COLLOQUIUM MATHEMATICUM

VOL. 129 2012 NO. 1

POTENTIAL THEORY OF ONE-DIMENSIONAL GEOMETRIC STABLE PROCESSES

BY

TOMASZ GRZYWNY and MICHAŁ RYZNAR (Wrocław)

Abstract. The purpose of this paper is to find optimal estimates for the Green function and the Poisson kernel for a half-line and intervals of the geometric stable process with parameter $\alpha \in (0,2]$. This process has an infinitesimal generator of the form $-\log(1+(-\Delta)^{\alpha/2})$. As an application we prove the global scale invariant Harnack inequality as well as the boundary Harnack principle.

1. Introduction. Let $B = (B_t : t \geq 0)$ be a Brownian motion in $\mathbb{R}^d$ and $T = (T_t : t \geq 0)$ be a subordinator independent of B. The process $X = (X_t : t \geq 0)$ defined by $X_t = B_{T_t}$ is a rotationally invariant Lévy process in $\mathbb{R}^d$ and is called a subordinate Brownian motion. The subordinator T used to define the subordinate Brownian motion X can be interpreted as *operational* time or *intrinsic* time. For this reason, subordinate Brownian motions have been used in mathematical finance and other applied fields.

Let ψ denote the *Laplace exponent* of the subordinator T, that is,

$$E \exp\{-\lambda T_t\} = \exp\{-t\psi(\lambda)\}.$$

Then the characteristic exponent Φ of the subordinate Brownian motion X takes on a very simple form, $\Phi(x) = \psi(|x|^2)$ (our Brownian motion B runs at twice the usual speed). Therefore, properties of X should follow from properties of the Laplace exponent of the subordinator.

Much progress has been made in recent years in the study of the potential theory of subordinate Brownian motions: see, for instance [RSV, SV, KSV1, B–V, KSV2, KSV3]. At first, the focus was on the potential theory of the process X in the whole of $\mathbb{R}^d$, and basic results about the behaviour of the potential kernel and Lévy measures were established for many particular examples of subordinators including geometric stable ones (see [RSV, SV]). Then in a natural path of investigation the (killed) subordinate Brownian motion in an open subset was explored. In the last few years significant progress has been made in studying the potential theory of subordinate Brownian motion

2010 *Mathematics Subject Classification*: Primary 60J45.
Key words and phrases: geometric stable process, Green function, first exit time, tail function, Poisson kernel, Lévy process, ladder height process, renewal function.

DOI: 10.4064/cm129-1-2

killed upon exiting an open subset of $\mathbb{R}^d$ (see the survey [KSV3]). The main results include the Harnack inequality, the boundary Harnack principle and sharp Green function estimates. However, such results were confined to subordinate Brownian motions obtained by using ψ not only being a complete Bernstein function but also satisfying a certain property,

$$\psi(\lambda) \approx \lambda^\beta l(\lambda), \qquad \lambda \to \infty, \tag{1.1}$$

where $0 < \beta < 1$, and l is a slowly varying function at ∞. Moreover, an extra assumption was imposed on ψ to avoid the process X being recurrent. In a recent paper [KSV3] the condition (1.1) was relaxed to comparability at ∞.

A natural question about the Harnack inequality, the boundary Harnack principle and sharp Green function estimates arises in the case $\beta = 0$ and without the transience assumption. In this note we do not attempt to investigate a general such case (i.e. $\beta = 0$), but we rather consider an important particular process, namely a geometric α-stable process on the real line. For this process the corresponding subordinator has the Laplace exponent $\psi(\lambda) = \log(1 + \lambda^{\alpha/2})$, $0 < \alpha \leq 2$. For $\alpha = 2$ it is also called the gamma variance process. Geometric α-stable processes have been treated in the literature and play an important role in theory and applications (see e.g. [MR]). Some potential theory of them was established in [SV], but to the best of our knowledge no sharp estimates of the Green functions and Poisson kernels of open subsets are known, even in the one-dimensional case.

Our main results are sharp estimates of the Green functions and Poisson kernels of intervals (including a half-line), global scale invariant Harnack inequality and the boundary Harnack inequality for harmonic functions on intervals. It is worth mentioning that our estimates take into account the size of the intervals, and the constants depend only on the characteristics of the process, when Green functions and Poisson kernels are considered. For example, we show that Poisson kernels for half-lines for α-stable and geometric α-stable processes are of the same order provided the starting point and the exit point are away from the boundary, if $0 < \alpha < 2$. On the other hand for starting points and exit points close to the boundary we have the same type of behaviour of the Poisson kernels for all $0 < \alpha \leq 2$.

A few months after the first version of this article was posted on arXiv we found a paper by P. Kim and A. Mimica [KM], who proved the scale invariant Harnack inequality for a very large class of subordinate Brownian motions in every dimension provided the process is transient. Our one-dimensional case is included in the class of processes they treated, but their results are local and ours are global.

2. Preliminaries. Throughout the paper we denote by $c, c_1, \ldots$ nonnegative constants which may depend on other constant parameters only.

The value of c or $c_1, \ldots$ may change from line to line in a chain of estimates. If we use C or $C_1, \ldots$ then they are fixed constants.

The notation $p(u) \approx q(u)$, $u \in A$, means that the ratio $p(u)/q(u)$, $u \in A$, is bounded from below and above by positive constants which may depend on other constant parameters only but do not depend on the set A.

In this section we present some basic material regarding the geometric stable process. For more detailed information, see [SV]. For questions regarding the Markov and strong Markov properties, semigroup properties, Schrödinger operators and basic potential theory, the reader is referred to [CZ] and [BG].

