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WEAKER FORMS OF CONTINUITY AND VECTOR-VALUED RIEMANN INTEGRATION

BY

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Abstract. It was proved by Kadets that a weak*-continuous function on $[0,1]$ taking values in the dual of a Banach space X is Riemann-integrable precisely when X is finite-dimensional. In this note, we prove a Fréchet-space analogue of this result by showing that the Riemann integrability holds exactly when the underlying Fréchet space is Montel.

1. Introduction. It is folklore that given a Banach space X, each continuous function $f : [0,1] \to X$ is Riemann-integrable. More generally, for a given (sequentially complete) topological vector space X, Riemann integrability of each continuous function $f : [0,1] \to X$ forces X to be locally convex (and conversely) (see [6, Theorem 3.5.1]). On the other hand, weak continuity of f is not sufficient to guarantee the Riemann integrability of f, unless X is a Schur space, i.e. weakly convergent sequences in X are already norm-convergent ([7]).

For functions taking values in a dual Banach space, it was shown by V. M. Kadets [3] that for each infinite-dimensional Banach space X, there always exists an X^*-valued function on $[0,1]$ which is weak*-continuous but not Riemann-integrable. We produce an alternative proof of this statement which follows as a special case of our main result stating that for each weak*-continuous function into X^*, the (strong) dual of the Fréchet space X, to be Riemann-integrable, it is both necessary and sufficient that X be a Montel space.

The main idea of the proof of the sufficiency part of the main theorem is inspired by the argument of Wang and Yang [7] which is suitably modified to fit the framework of Fréchet spaces as treated in this paper. Recalling that the classes of Banach and Montel spaces intersect precisely in the class of finite-dimensional spaces, Kadets' result quoted above follows as an easy consequence.

In what follows, we shall let X denote a Fréchet space and shall denote by I the closed interval $[0,1]$. By a *tagged partition* of $[0,1]$, we shall mean

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a set of the type $\{(s_i, [t_{i-1}, t_i]); 1 \le i \le j\}$ where $0 = t_0 < t_1 < \cdots < t_j = 1$ and $s_i \in [t_{i-1}, t_i]$, $1 \le i \le j$.

We shall say that a function $f : I \to X$ is *Riemann-integrable* (R-integrable) if the following holds: there exists $x \in X$ such that for all $\epsilon > 0$ and $n \ge 1$, there exists $\delta = \delta(\epsilon, n) > 0$ such that for each tagged partition $P = \{(s_i, [t_{i-1}, t_i]); 1 \le i \le j\}$ of $[0, 1]$ with

$$\|P\| = \max_{1 \le i \le j}(t_i - t_{i-1}) < \delta,$$

we have $p_n(S(f, P) - x) < \epsilon$, where $S(f, P)$ is the *Riemann sum* of f corresponding to the partition P,

$$S(f, P) = \sum_{i=1}^{j} f(s_i)(t_i - t_{i-1}).$$

Here $\{p_n\}_{n=1}^{\infty}$ denotes a sequence of seminorms generating the (Fréchet) topology of X. The (unique) vector x, to be denoted by $\int_0^1 f(t)\,dt$, will be called the *Riemann integral* of f on $[0, 1]$. For locally convex spaces X which are not metrizable, the definition of Riemann integrability of X-valued functions will be the same as given above except that the sequence $\{p_n\}_{n=1}^{\infty}$ will now be replaced by a family $\{p_\alpha\}_{\alpha \in \lambda}$ of seminorms generating the topology of X.

We shall also say that a sequence $x_n^* \subset X^*$ is *weak*-convergent* to $x^* \in X^*$ (in symbols, $x_n^* \xrightarrow{w^*} x^*$) if $x_n^*(x) \to x^*(x)$ for all $x \in X$. Let us call a (metrizable) locally convex space X *Montel* if closed and bounded subsets of X are compact. Besides finite-dimensional spaces, the class of Fréchet–Montel spaces includes all nuclear Fréchet spaces and, more generally, all Fréchet Schwartz spaces. In particular, the space ω (the countable product of the line), s (the space of rapidly decreasing sequences) and $H(\mathbb{C})$ (the space of entire functions with compact-open topology) are some of the well-known examples of Fréchet–Montel spaces. For basic definitions and a comprehensive treatment of these and related issues involving locally convex spaces and nuclear Fréchet spaces, see [4] and [8].

