ at most $k$ in every $m_k + k$ times. Thus $\nu(C_k) \le k/m_k$, and hence $\nu(F) = 1 - \nu(C) > 1 - \varepsilon$, as claimed. The conjugacy between the action induced on $F$ and that on $X$ is established by the map $\phi(y) := y_0$, verification of which is immediate (we need to additionally define the induced map at one point: $S_F(y'') := y'$). ■

It is easy to see that if $(X, T)$ is expansive then the above leads to an expansive (i.e., isomorphic to symbolic) system $(Y, S)$.

REMARK. Clearly, the lengths of the inserted blocks $c^k$ can be slightly changed. By playing with these lengths it seems possible to obtain $(Y, S)$ totally transitive (= weakly mixing here) or even topologically mixing.

## 3. $\mathbb{N}$-actions.

We now present a much more complicated construction for $\mathbb{N}$-actions.

THEOREM 2. *Let $(X, \mu, T)$ be an arbitrary non-periodic ergodic measure-preserving system and let $\varepsilon > 0$ be given. Then there exists a ToP-system $(Y, S)$, a Borel set $F \subset Y$, and an invariant measure $\nu$ on $Y$ such that $\nu(F) > 1 - \varepsilon$ and $(F, \nu_F, S_F)$ is a measure-theoretic extension of $(X, \mu, T)$, where $S_F$ is the first return map induced on $F$ and $\nu_F$ is the conditional probability on $F$ induced by $\nu$.*

COROLLARY. *There exists a ToP $\mathbb{N}$-action with positive entropy.*

*Proof of Theorem 2.* The proof will be divided into several stages. In Stage 1 we embed $X$ in a convex (hence also connected) set (so that gradual passages from one element to another become possible), and then we represent our system as a shift system over this convex set of symbols. In Stage 2 we inductively construct a family of blocks $A_t(\alpha)$ and an auxiliary family $B_{t+1}(\alpha)$ (both indexed by $t \in \mathbb{N}$), depending continuously on the real parameter $\alpha$ ranging between 0 and $2^t$. The block $A_t(0)$ is almost identical to the initial block of the future transitive point $y'$ of $(Y, S)$, and as $\alpha$ rises to $2^t$ it changes gradually toward a constant block passing through periodic concatenations of $A_s(0)$ for all indices $s < t$. The essential properties of the blocks $A_t(\alpha)$ and $B_{t+1}(\alpha)$ are listed and proved in Stage 3. These properties will ensure (which we verify in Stage 4) that every element $y = \lim_j S^{n_j} y'$ of $Y$ is either periodic (if the coordinates $n_j$ fall in $y'$ into blocks $A_t(\alpha)$ (with larger and larger $t$) "similar" to periodic repetitions of $A_s(\beta)$ for some fixed $s$ and $\beta$) or transitive (in the opposite case). Stage 5 contains the definition of the set $F$ and the discussion of the induced transformation.

STAGE 1: *Preliminary reshaping.* By the Jewett–Krieger theorem, we can think of $(X, \mu, T)$ as of a uniquely ergodic minimal topological system. Next, we add to $X$ an isolated point $c$ and we realize $X \cup \{c\}$ as the set of extreme points in a compact simplex $K$. For instance, $K$ can be identified with the set of all probability measures on $X \cup \{c\}$ (endowed with the weak* topology), where each point $x$ of $X$ (or $c$) corresponds to the point mass at $x$ (or at $c$). The points $x \in X$ will be called *pure elements* of $K$. To *spoil* a pure element $x$ by $\varepsilon$ means to replace it by the convex combination $(1-\varepsilon)x + \varepsilon c$. As before, we let $(\widetilde{X}, S)$ be the subshift conjugate to $(X, T)$ obtained as the set of trajectories $\widetilde{x} := \widetilde{x}_1, \widetilde{x}_2, \ldots$, where $\widetilde{x}_n := T^{n-1}x$. This time, however, $\widetilde{X} \subset X^{\mathbb{N}} \subset K^{\mathbb{N}}$.