We first introduce an appropriate class of subordinating processes. As mentioned in the Introduction the geometric α-stable process is obtained by subordination of the Brownian motion with a subordinator having the Laplace exponent $\psi(\lambda) = \log(1 + \lambda^{\alpha/2})$, $0 < \alpha \le 2$. The resulting process has the Lévy–Khinchin exponent $\Psi(x) = \psi(|x|^2) = \log(1 + |x|^\alpha)$. Another way of constructing the geometric α-stable process is to subordinate the rotational invariant α-stable process with the Gamma subordinator. Let $g_t(u) = \Gamma(t)^{-1} e^{-u} u^{t-1}$, $u, t > 0$, denote the density function of the Gamma subordinator T_t, with the Laplace transform

$$Ee^{-\lambda T_t} = e^{-t\log(1+\lambda)}. \tag{2.1}$$

Let Y_t^α be the isotropic α-stable process in $\mathbb{R}^d$ with the characteristic function of the form

$$E^0 e^{i\xi\cdot Y_t} = e^{-t|\xi|^\alpha}. \tag{2.2}$$

Assume that the processes T_t and Y_t are stochastically independent. Then the process $X_t^\alpha = Y_{T_t}^\alpha$ is called the *geometric stable process*. In what follows we use the generic notation X_t instead of X_t^α. From (2.1) and (2.2) it is clear that the characteristic function of X_t is of the form

$$E^0 e^{i\xi\cdot X_t} = e^{-t\log(1+|\xi|^\alpha)}.$$

In the case $\alpha = 2$, i.e. Y_t^2 is a Brownian motion running twice the usual speed, the corresponding process is the symmetric gamma variance process.

X_t is a Lévy process (i.e. homogeneous, with independent increments). We always assume that sample paths of the process X_t are right-continuous and have left-hand limits ("cadlag"). Then X_t is Markov and has the strong Markov property under the so-called standard filtration.

The geometric stable density can now be computed in the following way:

$$p_t(x) = \int_0^\infty s_u(x) g_t(u)\, du,$$

where

$$s_u(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix\xi - u|\xi|^\alpha} \, d\xi$$

is the density of the isotropic α-stable process, defined by (2.2). In general potential theory a very important role is played by potential kernels, which are defined as

$$U(x,y) = \int_0^\infty p_t(x-y)\, dt, \qquad x, y \in \mathbb{R}^d,$$

if the defining integral above is finite. For the geometric process the potential kernel is well defined for $d > \alpha$ but in contrast to the stable case it is not expressible as an elementary function. Recall that for the isotropic α-stable process the potential kernel is equal to $C|x-y|^{\alpha-d}$ for $d > \alpha$, where C is an appropriate constant depending on α and d. Nevertheless the asymptotic behaviour of the potential kernel was established in [SV]:

$$U(x-y) \approx \frac{1}{|x-y|^d \log^2(1+|x-y|^{-\alpha/2})}, \qquad x, y \in \mathbb{R}^d. \tag{2.3}$$

Note that (2.3) suggests that the process globally behaves like a stable one since its potential kernel is asymptotically equivalent to the stable process, when $|x-y|$ is large.

We also recall the form of the density function $\nu(x)$ of the Lévy measure of the geometric stable process:

$$\nu(x) = \int_0^\infty s_u(x) u^{-1} e^{-u} \, du.$$

The behaviour of the Lévy measure was investigated in [SV]. We recall that result for the d-dimensional case, but we need it only for $d = 1$ in the present paper. For $\alpha = 2$ we have

$$\nu(x) \approx \frac{1 + |x|^{(d-1)/2}}{|x|^d} e^{-|x|},$$

and for $0 < \alpha < 2$,

$$\nu(x) \approx \frac{1}{|x|^d (1 + |x|^\alpha)}.$$

For $d = 1$, which is the case investigated in this paper, for $\alpha = 2$ we even have an exact formula

$$\nu(x) = |x|^{-1} e^{-|x|}. \tag{2.4}$$

The *first exit time* of an (open) set $D \subset \mathbb{R}^d$ by the process X_t is defined by the formula

$$\tau_D = \inf\{t > 0 : X_t \notin D\}.$$

The fundamental object of potential theory is the *killed process* X_t^D when exiting the set D. It is defined in terms of sample paths up to time τ_D. More precisely, we have the following "change of variables" formula:

$$E^x f(X_t^D) = E^x[t < \tau_D;\, f(X_t)], \qquad t > 0.$$

The density function of the transition probability of the process X_t^D is denoted by p_t^D. We have

$$p_t^D(x,y) = p_t(x-y) - E^x[t > \tau_D;\, p_{t-\tau_D}(X_{\tau_D} - y)], \qquad x, y \in \mathbb{R}^d.$$

Obviously, we obtain

$$p_t^D(x,y) \le p_t(x,y), \qquad x, y \in \mathbb{R}^d.$$

$(p_t^D)_{t>0}$ is a strongly contractive semigroup (under composition) and shares most of the properties of the semigroup p_t. In particular, it is strongly Feller and symmetric: $p_t^D(x,y) = p_t^D(y,x)$.

The potential kernel of the process X_t^D is called the *Green function* of the set D and is denoted by G_D. Thus, we have

$$G_D(x,y) = \int_0^\infty p_t^D(x,y)\, dt.$$

Another important object in the potential theory of X_t is the *harmonic measure* of the set D. It is defined by the formula

$$P_D(x,A) = E^x[\tau_D < \infty;\, \mathbf{1}_A(X_{\tau_D})].$$

The density kernel (with respect to the Lebesgue measure) of the measure $P_D(x,A)$ (if it exists) is called the *Poisson kernel* of the set D. The relationship between the Green function of D and the harmonic measure is provided by the Ikeda–Watanabe formula [IW],

$$P_D(x,A) = \int_A \int_D G_D(x,y)\nu(y-z)\, dy\, dz, \qquad A \subset (\bar{D})^c.$$

In the case which we investigate in this paper, that is, when D is an open interval or a half-line, the above formula holds for any Borel $A \subset D^c$.