2. Main Theorem

THEOREM 2.1. *A Fréchet space X is Montel if and only if each weak*-continuous function $f : I \to X^*$ is R-integrable.*

In our proof of this theorem, we shall make use of the so-called 'fat' Cantor set in $[0, 1]$ which is nowhere dense and has a positive Lebesgue measure. It is constructed exactly as Cantor's ternary set except that after the first step when the middle third part of the interval $[0, 1]$ has been taken away, the construction at the nth stage consists in knocking out 2^{n-1} open subintervals from the middle of the remaining 2^{n-1} closed subintervals so that the length of each of the subintervals is equal to $\frac{1}{2^{n-1}}\frac{1}{3^n}$. Precisely, the *fat Cantor set* C is defined by

$$C = [0,1] \setminus G, \quad \text{where} \quad G = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{2^{k-1}} A_k^{(i)}$$

and $A_k^{(i)} = (a_k^{(i)}, b_k^{(i)})$ is the ith subinterval taken away from $B_k^{(i)}$, the middle of the ith closed subinterval left behind at the $(k-1)$th stage. Let us write $B_1^{(1)} = [0,1]$. Clearly, $A_k^{(i)} \subset B_k^{(i)}$ and for $i = 1, 2, \ldots, 2^{k-1}, k \geq 1$, we have

$$d(B_k^{(i)}) = \frac{1}{2^{k-1}}\left(1 - \sum_{j=1}^{k-1} \frac{1}{3^j}\right) \quad \text{and} \quad d(A_k^{(i)}) = \frac{1}{2^{k-1}}\,\frac{1}{3^k}.$$

It is easily seen that C has Lebesgue measure equal to $1/2$. Let the midpoint of $A_k^{(i)}$ be denoted by $c_k^{(i)}$. With this background, we are now ready for

Proof of Theorem 2.1. In our proof, we shall make use of the following characterisation of Fréchet–Montel spaces which is the Fréchet analogue of the Josefson–Nissenzweig theorem for Banach spaces.

THEOREM 2.2 (Bonet, Lindström and Valdivia [1]). *A Fréchet space X is Montel if and only if each weak*-null sequence in X^* is strong*-null.*

Let us recall that the strong*-topology (denoted by $\beta^*(X^*, X)$) on the dual X^* of the locally convex space X is defined by the family of seminorms $\{p_B;\ B \subset X$ an absolutely convex bounded set$\}$ where

$$p_B(f) = \sup_{x \in B} |f(x)|, \quad f \in X^*.$$

For a Banach space X, the locally convex topology $\beta^*(X^*, X)$ coincides with the norm topology given by the dual norm on X^*:

$$\|f\| = \sup_{\|x\| \leq 1} |f(x)|, \quad f \in X^*.$$

Unless otherwise stated, X^* shall denote the topological dual of X equipped with the $\beta^*(X^*, X)$-topology. We shall also write X^*_σ for $(X^*, \sigma(X^*, X))$ where $\sigma(X^*, X)$ is the weak*-topology on X^* generated by the family of seminorms $\{p_x;\ x \in X\}$ where

$$p_x(f) = |f(x)|, \quad f \in X^*.$$

Necessity. Assume that X is Fréchet–Montel and let $f : I \to X^*_\sigma$ be a continuous function. By sequential completeness of X^*_σ as the weak*-dual of a barrelled space (see [8, Chapter 10]), f is weak*-Riemann-integrable (with X^* equipped with its weak*-topology). Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of partitions of I such that $\|P_n\| \to 0$ as $n \to \infty$. Now weak*-Riemann integrability of f implies that $S(f, P_n) \xrightarrow{w^*} x^*$ for some $x^* \in X^*$, where, as usual, $S(f, P_n)$ denotes the Riemann sum of f corresponding to the partition P_n. Recalling that weak* and strong*-convergence of sequences coincide in the dual of a Fréchet–Montel space (Theorem 2.1), it follows that $S(f, P_n) \to x^*$ in X^*. In other words, f is Riemann-integrable (with respect to the strong*-topology of X^*).