STAGE 2: *Construction.* If $y \in K^{\mathbb{N}}$ and $B \in K^n$ ($n \in \mathbb{N}$) then by *inserting* the block $B$ in $y$ *after* positions $n_1, n_2, \ldots$ we mean producing the sequence

$$y' = y[1, n_1]\ B\ y(n_1, n_2]\ B\ y(n_2, n_3] \ldots$$

We fix some $x' \in X$. We will define an element $y' \in K^{\mathbb{N}}$ by applying inductively a countable series of insertions in $\widetilde{x}'$. Later we will define $Y$ as the shift orbit closure of $y'$.

We will be using the following notation: For two blocks $B$ and $C$ of the same length with symbols belonging to a convex set and an $\alpha \in [0, 1]$ we define

$$\langle\!\langle \alpha, B, C \rangle\!\rangle = (1-\alpha)B + \alpha C$$

(the block obtained as a coordinatewise convex combination). Suppose we have defined a family of blocks of the same length $p$, $\{A(\alpha) : \alpha \in [0,k]\}$ ($k \in \mathbb{N}$). Then, for a positive integer $q = (2m+1)p$ ($m \in \mathbb{N}$) and a real parameter $\beta \in [0,k]$, we denote by $A^q[0 \to \beta \to 0]$ the concatenation

$$A(0)A\left(\frac{\beta}{m}\right)A\left(\frac{2\beta}{m}\right)\ldots A\left(\frac{(m-1)\beta}{m}\right)A(\beta)A\left(\frac{(m-1)\beta}{m}\right)\ldots$$
$$\ldots A\left(\frac{2\beta}{m}\right)A\left(\frac{\beta}{m}\right)A(0).$$

Note that, regardless of $\beta$, the length of $A^q[0 \to \beta \to 0]$ is $q$. The block $A^q[0 \to 0 \to 0]$ consists of $2m+1$ repetitions of $A(0)$. Such an object will also be denoted briefly by $A^q(0)$ (here the index $q$ does not denote the number of repetitions, but the total length).

Each inductive step below contains three parts: (a) the definition of $A_t(\alpha)$, (b) the definition of $B_{t+1}(\alpha)$ and (c) the description of the insertion of $B_{t+1}(0)$ in $y$.

Step 0. Recall that the first entry of $\widetilde{x}'$ is $x'$. Let $x^* = 0.9x' + 0.1c$ (we spoil $x'$ by 0.1). For $\alpha \in [0,1]$ set

(a) $$A_0(\alpha) := \langle\!\langle \alpha, x^*, c \rangle\!\rangle$$

(in particular, $A_0(0) = x^*$, $A_0(1) = c$). Choose some odd $q_1 \in \mathbb{N}$ and, for each $\alpha$ in the same interval as above, let

(b) $$B_1(\alpha) := A_0^{q_1}[0 \to 1-\alpha \to 0].$$

Note that $B_1(1) = A_0^{q_1}(0)$. Clearly, the assignments $\alpha \mapsto A_0(\alpha)$ and $\alpha \mapsto B_1(\alpha)$ are continuous for coordinatewise convergence. Observe that for any $\alpha \in [0,1]$, neither $A_0(\alpha)$ nor any entries of $B_1(\alpha)$ are pure.

(c) Now, we fix a positive integer $p_1$ (larger than $q_1$) and we produce a new sequence $y$ by inserting in $\widetilde{x}'$ the block $B_1(0)$ after positions $(2k-1)p_1$ ($k \in \mathbb{N}$). The first of the inserted blocks starts at position $p_1 + 1$, then they appear periodically with period $r_1 := 2p_1 + q_1$, or, in other words, with gap $2p_1$ (see figure below).

$$\ldots\boxed{B_1(0)}\ldots\ldots\boxed{B_1(0)}\ldots\ldots\boxed{B_1(0)}\ldots\ldots\boxed{B_1(0)}\ldots$$

(The dots represent the consecutive (pure) entries of $\widetilde{x}'$, $p_1 = 3$; $q_1$ is not specified.)