Now we define harmonic and regular harmonic functions. Let u be a Borel measurable function on $\mathbb{R}^d$. We say that u is a *harmonic* function in an open set $D \subset \mathbb{R}^d$ if

$$u(x) = E^x u(X_{\tau_B}), \qquad x \in B,$$

for every bounded open set B with the closure $\overline{B} \subset D$. We say that u is *regular harmonic* if

$$u(x) = E^x[\tau_D < \infty;\, u(X_{\tau_D})], \qquad x \in D.$$

The following lemma provides a very useful lower bound for the Green function. Its proof closely follows the approach used in [RSV], where the bounds on the potential kernels (Green functions for the whole $\mathbb{R}^d$) were established for some special subordinate Brownian motions (in particular for our process for $d > \alpha$). We omit the proof, since one can proceed exactly in the same way as in the proof of Lemma 2.11 in [GR].

LEMMA 2.1. *For any open set $D \subset \mathbb{R}^d$ we have*

$$G_D(x,y) \geq G_D^{(\alpha)}(x,y),$$

where $G_D^{(\alpha)}(x,y)$ is the Green function of D for the isotropic α-stable process.

3. Properties of the exit time from interval. Now, we briefly recall the basic notions of the fluctuation theory for Lévy processes. For a general account of this theory we refer the reader to [D]. Suppose that X_t is a general one-dimensional Lévy process. Let L_t be the local time of the process X_t reflected at its supremum $M_t = \sup_{s\leq t} X_s$, and denote by L_s^{-1} the right-continuous inverse of L_t, the *ascending ladder time process* for X_t. This is a (possibly killed) subordinator, and $H_s = X_{L_s^{-1}} = M_{L_s^{-1}}$ is another (possibly killed) subordinator, called the *ascending ladder height process.* The Laplace exponent of the increasing ladder process, that is, the (possibly killed) bivariate subordinator (L_s^{-1}, H_s) $(s < L(\infty))$, is denoted by $\kappa(z,\xi)$,

$$\kappa(z,\xi) = c\exp\Big(\int_0^\infty \int_{[0,\infty)} (e^{-t} - e^{-zt-\xi x})t^{-1}P(X_t \in dx)\,dt\Big),$$

where c is a normalization constant of the local time. Since our results are not affected by the choice of c we assume that $c = 1$.

Moreover, if X_t is not a compound Poisson process, then by [F, Corollary 9.7],

$$\kappa(0,\xi) = \exp\bigg(\frac{1}{\pi}\int_0^\infty \frac{\xi\log\Psi(\zeta)}{\xi^2+\zeta^2}\,d\zeta\bigg) = \psi^\dagger(\xi),$$

where $\Psi(\xi)$ is the Lévy exponent of X_t. By $V(x) = \int_0^\infty \mathbf{P}(H_s \leq x)\,ds$ we denote the renewal function of the process H_s. It is well known that V is subadditive, that is,

$$V(x+y) \leq V(x) + V(y), \qquad x, y \geq 0.$$

The behaviour of the renewal function and its derivative is crucial for our purposes. The following result was obtained independently in [KMR] and [KSV3]. In [KSV3] the assumption on the process X_t was a bit more restrictive.

PROPOSITION 3.1. *Let X_t be a symmetric Lévy process whose Lévy–Khinchin exponent $\Psi(\theta)$ has the property that $\Psi(\theta)$ and $\theta^2/\Psi(\theta)$ are increas-*

ing functions. Then

$$\psi^\dagger(\xi) \approx \sqrt{\Psi(\xi)} \quad \textit{and} \quad V(x) \approx \frac{1}{\sqrt{\Psi(1/x)}}.$$

Suppose additionally that $\Psi(\theta)$ is regularly varying at zero (resp. at ∞) with positive exponent and $V'(x)$ is eventually monotone at infinity (resp. at zero). Then

$$V'(x) \approx \frac{1}{x\sqrt{\Psi(1/x)}}, \qquad x \to \infty \qquad (\textit{resp. } x \to 0).$$

In the case when $\Psi(\theta)$ is slowly varying at ∞ or 0 the above proposition is of little help in estimating $V'(x)$ and we need to use another tool. We will take advantage of the following result proved recently in [KMR].

Proposition 3.2. *Let $\Psi(\xi)$ be the Lévy–Khinchin exponent of a symmetric Lévy process X_t which is not a compound Poisson process, and suppose that $\Psi(\xi) = \psi(\xi^2)$ for a complete Bernstein function ψ. Then V is a Bernstein function, and*

$$V(x) = bx + \frac{1}{\pi}\int_{0^+}^{\infty} \operatorname{Im}\left(-\frac{1}{\psi^+(-\xi^2)}\right)\frac{\psi^\dagger(\xi)}{\xi}(1 - e^{-x\xi})\, d\xi, \quad x > 0. \tag{3.1}$$

Moreover, V' is a completely monotone function and

$$V'(x) = b + \frac{1}{\pi}\int_{0^+}^{\infty} \operatorname{Im}\left(-\frac{1}{\psi^+(-\xi^2)}\right)\psi^\dagger(\xi)e^{-x\xi}\, d\xi, \qquad x > 0, \tag{3.2}$$

where $b = \lim_{\xi\to 0^+} \xi/\sqrt{\psi(\xi^2)}$.

Here the expression $\operatorname{Im}(-1/\psi^+(-\xi^2))\,d\xi$ should be understood in the distributional sense, as the weak limit of the measures $\operatorname{Im}(-1/\psi(-\xi^2+i\varepsilon))\,d\zeta$ on $\xi \in (0,\infty)$ as $\varepsilon \to 0^+$. The measure $\operatorname{Im}(-1/\psi^+(-\xi^2))\,d\xi$ has an atom of mass πb at 0, and this atom is not included in the integrals from 0^+ to ∞ in (3.1) and (3.2).

For the rest of this section we assume that X_t is a symmetric Lévy process (which is not a compound Poisson process) with renewal function V corresponding to a choice of the local time such that the Laplace exponent of the ladder time process is $\kappa(z,0) = \sqrt{z}$. We start with an estimate of the distribution function of the exit time τ from a half-line $(0,\infty)$, which was obtained in [KMR, Corollary 2].