Sufficiency. For each $k \geq 1$ and $i = 1, 2, \ldots, 2^{k-1}$, define

$$\phi_k^{(i)} : [0,1] \to \mathbb{R}$$

so that it vanishes outside of $A_k^{(i)} = [a_k^{(i)}, b_k^{(i)}]$ and is piecewise linear on $A_k^{(i)}$. More precisely, we can choose $\phi_k^{(i)}$ so that

$$\phi_k^{(i)}(t) = \begin{cases} (1/c_k^{(i)})(t - a_k^{(i)}), & t \in [a_k^{(i)}, c_k^{(i)}], \\ (1/c_k^{(i)})(b_k^{(i)} - t), & t \in [c_k^{(i)}, b_k^{(i)}], \\ 0, & \text{otherwise.} \end{cases}$$

Noting that $c_k^{(i)} = a_k^{(i)} + (b_k^{(i)} - a_k^{(i)})/2$, it follows that $\phi_k^{(i)}$ is continuous and so is $h_k : [0,1] \to \mathbb{R}$ where $h_k(t) = \sum_{i=1}^{2^{k-1}} \phi_k^{(i)}(t)$. To show that X is a Fréchet–Montel space, it suffices, by the Bonet–Lindström–Valdivia theorem quoted above, to prove that each weak*-null sequence in X^* is strong*-null (see also [2]).

Assume, on the contrary, that there exists a sequence $\{x_n^*\} \subset X^*$ which is weak*-null but not strong*-null. Let us define

$$(*) \qquad f(t) = \sum_{k=1}^{\infty} h_k(t) x_k^*, \qquad t \in [0,1].$$

The above formula gives a well-defined function $f : [0,1] \to X^*$; after all the series defining f is actually a finite sum in X^*. This follows from the observation that $f(t) = 0$ for $t \in C$ and and that for $t \notin C$, there exists $k_0 \geq 1$ such that $t \in (a_{k_0}^{(i)}, b_{k_0}^{(i)})$ for some i where $1 \leq i \leq 2^{k_0-1}$. In this case, the above series reduces to $h_{k_0}(t) x_{k_0}^*$.

Claim 1. *f is weak*-continuous.*

Since h_k is continuous for all $k \geq 1$, it suffices to show that the series defining f is uniformly convergent in $(X^*, \sigma(X^*, X))$.

To this end, fix $\epsilon > 0$ and $x \in X$. We can choose K_0 such that $|\langle x_k^*, x\rangle| < \epsilon$ for all $k \geq K_0$. By the definition of $h_k(t)$, it follows that

$$\sum_{k=K_0+1}^{\infty} h_k(t) x_k^* = 0 \quad \text{ for } t \in \bigcup_{k=1}^{K_0} \bigcup_{i=1}^{2^{k-1}} A_k^{(i)}$$

and that we can choose $k_0 > K_0$ such that

$$\sum_{k=K_0+1}^{\infty} h_k(t)x_k^* = h_{k_0}(t)x_{k_0}^* \quad \text{for } t \in \bigcup_{k=K_0+1}^{\infty} \bigcup_{i=1}^{2^{k-1}} A_k^{(i)}.$$

Noting that $|\langle h_{k_0}(t)x_{k_0}^*, x\rangle| = |h_{k_0}(t)|\,|\langle x_{k_0}^*, x\rangle| < \epsilon$ (because $|h_k(t)| \leq 1$ for all $t \in [0,1]$ and $k \geq 1$), it follows that for all $t \in [0,1]$, we have

$$\Big|\Big\langle f(t) - \sum_{k=1}^{K_0} h_k(t)x_k^*, x\Big\rangle\Big| = \Big|\Big\langle \sum_{k=K_0+1}^{\infty} h_k(t)x_k^*, x\Big\rangle\Big| < \epsilon.$$