Step 1. After step 0, $y$ begins with

$$y[1, r_1] = y[1, p_1]\, B_1(0)\, y(r_1 - p_1, r_1],$$

where $y[1, p_1] = \widetilde{x}'[1, p_1]$ and $y(r_1 - p_1, r_1] = \widetilde{x}'(p_1, 2p_1]$. We spoil the pure entries of $y[1, r_1]$ by 0.01, and so we obtain a block of the form

$$y^*[1, p_1]\, B_1(0)\, y^*(r_1 - p_1, r_1].$$

This will be our $A_1(0)$. Generally, for $\alpha \in [0,1]$ we set

(a) $$A_1(\alpha) := \langle\!\langle \alpha, y^*[1,p_1], A_0^{p_1}(0)\rangle\!\rangle\, B_1(\alpha)\, \langle\!\langle \alpha, y^*(r_1-p_1,r_1], A_0^{p_1}(0)\rangle\!\rangle.$$

In particular, $A_1(1) = A_0^{r_1}(0)$. In other words, with $\alpha$ varying from 0 to 1, $A_1(\alpha)$ changes continuously from a block similar to $y[1,r_1]$ to periodic repetitions of $A_0(0)$; the end-sections of length $p_1$ change linearly while the center uses the path of the blocks $B_1(\alpha)$.

Additionally, we can extend continuously the definition for $\alpha \in [1,2]$ by letting

$$A_1(\alpha) := A_0^{r_1}(\alpha-1).$$

In this manner $A_1(\alpha)$ is defined for every $\alpha \in [0,2]$.

Next, we choose $q_2$ to be a (large) odd multiple of $r_1$, and for $\alpha$ in the same interval as above we set

(b) $$B_2(\alpha) := A_1^{q_2}[0 \to 2-\alpha \to 0].$$

Again, note that $B_2(2) = A_1^{q_2}(0)$ and the assignment $\alpha \mapsto B_2(\alpha)$ is continuous. As before, the blocks constructed in this step do not contain any pure entries.

(c) We pick a positive integer $p_2$ which is a multiple of $r_1$ (much larger than $q_2$) and we produce a new sequence $y'$ by inserting in $y$ the block $B_2(0)$ after positions $(2k-1)p_2$ ($k \in \mathbb{N}$). As before, note that the first of the inserted blocks starts at position $p_2+1$, then they appear periodically with period $r_2 := 2p_2 + q_2$, or, in other words, with gap $2p_2$. To reduce the number of subscripts, the sequence $y'$ produced in this step will again be denoted by $y$.

The structure of $y[1,r_2]$ is sketched below:

$$\ldots\boxed{B_1(0)}\ldots\ldots\boxed{B_1(0)}\ldots\boxed{\qquad B_2(0)\qquad}\ldots\boxed{B_1(0)}\ldots\ldots\boxed{B_1(0)}\ldots$$

($p_1 = 3$, $p_2 = 2r_1$; $q_1$ and $q_2$ are not specified; for compactness of the picture we violated the assumption that $p_1$ and $p_2$ are much larger than $q_1$ and $q_2$, respectively).

Step $t$. Suppose we have defined a continuous assignment $\alpha \mapsto B_t(\alpha)$ for $\alpha \in [0, 2^{t-1}]$ into blocks of length $q_t$ so that $B_t(2^{t-1}) = A_{t-1}^{q_t}(0)$ ($q_t$ is a large odd multiple of the length $r_{t-1}$ of $A_{t-1}(0)$). We assume that these blocks contain no pure entries.

After the insertion of $B_t(0)$, the sequence $y$ begins with

$$y[1,r_t] = y[1,p_t]\, B_t(0)\, y(r_t-p_t, r_t]$$

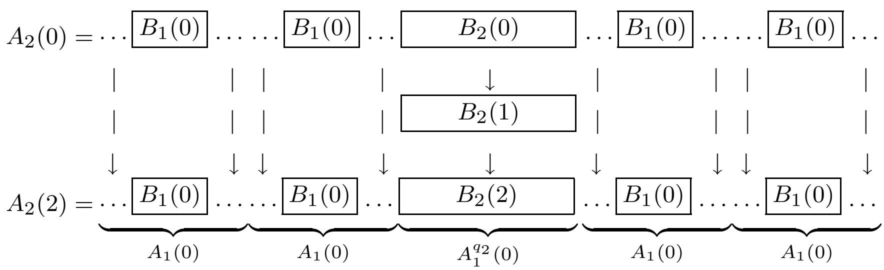
($r_t = 2p_t + q_t$, $p_t$ is a multiple of $r_{t-1}$ much larger than $q_t$). Spoiling (only) the pure entries of $y[1,r_t]$ by $10^{-t-1}$ we obtain a block of the form