Lemma 3.3. *Let τ be the exit time from $(0,\infty)$. There is an absolute constant C_1 such that*

$$P^x(\tau > t) \ge C_1\left(1 \wedge \frac{V(x)}{\sqrt{t}}\right), \qquad x, t > 0. \tag{3.3}$$

LEMMA 3.4. *Let* $0 < x < R$ *and* $\tau_{(0,R)}$ *be the exit time from the interval* $(0, R)$. *Then*

$$P^x(\tau_{(0,R)} < \tau) \le \frac{V(x)}{V(R)}. \tag{3.4}$$

Proof. The inequality (3.4) was shown in [KSV2] for the case when the resolvent kernels of the Lévy process are absolutely continuous with respect to the Lebesgue measure and 0 is regular for $(0, \infty)$. Let $Y_t^\epsilon = X_t + \epsilon B_t$, where B_t is a Brownian motion independent of X_t. Obviously Y^ϵ satisfies the above conditions. Furthermore it is easy to see that the renewal function of the Y^ϵ converges pointwise to V. Moreover, since Y^ϵ converges a.s. to X, uniformly on bounded intervals, the result follows by a limit passage. ∎

PROPOSITION 3.5. *Let* $0 < x < R$. *Then*

$$\frac{C_1^4}{16} V(x \wedge (R - x))V(R) \le E^x \tau_{(0,R)} \le V(x \wedge (R - x))V(R),$$

where C_1 *is the constant from Lemma* 3.3.

Proof. By symmetry it is enough to consider $x \le R/2$. According to [Be, p. 176, Theorem 20], for any measurable function $f : [0, \infty) \to [0, \infty)$, we have

$$E^x\Big[\int_0^\tau f(X_t)\,dt\Big] = \int_{[0,\infty)} V(dy) \int_{[0,x]} V(dz)\, f(x + y - z).$$

We take $f = I_{[0,R]}$. Then

$$\begin{aligned} E^x\tau_{(0,R)} &= E^x\Big[\int_0^{\tau_{(0,R)}} f(X_t)\,dt\Big] \le E^x\Big[\int_0^\tau f(X_t)\,dt\Big] \\ &= \int_{[0,\infty)} V(dy) \int_{[0,x]} V(dz)\, f(x + y - z) \\ &\le \int_{[0,R]} V(dy) \int_{[0,x]} V(dz) = V(R)V(x), \end{aligned}$$

which completes the proof of the upper bound.

To prove the lower bound we observe that

$$P^x(\tau > t) \le P^x(\tau_{(0,R)} > t) + P^x(\tau_{(0,R)} < \tau) \le \frac{E^x\tau_{(0,R)}}{t} + P^x(\tau_{(0,R)} < \tau).$$

Hence from (3.3) for $\sqrt{t} > V(x)$, and from (3.4), we obtain

$$\begin{aligned} E^x\tau_{(0,R)} &\ge t(P^x(\tau > t) - P^x(\tau_{(0,R)} < \tau)) \ge t\Big(C_1\frac{V(x)}{\sqrt{t}} - \frac{V(x)}{V(R)}\Big) \\ &= V(x)\sqrt{t}\Big(C_1 - \frac{\sqrt{t}}{V(R)}\Big). \end{aligned}$$

Let $\sqrt{t} = (C_1/2)V(R)$. Then, for $2V(x) \le C_1 V(R)$, we have

$$E^x\tau_{(0,R)} \ge \frac{C_1^2}{4}V(x)V(R).$$

Next, we deal with $(C_1/2)V(R) \le V(x) \le V(R/2)$. Then for x_0 such that $V(x_0) = (C_1/2)V(R)$, using the already proved lower bound, we obtain

$$E^R\tau_{(0,2R)} \ge E^{x_0}\tau_{(0,R)} \ge \frac{C_1^2}{4}V(x_0)V(R) = \frac{C_1^3}{8}V(R)V(R).$$

Finally, let $x_0 \le x \le R/2$. Then $V(x) \ge V(x_0) = (C_1/2)V(R)$, which implies

$$E^x\tau_{(0,R)} \ge E^x\tau_{(0,2x)} \ge \frac{C_1^3}{8}V(x)V(x) \ge \frac{C_1^4}{16}V(x)V(R). \ \blacksquare$$

REMARK 3.6. Assume that the Green function of the half-line exists. Then, for $x \le R/2$,

$$\int_0^R G_{(0,\infty)}(x,y)\,dy \le V(x)V(R).$$

Another consequence of Lemma 3.3 is the following two-sided bound on the exit probability, which is interesting in its own right. There is a huge literature on the subject of so called *scale* functions which describe the probability that the process leaves a given interval through its right end. The scale function has been found for numerous examples of spectrally negative processes (see the survey [KKR] and references therein). To the best of our knowledge, for symmetric processes, except the Brownian motion and α-stable motions, exact formulas are not known, hence optimal estimates seem important.

PROPOSITION 3.7. *Let* $0 < x < R$ *and* $\tau_{(0,R)}$ *be the exit time from the interval* $(0,R)$. *Then*

$$\frac{C_1^2}{4}\frac{V(x)}{V(R)} \le P^x(\tau_{(0,R)} < \tau) \le \frac{V(x)}{V(R)},$$

where C_1 *is the constant from Lemma* 3.3.