Claim 2. *f is not Riemann-integrable.*

By the Cauchy criterion for R-integrability, it suffices to verify the following: there exists a bounded set $B \subset X$ such that for all $\delta > 0$, there exist partitions P_1, P_2 of $[0,1]$ with $\|P_1\| < \delta$, $\|P_2\| < \delta$, such that

$$p_B(S(f,P_1) - S(f,P_2)) > \frac{1}{2}.$$

Since $x_n^* \nrightarrow 0$ in $(X^*, \beta(X^*, X))$, there exists a bounded set $B \subset X$ such that, passing to a subsequence if necessary, $p_B(x_n^*) > 1$ for all $n \geq 1$. Now fix $\delta > 0$ and choose $m \geq 1$ such that $1/2^{m-1} < \delta$. Note that $d(B_m^{(i)}) < 1/2^{m-1}$ for $m \geq 1$ and $1 \leq i \leq 2^{m-1}$. Let us choose partitions P_1, P_2 of $[0,1]$ with $\|P_1\| < \delta, \|P_2\| < \delta$ where $P_1 = \{(s_j, [t_{j-1}, t_j]);\ 1 \leq j \leq N_m\}$, $P_2 = \{(s'_j, [t_{j-1}, t_j]);\ 1 \leq j \leq N_m\}$, satisfying the following properties:

(a) Both P_1 and P_2 contain the sets $B_m^{(i)}$ for $i = 1, \dots, 2^{m-1}$.
(b) $t_j - t_{j-1} < 1/2^{m-1}$, for $1 \leq j \leq N_m$.
(c) $s_j = s'_j$ if $[t_{j-1}, t_j] \neq B_m^{(i)}$, $i = 1, \dots, 2^{m-1}$.
(d) $s_j = c_m^{(i)}$ and $s'_j = a_m^{(i)}$ if $[t_{j-1}, t_j] = B_m^{(i)}$, $i = 1, \dots, 2^{m-1}$.

Now (c) gives $f(s_j) = f(s'_j)$ if $[t_{j-1}, t_j] \neq (B_m^{(i)})$, and (d) yields

$$f(s_j) = h_m(s_j)x_m^*, \quad f(s'_j) = 0 \quad \text{if } [t_{j-1}, t_j] = B_m^{(i)}.$$

Further, since $h_m(s_j) = \phi_m^{(i)}(c_m^{(i)}) = 1$, we get

$$\begin{aligned} p_B(S(f,P_1) - S(f,P_2)) &= p_B\Big(\sum_{j=1}^{N_m}(f(s_j) - f(s'_j))(t_j - t_{j-1})\Big) \\ &= p_B\Big(\sum_{i=1}^{2^{m-1}} h_m(s_j)x_m^* d(B_m^{(i)})\Big) = p_B(x_m^*)\sum_{i=1}^{2^{m-1}} d(B_m^{(i)}) \\ &= p_B(x_m^*)2^{m-1}\left[\frac{1}{2^{m-1}}\Big(1 - \sum_{j=1}^{k-1}\frac{1}{3^j}\Big)\right] > \frac{1}{2}, \end{aligned}$$

which means that f is not Riemann-integrable. This contradicts the given hypothesis and completes the proof.

Since a Banach space is Montel if and only if it is finite-dimensional, we recover Kadets' theorem [3] mentioned in the Introduction as a simple consequence:

Corollary 2.3. *Given an infinite-dimensional Banach space X, there always exists a weak*-continuous function $f : [0,1] \to X^*$ which is not Riemann-integrable.*

We conclude with the following problem which appears to be open.

Problem. Does the set of functions as guaranteed by Corollary 2.3 contain an infinite-dimensional vector space of dimension at least c, the cardinality of the continuum?

Some of the problems belonging to this circle of ideas arising in the theory of vector-valued measurability and integration have been treated in a recent joint work of the author with F. J. G. Pacheco [5].

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