$$y^*[1,p_t]\, B_t(0)\, y^*(r_t-p_t, r_t]$$

(note that we do not spoil the previously inserted blocks). This will be our $A_t(0)$. Generally, for $\alpha \in [0, 2^{t-1}]$, we let

(a) $A_t(\alpha) := \langle\!\langle \frac{\alpha}{2^{t-1}}, y^*[1, p_t], A_{t-1}^{p_t}(0) \rangle\!\rangle\, B_t(\alpha)\, \langle\!\langle \frac{\alpha}{2^{t-1}}, y^*(r_t - p_t, r_t], A_{t-1}^{p_t}(0) \rangle\!\rangle$.

In other words, with $\alpha$ varying from 0 to $2^{t-1}$, $A_t(\alpha)$ changes continuously from a block very similar to $y[1, r_t]$ to periodic repetitions of $A_{t-1}(0)$; the end-sections of length $p_t$ change linearly while the center uses the path of the blocks $B_t(\alpha)$. Notice that the previously inserted blocks $B_s(0)$ ($s \leq t$) appear in both end-sections of $y[1, r_t]$ at the same places as in $A_{t-1}^{p_t}(0)$, hence they remain fixed and do not vary with $\alpha$ (see figure below).




Since $A_t(2^{t-1}) = A_{t-1}^{r_t}(0)$, we can extend continuously the definition for $\alpha \in [2^{t-1}, 2^t]$ by letting

$$A_t(\alpha) := A_{t-1}^{r_t}(\alpha - 2^{t-1}).$$

In this manner $A_t(\alpha)$ is defined for every $\alpha \in [0, 2^t]$. Note that if $\alpha = 2^{t-1} + 2^{t-2} + \ldots + 2^s$ for some $s < t$ then $A_t(\alpha)$ becomes periodic repetitions of $A_s(0)$.

Then we choose a (very large) odd multiple $q_{t+1}$ of $r_t$, and for $\alpha$ in the same interval as above we let

(b) $$B_{t+1}(\alpha) := A_t^{q_{t+1}}[0 \to 2^t - \alpha \to 0].$$

As before, we note that $B_{t+1}(2^t) = A_t^{q_{t+1}}(0)$, and that the assignment $\alpha \mapsto B_{t+1}(\alpha)$ is continuous. By induction, there are no pure entries in any blocks constructed in this step.

(c) We now pick a positive integer $p_{t+1}$ which is a multiple of $r_t$ (much larger than $q_{t+1}$) and we modify the sequence $y$ by inserting the block $B_{t+1}(0)$ after positions $(2k-1)p_{t+1}$ ($k \in \mathbb{N}$). The first of the inserted blocks starts at position $p_{t+1} + 1$, then they appear periodically with period $r_{t+1} := 2p_{t+1} + q_{t+1}$, or, in other words, with gap $2p_{t+1}$.

This concludes the induction. From now on $y'$ denotes the sequence obtained after applying all steps. By choosing $q_{t+1}$ large enough in comparison with $r_t$, we can assume that the fractions

$$\delta_{t+1} = 2\,\frac{2^t r_t}{q_{t+1} - 1}$$

decrease to zero as $t \to \infty$.

STAGE 3: *Additional observations.* We point out several properties of the blocks $A_t(\alpha)$, $B_t(\alpha)$ and of the sequence $y'$.

($*$) For every $t$ the positions occupied in $y'$ by the blocks $B_{t'}(0)$ with all $t' \geq t$ form a syndetic set with maximal gap $2p_t$. Hence, if a position $m$ does not fall in such a block, then at most $2p_t$ positions to the right of $m$ we find the beginning of a block of the form $B_{t'}(0)$.