Proof. We only deal with the lower bound. From Proposition 3.5 we infer

$$P^x(\tau_{(0,R)} > t) \le \frac{E^x\tau_{(0,R)}}{t} \le \frac{V(x)V(R)}{t}.$$

Next, observe that

$$P^x(\tau > t) \le P^x(\tau_{(0,R)} > t) + P^x(\tau_{(0,R)} < \tau) \le \frac{V(x)V(R)}{t} + P^x(\tau_{(0,R)} < \tau).$$

Hence, from (3.3) for $\sqrt{t} \ge V(x)$, we have

$$C_1\frac{V(x)}{\sqrt{t}} - \frac{V(x)V(R)}{t} \le P^x(\tau_{(0,R)} < \tau).$$

If we choose $\sqrt{t} = (2/C_1)V(R) \geq (2/C_1)V(x) \geq V(x)$ then

$$C_1 \frac{V(x)}{\sqrt{t}} - \frac{V(x)V(R)}{t} = \frac{C_1^2}{4} \frac{V(x)}{V(R)}.$$

This yields

$$P^x(\tau_{(0,R)} < \tau) \geq \frac{C_1^2}{4} \frac{V(x)}{V(R)}, \qquad x < R. \blacksquare$$

4. Green function and Poisson kernel of the half-line. From now on, we assume that X_t is the one-dimensional geometric stable process. In order to find precise estimates of the Green function and the Poisson kernel we need to have nice estimates of the renewal function and its derivative for the ladder height process of X_t. Note that the Laplace exponent $\phi(\lambda) = \log(1 + \lambda^{\alpha/2})$ is a complete Bernstein function, therefore we can use Proposition 3.2. It is well known (see e.g. [B–V]) that the derivative $V'(x)$ of the renewal function is decreasing. Monotonicity of V' together with subadditivity of V is frequently used below.

LEMMA 4.1. *Let $\alpha \in (0, 2]$ and $x > 0$. Then*

$$V'(x) \approx \frac{1}{x \log^{3/2}(1 + x^{-\alpha/3})} \quad \textit{and} \quad V(x) \approx \frac{1}{\log^{1/2}(1 + x^{-\alpha})}.$$

Proof. The estimates of the renewal function $V(x)$, $x > 0$, as well as its derivative $V'(x)$ for $x > 1$ follow from Proposition 3.1. To deal with $V'(x)$ for $x \leq 1$, we apply Proposition 3.2 with $\psi(\xi) = \log(1 + \xi^{\alpha/2})$. Then it is evident that $b = 1$ for $\alpha = 2$ and $b = 0$ otherwise. Moreover,

$$\operatorname{Im}\left(-\frac{1}{\psi^+(-\xi^2)}\right) = \begin{cases} \dfrac{\pi}{\pi^2 + \log^2(\xi^2 - 1)} \mathbf{1}_{(1,\infty)}(\xi), & \alpha = 2, \\ \dfrac{\operatorname{Arg}(z)}{\operatorname{Arg}(z)^2 + \frac{1}{4}\log^2(1 + 2\xi^\alpha \cos\frac{\alpha\pi}{2} + \xi^{2\alpha})}, & \alpha < 2, \end{cases}$$

where $z = 1 + \xi^\alpha \cos\frac{\alpha\pi}{2} + i\xi^\alpha \sin\frac{\alpha\pi}{2}$. Next, by Proposition 3.1, we have $\psi^\dagger(\xi) \approx \sqrt{\psi(\xi^2)}$. Let

$$\mu(\xi) = \psi^\dagger(\xi) \operatorname{Im}\left(-\frac{1}{\psi^+(-\xi^2)}\right).$$

Note that $\mu(\xi) \approx \log^{-3/2} \xi$ for $\xi \geq 2$.

If $x \leq 1$ then by (3.2) we have

$$V'(x) = b + \int_0^2 e^{-x\xi}\mu(\xi)\, d\xi + \int_2^\infty e^{-x\xi}\mu(\xi)\, d\xi \approx 1 + \int_2^\infty \frac{e^{-x\xi}}{\log^{3/2}\xi}\, d\xi$$

$$\approx \int_{4x^{-1}}^\infty \frac{e^{-x\xi}}{\log^{3/2}\xi}\, d\xi + \int_2^{4x^{-1}} \frac{e^{-x\xi}}{\log^{3/2}\xi}\, d\xi \approx \int_2^{4x^{-1}} \frac{1}{\log^{3/2}\xi}\, d\xi \approx \frac{1}{x\log^{3/2}(4x^{-1})}.$$

In the last line we use the inequality

$$0 < \int_{4x^{-1}}^{\infty} \frac{e^{-x\xi}}{\log^{3/2}\xi}\,d\xi < \frac{e^{-4}}{x\log^{3/2}(4x^{-1})}. \quad \blacksquare$$

As an immediate consequence we obtain the estimate

$$V'(x) \approx \frac{V(x)}{x\log(2+x^{-1})}, \qquad x > 0. \tag{4.1}$$

The next lemma provides useful estimates for some integrals involving V.

LEMMA 4.2. *We have the following estimates:*

$$\int_0^x V(y)\,dy \approx xV(x), \qquad x > 0, \tag{4.2}$$

$$\int_1^x V(y)\,\frac{dy}{y} \approx V(x), \qquad x \ge 2, \tag{4.3}$$

$$\int_x^1 V(y)\,\frac{dy}{y} \approx \frac{1}{V(x)}, \qquad x \le 1/2, \tag{4.4}$$

$$\int_x^1 V^\beta(y)\,\frac{dy}{y^2} \approx \frac{V^\beta(x)}{x}, \qquad x \le 1/2,\ \beta > 0. \tag{4.5}$$

Proof. The first approximation is true for all Lévy processes. Indeed, by monotonicity and subadditivity of V we have

$$\frac{1}{4}xV(x) \le \frac{x}{2}V\left(\frac{x}{2}\right) \le \int_{x/2}^{x} V(y)\,dy \le \int_0^x V(y)\,dy \le xV(x).$$

For $y \ge 1$, by Lemma 4.1, we get $V(y)/y \approx V'(y)$, which leads to (4.3). Next, Lemma 4.1 implies

$$\int_x^1 V(y)\,\frac{dy}{y} \approx \int_x^1 \frac{dy}{y\log^{1/2}(1+1/y)} \approx \log^{1/2}\left(1+\frac{1}{x}\right) \approx \frac{1}{V(x)}.$$

Moreover, by (4.1),

$$\lim_{x\to 0^+} \frac{\frac{V^\beta(x)}{x^2}}{\frac{V^\beta(x)}{x^2} - \frac{\beta V^\beta(x)V'(x)}{V(x)x}} = 1,$$

which yields (4.5) by applying l'Hospital's rule. $\blacksquare$

By [Be, Theorem 20, p. 176] we have a basic and very useful formula for the Green function of the half-line.