($**$) For $\alpha \in [0, 2^{t-1}]$ the blocks $B_s(0)$ ($s < t$) occur in the block $A_t(\alpha)$ at the same places as the inductively inserted blocks in the initial part of $y'$. Moreover, in the center of $A_t(\alpha)$ we have $B_t(\alpha)$, while at the corresponding place in $y'$ we have $B_t(0)$.

We call the above occurrences of the blocks $B_s(0)$ and $B_t(\alpha)$ in $A_t(\alpha)$ *inserted blocks* in contrast to possible "accidental" occurrences of these blocks not resulting from the induction. (It can be proved that there are no such "accidental" occurrences, but it is not necessary to do it.) In particular,

($***$) any inserted block $B_s(0)$ ($s < t$) or $B_t(\alpha)$ ($s = t$) starts in $A_t(\alpha)$ not sooner than at a position farther than $p_s$. By construction, the structure of inserted blocks is symmetric, hence $A_t(\alpha)$ extends at least $p_s$ positions to the right of the right end of that block.

Consider a block $B_{t+1}(\alpha)$ ($\alpha \in [0, 2^t]$). It is a concatenation of blocks of the form $A_t(\beta)$ with $\beta \in [0, 2^t - \alpha]$. The parameter $\beta$ first grows then decreases by a constant increment which is largest in the case of $\alpha = 0$ and then equal to $\delta_{t+1}$ (see above). If $\beta \geq 2^{t-1}$ then the block $A_t(\beta)$ equals $A_{t-1}^{r_t}(\beta - 2^{t-1})$. Further, if $\beta \geq 2^{t-1} + 2^{t-2}$, then $A_{t-1}^{r_t}(\beta - 2^{t-1})$ can be written as $A_{t-2}^{r_t}(\beta - 2^{t-1} - 2^{t-2})$. And so on. Eventually $B_{t+1}(\alpha)$ can be viewed as a concatenation of groups of blocks of the form $A_s^{r_t}(\beta)$ with $s$ varying first non-increasingly from $t$ to some minimal value and then returning symmetrically to $t$, and $\beta \in [0, 2^{s-1})$ (exception: for $s = 0$ we admit $\beta \in [0, 1]$). Such blocks do not further decompose. We call this representation of $B_{t+1}(\alpha)$ the *explicit representation.*

($****$) If $A_s^{r_t}(\beta)$ and $A_{s'}^{r_t}(\beta')$ are two neighboring components in the explicit representation of $B_{t+1}(\alpha)$, then either $s = s'$ and $|\beta - \beta'| \leq \delta_{t+1}$, or the indices $s$ and $s'$ differ by 1, e.g. $s' = s + 1$. In the latter case we can view the block $A_s^{r_t}(\beta)$ as $A_{s'}^{r_t}(\beta'')$ with $\beta'' > 2^{s'-1}$ and then again $|\beta' - \beta''| \leq \delta_{t+1}$.

Fix a positive integer $m$ regarded as a coordinate in $y'$. The corresponding entry $y'_m$ may fall in one of the inserted blocks $B_t(0)$ ($t \geq 1$) or not. If it does, then in the explicit representation of $B_t(0)$ it falls in some block of the form $A_s^{r_t}(\beta)$ ($\beta \in [0, 2^{s-1}]$). In that case we record the pair of indices $t_1 := t$ (for $B$) and $s_1 := s$ (for $A$). Further, by ($**$), within $A_{s_1}(\beta)$, $m$ may fall in some inserted block $B_t(0)$ ($t < s_1$) or $B_t(\beta)$ ($t = s_1$), or not. If it does, then a new pair of parameters $t_2 := t$ and a corresponding $s_2$ arise. In this manner, to each $m$ we associate a finite (or empty) sequence of pairs $\langle t_i, s_i\rangle$ (of some length $k \ge 0$) satisfying $t_1 > s_1 \ge t_2 > s_2 \ge \ldots \ge t_k > s_k \ge 0$. This sequence will play a crucial role in the next argument. We denote it by $\mathcal{TS}(m)$.

STAGE 4: *The ToP property.* We now prove that the orbit closure $Y$ of $y'$ is ToP. Let $y$ be any element of $Y$, $y = \lim_j S^{m_j} y'$, and suppose that $y$ is not transitive (in particular, $y$ cannot belong to the orbit of $y'$ and we can assume that $m_j$ grows to infinity). Then there exists an $s_0$ such that for every $s \ge s_0$, $A_s(0)$ does not occur in $y$ (recall that for large $s$, $A_s(0)$ is very similar to a long initial block of $y'$). For each $j$, let $n_j$ denote the distance from $m_j$ to the beginning of the nearest (to the right of $m_j$) occurrence of a block $A_s(0)$ with $s \ge s_0$ in $y'$. Our considerations imply that $n_j \to \infty$. By choosing a subsequence, we may assume that $n_j \ge 2r_{s_0+j}$.

CLAIM. *For each $j \ge 2$, $\mathcal{TS}(m_j)$ contains a pair $\langle t_j, s_j\rangle$ with $t_j \ge s_0 + j$ and $s_j \le s_0 + 2$.*

We will consider (and eliminate) several cases in the negation of our claim. Firstly, if $m_j$ falls outside all blocks of the form $B_t(0)$ with $t \ge s_0 + j$, then, by $(*)$, at most $2p_{s_0+j}$ to the right of $m_j$ we find a block $B_t(0)$ with $t \ge s_0 + j$ (which begins with $A_{t-1}(0)$ and $t - 1 \ge s_0$). Thus $n_j \le 2p_{s_0+j} < 2r_{s_0+j}$, a contradiction. This implies that the sequence $\mathcal{TS}(m_j)$ is non-empty and contains at least one pair with $t \ge s_0 + j$. Let $\langle t_j, s_j\rangle$ denote the smallest such pair. Then $m_j$ falls into an inserted block $B_{t_j}(\alpha_j)$ (perhaps inside a larger inserted block) and, therein, into $A_{s_j}(\beta_j)$ which is a part of the component $A_{s_j}^{r_{t_j}}(\beta_j)$ of the explicit representation of $B_{t_j}(\alpha_j)$. Further, within $A_{s_j}(\beta_j)$, $m_j$ falls outside any inserted block with an index $t' \ge s_0 + j$ ($t_j$ is minimal). Two cases are possible: the block $A_{s_j}(\beta_j)$ extends at least $r_{s_0+j}$ positions to the right of $m_j$ or not. If it does, then, by $(**)$ and $(*)$, at most $2p_{s_0+j}$ positions to the right of $m_j$ we will find the block $B_{s_0+j}(0)$ (contained completely inside $A_{s_j}(\beta_j)$; recall that $r_{s_0+j} = 2p_{s_0+j} + q_{s_0+j}$), which begins with $A_{s_0+j-1}(0)$. Thus $n_j \le r_{s_0+j}$, again a contradiction. In the second case we assume (contrary to our claim) that $s_j \ge s_0 + 3$. Consider two subcases:

(a) our $A_{s_j}(\beta_j)$ is the terminal part of the terminal component $A_{s_j}^{r_{t_j}}(\beta_j)$ in the explicit representation of the block $B_{t_j}(\alpha_j)$ (which is only possible if $s_j = t_j - 1$ and $\beta_j = 0$);

(b) our $A_{s_j}(\beta_j)$ is followed (inside $B_{t_j}(\alpha_j)$) by some $A_{s'}(\beta')$ (then $s_j - 1 \le s' \le s_j + 1$).

In case (b) we have $s' \ge s_0 + 2$, hence by $(**)$, at position $p_{s_0+1}$ (relative to $A'_s(\beta')$) we find an occurrence of $B_{s_0+1}(0)$ (which begins with $A_{s_0}(0)$). Thus $n_j \le r_{s_0+j} + p_{s_0+1} < 2r_{s_0+j}$, a contradiction. Case (a) splits again: either $\alpha_j = 0$ and our block $B_{t_j}(0)$ is inserted directly in $y'$ (i.e., $t_j$ is maximal in $\mathcal{TS}(m_j)$), or our block $B_{t_j}(\alpha_j)$ is inserted in a larger $A_u(\gamma)$. In the first case, the position following $B_{t_j}(0)$ is outside of any inserted blocks, thus, by $(*)$ at most $2p_{s_0+1}$ positions farther we will find a block $B_{s_0+1}$ (beginning with $A_{s_0}(0)$), hence $n_j \le r_{s_0+j} + 2p_{s_0+1} < 2r_{s_0+j}$, another contradiction. In the remaining (and last) case, by $(***)$, $A_u(\gamma)$ extends at least $p_{t_j}$ positions to the right of the right end of $B_{t_j}(\alpha_j)$. For $j \ge 2$ we have $p_{t_j} \ge p_{s_0+j} > r_{s_0+1}$, hence, by $(*)$, we can argue as in the preceding case (relative to $A_u(\gamma)$). This concludes the proof of the Claim.