Lemma 4.3. *For $0 < x < y$ we have*

$$G_{(0,\infty)}(x,y) = \int_0^x V'(u)V'(y-x+u)\,du.$$

At this point let us recall that the exact formulas for the Brownian Green functions are well known for several regular sets, like intervals or half-lines (see e.g. [B]). Since some of them will be useful later, we list them for future reference. Recall that the Brownian motion we refer to in this paper has its clock running twice as fast as the usual Brownian motion. Denote the renewal function for the symmetric α-stable process (properly normalized) by

$$V^{(\alpha)}(x) = x^{\alpha/2}, \qquad \text{where } \alpha \in (0,2].$$

For the half-line we have

$$G^{(2)}_{(0,\infty)}(x,y) = x \wedge y = V^{(2)}(x \wedge y), \qquad x, y > 0,$$

while for the finite interval $(0,R)$,

$$\begin{aligned} G^{(2)}_{(0,R)}(x,y) &= \frac{x(R-y) \wedge y(R-x)}{R} \\ &= \frac{(V^{(2)}(x)V^{(2)}(R-y)) \wedge (V^{(2)}(R-x)V^{(2)}(y))}{R}, \quad x, y \in (0,R). \end{aligned}$$

We also recall known estimates for the stable case (see e.g. [BB]):

$$G^{(\alpha)}_{(0,R)}(x,y)$$

$$\approx \begin{cases} \min\left\{\dfrac{1}{|x-y|^{1-\alpha}}, \dfrac{(\delta_R(x)\delta_R(y))^{\alpha/2}}{|x-y|}\right\} \\ \quad = \min\left\{\dfrac{1}{|x-y|^{1-\alpha}}, \dfrac{V^{(\alpha)}(\delta_R(x))V^{(\alpha)}(\delta_R(y))}{|x-y|}\right\}, & \alpha < 1, \\ \log\left(1 + \dfrac{(\delta_R(x)\delta_R(y))^{1/2}}{|x-y|}\right) \\ \quad = \log\left(1 + \dfrac{V^{(1)}(\delta_R(x))V^{(1)}(\delta_R(y))}{|x-y|}\right), & \alpha = 1, \\ (\delta_R(x)\delta_R(y))^{(\alpha-1)/2} \wedge \dfrac{(\delta_R(x)\delta_R(y))^{\alpha/2}}{|x-y|} \\ \quad = \dfrac{V^{(\alpha)}(\delta_R(x))V^{(\alpha)}(\delta_R(y))}{(\delta_R(x)\delta_R(y))^{1/2}} \wedge \dfrac{V^{(\alpha)}(\delta_R(x))V^{(\alpha)}(\delta_R(y))}{|x-y|}, & \alpha > 1, \end{cases}$$

where $\delta_R(x) = x \wedge (R-x)$ for $R < \infty$ and $\delta_\infty(x) = x$.

Define

$$\hat{G}^{(\alpha)}_{(0,\infty)}(x,y)=\begin{cases} V(x\wedge y), & \alpha=2,\\ \min\left\{(xy)^{(\alpha-1)/2},\dfrac{V(x)V(y)}{|x-y|}\right\}, & 1<\alpha<2,\\ \log\left(1+\dfrac{V(x)V(y)}{V^2(|x-y|)}\right), & \alpha=1,\\ \min\left\{1,\dfrac{V(x)V(y)}{V^2(|x-y|)}\right\}\dfrac{1}{|x-y|^{1-\alpha}}, & \alpha<1.\end{cases}$$

Note that

$$\hat{G}^{(\alpha)}_{(0,\infty)}(x,y)\approx G^{(\alpha)}_{(0,\infty)}(x,y) \tag{4.6}$$

if $\alpha\neq 1$ for $x,y>1/2$, and if $\alpha=1$ for $x,y>1/2$ and $|x-y|>1/2$.

Now, we are in a position to prove the optimal estimates of the Green function of $(0,\infty)$, which are crucial for the rest of the paper.

THEOREM 4.4. *Let $x,y>0$. Then*

$$G_{(0,\infty)}(x,y)\approx\left(1\wedge\frac{V(x)V(y)}{V^2(|y-x|)}\right)\frac{1}{|y-x|\log^2(2+|y-x|^{-1})}+\hat{G}^{(\alpha)}_{(0,\infty)}(x,y).$$

Proof. Note that by monotonicity V' and Lemma 4.1, for $0<u\le w$,

$$V'(2w)\le V'(w+u)\le V'(w)\approx V'(2w). \tag{4.7}$$

Assume that $0<x<y$. We split the proof into several cases.

CASE 1: $2x\le y$. In this region $y/2\le y-x<y$ so, by subadditivity of V, $V(x)V(y)/V^2(|y-x|)\le 4$. Hence, by Lemma 4.3 and (4.7),

$$\begin{aligned} G_{(0,\infty)}(x,y)&\approx\int_0^x V'(u)V'(y)\,du=V(x)V'(y)\\ &\approx\frac{V(x)V(y)}{|y-x|}\,\frac{1}{\log(2+|y-x|^{-1})}. \end{aligned} \tag{4.8}$$

For $|y-x|\le 1$, by Lemma 4.1 we get $V^2(|y-x|)\approx\log^{-1}(2+|y-x|^{-1})$, which leads to

$$\begin{aligned} G_{(0,\infty)}(x,y)&\approx\frac{V(x)V(y)}{V^2(|y-x|)}\,\frac{1}{|y-x|\log^2(2+|y-x|^{-1})}\\ &\approx\frac{V(x)V(y)}{V^2(|y-x|)}\,\frac{1}{|y-x|\log^2(2+|y-x|^{-1})}+\hat{G}^{(\alpha)}_{(0,\infty)}(x,y), \end{aligned}$$

where the last step follows from the inequality

$$\frac{V(x)V(y)}{|y-x|}\,\frac{1}{\log(2+|y-x|^{-1})}\ge\hat{G}^{(\alpha)}_{(0,\infty)}(x,y).$$

Next, for $y-x>1$,

$$\hat{G}^{(\alpha)}_{(0,\infty)}(x,y) \approx \frac{V(x)V(y)}{y-x}.$$

Again, by Lemma 4.1, we have $V^2(y-x)\log^2(2+|y-x|^{-1}) \approx (y-x)^\alpha$ for $|y-x|>1$. Hence,

$$\begin{aligned} G_{(0,\infty)}(x,y) &\approx \frac{V(x)V(y)}{|y-x|} \approx \hat{G}^{(\alpha)}_{(0,\infty)}(x,y) \\ &\approx \frac{V(x)V(y)}{V^2(|y-x|)}\,\frac{1}{|y-x|\log^2(2+|y-x|^{-1})} + \hat{G}^{(\alpha)}_{(0,\infty)}(x,y). \end{aligned}$$

Case 2: $x+1/2<y<2x$. Note that $x>1/2$. By (4.6), $G^{(\alpha)}_{(0,\infty)}(x,y) \approx \hat{G}^{(\alpha)}_{(0,\infty)}(x,y)$ and

$$G^{(\alpha)}_{(0,\infty)}(x,y) \geq c\frac{1}{|y-x|\log^2(2+|y-x|^{-1})}.$$

By Lemma 4.3 and (4.7),

$$\begin{aligned} G_{(0,\infty)}(x,y) &= \int_0^{1/2} V'(u)V'(y-x+u)\,du + \int_{1/2}^{x} V'(u)V'(y-x+u)\,du \\ &\approx V'(y-x)V(1/2) + \int_0^{1/2} V'(u)V'(y-x+u)\,du. \end{aligned}$$

Similarly,

$$G^{(\alpha)}_{(0,\infty)}(x,y) \approx (V^{(\alpha)})'(y-x)V^{(\alpha)}(1/2) + \int_{1/2}^{x} (V^{(\alpha)})'(u)(V^{(\alpha)})'(y-x+u)\,du.$$

It follows from Lemma 4.1 that $V'(u) \approx (V^{(\alpha)})'(u)$ and $V(u) \approx V^{(\alpha)}(u)$ for $u \geq 1/2$. Hence

$$G_{(0,\infty)}(x,y) \approx G^{(\alpha)}_{(0,\infty)}(x,y) \approx \hat{G}^{(\alpha)}_{(0,\infty)}(x,y) + \frac{1}{|y-x|\log^2(2+|y-x|^{-1})}.$$

Case 3: $x<y<(x+1/2)\wedge 2x$. We use Lemma 4.3 and (4.7) to get

$$\begin{aligned} G_{(0,\infty)}(x,y) &= \int_0^{y-x} V'(u)V'(y-x+u)\,du + \int_{y-x}^{x} V'(u)V'(y-x+u)\,du \\ &\approx V'(y-x)V(y-x) + \int_{y-x}^{x} V'(u)V'(u)\,du. \end{aligned}$$

By Lemma 4.1 the first term is estimated in the following way:

$$V'(y-x)V(y-x) \approx \frac{1}{|y-x|\log^2(2+|y-x|^{-1})}. \tag{4.9}$$

It remains to estimate $\int_{y-x}^{x}(V')^2(u)\,du$. Note that, by Lemma 4.1, $V'(u) \approx V^3(u)/u$, $u \le 2$. Hence, for $x \le 2$, by (4.5),

$$\begin{aligned}\int_{y-x}^{x}(V')^2(u)\,du &\le c\int_{y-x}^{2}\frac{V^6(u)}{u^2}\,du \approx \frac{V^6(y-x)}{y-x} \\ &\approx \frac{1}{|y-x|\log^3(2+|y-x|^{-1})}.\end{aligned} \tag{4.10}$$

For $x > 2$, again by Lemma 4.1,

$$\int_1^x (V')^2(u)\,du \approx \begin{cases} x^{\alpha-1}, & \alpha>1 \\ \log x, & \alpha=1 \\ 1, & \alpha<1 \end{cases} \approx \begin{cases} (xy)^{(\alpha-1)/2}, & \alpha>1, \\ \log(1+x^{1/2}y^{1/2}), & \alpha=1, \\ 1, & \alpha<1. \end{cases}$$

Hence, for $\alpha > 1$,

$$\int_1^x (V')^2(u)\,du \approx \hat{G}^{(\alpha)}_{(0,\infty)}(x,y), \tag{4.11}$$

and for $\alpha \le 1$,

$$\int_1^x (V')^2(u)\,du \le c\hat{G}^{(\alpha)}_{(0,\infty)}(x,y). \tag{4.12}$$

Moreover, by (4.10),

$$\int_{y-x}^{x} (V')^2(u)\,du \ge c\hat{G}^{(\alpha)}_{(0,\infty)}(x,y). \tag{4.13}$$

Finally, combining (4.9)–(4.13) we get

$$G_{(0,\infty)}(x,y) \approx \frac{1}{|y-x|\log^2(2+|y-x|^{-1})} + \hat{G}^{(\alpha)}_{(0,\infty)}(x,y). \blacksquare$$

Remark 4.5. Let $|x-y| > A$. Then there exists a constant $C = C(A)$ such that

$$C^{-1}\hat{G}^{(\alpha)}_{(0,\infty)}(x,y) \le G_{(0,\infty)}(x,y) \le C\hat{G}^{(\alpha)}_{(0,\infty)}(x,y).$$

Moreover, if $x, y < 4$ then

$$G_{(0,\infty)}(x,y) \approx \left(1 \wedge \frac{V(x)V(y)}{V^2(|y-x|)}\right)|y-x|^{-1}\log^{-2}(1+|y-x|^{-1}).$$

In the rest of this section we prove the estimates of the Poisson kernel of $(0,\infty)$.