We now prove that $y$ is a periodic sequence. By choosing a subsequence of $m_j$, we can assume that $s_j$ assumes a constant value $s$, $t_j$ grows to infinity, and $\beta_j$ converges to some $\beta$ (note that $\beta_j \in [0, 2^{s-1}]$). We can also assume that the relative position of $m_j$ within the corresponding block $A_s(\beta_j)$ is fixed. Since eventually $s < t_j - 1$, the component $A_s^{r_{t_j}}(\beta_j)$ is not the terminal one in the explicit representation of $B_{t_j}(\alpha_j)$. Thus, by $(****)$, this component and the following one can be written as $A_{s'}^{r_{t_j}}(\beta')$ and $A_{s'}^{r_{t_j}}(\beta'')$ where either $s' = s$ and $|\beta' - \beta''| \le \delta_{t_j}$ and one of $\beta'$, $\beta''$ is $\beta_j$, or $s' = s+1$, and both $\beta'$ and $\beta''$ differ from $2^s$ by at most $\delta_{t_j}$. Choosing a subsequence and passing to the limit, we conclude that $y$ belongs to the shift orbit of the sequence obtained by periodic repetitions of $A_s(\beta)$ (or of $A_{s+1}(2^s) = A_s(0)$). The proof of the ToP property is now complete.

Stage 5: *Induced transformation on $F$.* We now consider the remaining part of the assertion of the theorem. We define $F \subset Y$ to be the set of the elements $y$ for which $y_1$ is pure, i.e.,

$$y = \widetilde{x}[1, k_1)\, B_{t_1}(0)\, \widetilde{x}[k_1, k_2)\, B_{t_2}(0)\, \widetilde{x}[k_2, k_3)\, B_{t_3}(0)\, \ldots$$

for some pair of sequences $(k_i)$ (strictly increasing, $k_1 \ge 2$) and $(t_i)$ of positive integers, and a trajectory $\widetilde{x}$ of some point $x \in X$. Clearly, $y' \in F$. Consider a pair of finite sequences $\mathbb{K} := k_1, \ldots, k_n$ and $\mathbb{T} := t_1, \ldots, t_{n-1}$ of positive integers ($n \ge 1$; $\mathbb{T}$ can be empty, $\mathbb{K}$ is increasing). Let

$$F_{\mathbb{K},\mathbb{T}} := \{y \in Y : (\exists x \in X)$$
$$y = \widetilde{x}[1, k_1)\, B_{t_1}(0)\, \widetilde{x}[k_1, k_2)\, B_{t_2}(0)\, \ldots B_{t_{n-1}}(0)\, \widetilde{x}[k_{n-1}, k_n)\, \ldots\}.$$

Since for any $t$ the block $B_t(0)$ contains no pure entries while every element of the trajectory $\widetilde{x}$ is pure, it is seen that the sets $F_{\mathbb{K},\mathbb{T}}$ and $F_{\mathbb{K}',\mathbb{T}'}$ are either disjoint or one is contained in the other (the latter can happen only when $\mathbb{K}'$ and $\mathbb{T}'$ are extensions of $\mathbb{K}$ and $\mathbb{T}$, respectively, or vice versa). Let $F_k$ denote the (disjoint) union of the sets $F_{\mathbb{K},\mathbb{T}}$ over all pairs $\mathbb{K}, \mathbb{T}$ of sequences such that the last entry of $\mathbb{K}$ (say $k_n$) is equal to $k$. We have

$$F = \bigcap_{k=2}^{\infty} F_k.$$

Let $F_{k,t}$ be the union of the sets $F_{\mathbb{K},\mathbb{T}}$ over all pairs $\mathbb{K},\mathbb{T}$ with $k_n = k$ and $\mathbb{T}$ bounded by $t$. Obviously, each $F_{k,t}$ is closed, and $F_k = \bigcup_t F_{k,t}$. This proves that $F$ is a Borel subset of $Y$.

Fix some $t_0$ and let $k$ be the number of pure entries in $y'[1, r_{t_0}]$. Observe that $S^{m-1}y' \in F_{k,t_0}$ whenever $m$ falls outside all inserted blocks (then $y'_m$ is in the orbit of $x'$) and it does not fall in the interval of length $r_{t_0}$ preceding an inserted block $B_t(0)$ with $t > t_0$. For given $s > t_0$, the frequency of such integers $m$ in the interval $[1, r_s]$ is at least

$$1 - \sum_{t=1}^{s} \frac{q_t}{r_t} - \sum_{t=t_0+1}^{s} \frac{r_{t_0}}{r_t} \geq 1 - 2\sum_{t=1}^{s} \frac{q_t}{r_t}.$$

Choosing $p_t$ sufficiently large compared to $q_t$ we can make the series of the fractions $q_t/r_t$ summable to less than $\varepsilon/2$ ($\varepsilon$ was chosen in the assertion of the theorem).

Consider the following sequence of measures on $Y$:

$$M_s := \frac{1}{r_s} \sum_{i=0}^{r_s - 1} S^i(\delta_{y'}),$$

where $\delta_{y'}$ is the point mass at $y'$. Let $\nu$ be a weak* accumulation point of this sequence. Clearly, $\nu$ is an invariant measure on $Y$. In the above discussion of frequencies we have shown that $M_s(F_{k,t_0}) > 1 - \varepsilon$. Since the characteristic function of a closed set is upper semicontinuous, the limit measure $\nu$ assigns to $F_{k,t_0}$ a non-smaller value. Obviously, $F_k$ is a larger set, hence $\nu(F_k) \geq 1 - \varepsilon$. Finally, note that the $k$ in the above argument is arbitrarily large, hence $F$, being the decreasing intersection of the sets $F_k$, is also of $\nu$-measure at least $1 - \varepsilon$.

It is rather immediate to see how the induced map $S_F$ acts on $F$: if $y$ starts with $x\,Tx\,\ldots$ then $S_F y = Sy$, if $y$ starts with $x\,B_t(0)\,Tx\,\ldots$ then $S_F y = S^{q_t+1}y$. In any case $S_F$ shifts the element $y$ (starting with some $x$) so that $S_F y$ starts with $Tx$ (skipping over the inserted blocks). Thus, the projection $\pi_1$ onto the first coordinate serves as a factor map from the system $(F, S_F)$ into $(X, T)$. Clearly, the conditional measure $\nu_F$ projects to some invariant measure on $X$ supported by the image $\pi(F)$. But $(X, T)$ is assumed to be uniquely ergodic, hence $\pi(F)$ is a full measure subset of $X$. We have proved that $(X, \mu, T)$ is a measure-theoretic factor of $(F, \nu_F, S_F)$. The proof of the theorem is now complete. ∎

QUESTION 1. In the above construction it is essential that the periods $r_t$ form a base of an odometer (i.e., $r_t \,|\, r_{t+1}$). This is why the system $(Y, S)$ is not totally transitive. It is natural to ask whether a totally transitive (hence topologically weakly mixing, see [B]) or mixing ToP $\mathbb{N}$-action exists. (The condition $r_t \,|\, r_{t+1}$ is necessary to produce paths of blocks continuously changing from $r_t$-periodic to $r_{t+1}$-periodic. By Theorem A, some kind of such passages must occur.) By [G-W] a ToM system cannot be almost equicontinuous (which implies uniformly rigid). Is there any uniformly rigid ToM system?

QUESTION 2. As already mentioned, every ToP $\mathbb{N}$-action is chaotic in the sense of Li–Yorke. Is this also true for a ToM system or more generally for a transitive system with a dense set of minimal points?

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