Recall that for $0 < \alpha < 2$ we know the Poisson kernel for the α-stable process (see e.g. [BGR]):

$$P^{(\alpha)}_{(0,\infty)}(x,z) = C_\alpha \frac{V^{(\alpha)}(x)}{V^{(\alpha)}(|z|)}\, \frac{1}{x-z}, \qquad z < 0 < x.$$

LEMMA 4.6. *Let $z < 0 < x$. Assume that $x \vee |z| \ge 1$. Then*

$$P_{(0,\infty)}(x,z) \approx \begin{cases} e^z \dfrac{V(x\wedge 1)}{V(|z|)}, & \alpha = 2, \\ \dfrac{V(x)}{V(|z|)}\, \dfrac{1}{x-z}, & \alpha < 2. \end{cases}$$

Proof. We consider several cases.

CASE 1: $\alpha = 2$ and $z < -1$, $x > 0$. Observe that

$$V(x\wedge 1)V(y\wedge 1) \le V(x \wedge y)V(1) \le V(x\wedge 1)V(y \vee 1).$$

Assume that $z \le -1$. Since by Remark 4.5, $G_{(0,\infty)}(x,y) \approx V(x\wedge y)$, $y \ge x+1$, using formula (2.4) and (3.6) we obtain

$$\begin{aligned} P_{(0,\infty)}(x,z) &\le c \int_0^\infty V(x\wedge y)\frac{e^{z-y}}{y-z}\,dy + c\int_{0\vee(x-1)}^{x+1} G_{(0,\infty)}(x,y)\frac{e^{z-y}}{y-z}\,dy \\ &\le \frac{ce^z}{|z|V(1)} V(x\wedge 1)\int_0^\infty V(y\vee 1)e^{-y}dy + c\frac{e^{z-x}}{x-z}\int_0^{2(x+1)} G_{(0,\infty)}(x,y)\,dy \\ &\le cV(x\wedge 1)\frac{e^z}{V(|z|)} + cV(x)V(2x+2)e^{-x}\frac{e^z}{V(|z|)} \\ &\le cV(x\wedge 1)\frac{e^z}{V(|z|)}. \end{aligned}$$

Similarly

$$\begin{aligned} P_{(0,\infty)}(x,z) &\ge c\int_0^1 V(x\wedge y)\frac{e^{z-y}}{y-z}\,dy \\ &\ge \frac{ce^z}{-2zV(1)} V(x\wedge 1)\int_0^1 V(y)e^{-y}\,dy \\ &\approx V(x\wedge 1)\frac{e^z}{V(|z|)}. \end{aligned}$$

CASE 2: $\alpha < 2$ and $z \le -1$, $x \ge 1$. For $y \le 1/2$ we have, by (4.8), $G_{(0,\infty)}(x,y) \approx V(x)V(y)/x$ and similarly $G^{(\alpha)}_{(0,\infty)}(x,y) \approx V^{(\alpha)}(x)V^{(\alpha)}(y)/x$. Observing that $\nu(y-z) \approx \nu^{(\alpha)}(y-z)$ for $y > 0$ and applying $V(x) \approx V^{(\alpha)}(x)$, which follows from Lemma 4.1, we obtain

$$\begin{aligned}\int_0^{1/2} G_{(0,\infty)}(x,y)\nu(y-z)\,dy &\approx \frac{V(x)}{x}\nu(z)\int_0^{1/2} V(y)\,dy \approx \frac{V(x)}{x}\nu(z)\\ &\approx \int_0^{1/2} \frac{V^{(\alpha)}(x)V^{(\alpha)}(y)}{x}\nu^{(\alpha)}(y-z)\,dy\\ &\approx \int_0^{1/2} G^{(\alpha)}_{(0,\infty)}(x,y)\nu^{(\alpha)}(y-z)\,dy.\end{aligned}$$

Note, by Lemma 2.1 and (4.6), that $G_{(0,\infty)}(x,y) \geq G^{(\alpha)}_{(0,\infty)}(x,y)$, and $G_{(0,\infty)}(x,y) \approx \hat{G}^{(\alpha)}_{(0,\infty)}(x,y) \approx G^{(\alpha)}_{(0,\infty)}(x,y)$ for $y \geq 1/2$, $|x-y| \geq 1$. We then infer that there is c such that

$$\begin{aligned}\int_0^{\infty} G^{(\alpha)}_{(0,\infty)}(x,y)\nu^{(\alpha)}(y-z)\,dy &\leq \int_0^{\infty} G_{(0,\infty)}(x,y)\nu(y-z)\,dy\\ &\leq c\int_0^{\infty} G^{(\alpha)}_{(0,\infty)}(x,y)\nu^{(\alpha)}(y-z)\,dy\\ &\quad + \int_{x-1}^{x+1} G_{(0,\infty)}(x,y)\nu(y-z)\,dy.\end{aligned}$$

Moreover, by (3.6),

$$\begin{aligned}\int_{x-1}^{x+1} G_{(0,\infty)}(x,y)\nu(y-z)\,dy &\leq \nu(x-1-z)\int_0^{2x} G_{(0,\infty)}(x,y)\,dy\\ &\leq c\nu(x-z)V(x)V(2x) \approx \frac{V^2(x)}{|x-z|^{1+\alpha}}\\ &\leq c\frac{V(x)}{|x-z|\,|z|^{\alpha/2}} \approx P^{(\alpha)}_{(0,\infty)}(x,z),\end{aligned}$$

which finally implies

$$P_{(0,\infty)}(x,z) \approx P^{(\alpha)}_{(0,\infty)}(x,z) \approx \frac{V(x)}{V(|z|)}\,\frac{1}{|x-z|}.$$

Case 3: $\alpha < 2$ and $z < -1$, $x \leq 1$. By (3.6),

$$\int_0^2 G_{(0,\infty)}(x,y)\nu(y-z)\,dy \leq \nu(|z|)\int_0^2 G_{(0,\infty)}(x,y)\,dy \leq c\nu(z)V(x) \approx \frac{V(x)}{|z|^{1+\alpha}}.$$

For $y \geq 2$, by (4.8), we have $G_{(0,\infty)}(x,y) \approx V(x)V(y)/y \approx V(x)/y^{1-\alpha/2}$, hence

$$\int_2^\infty G_{(0,\infty)}(x,y)\nu(y-z)\,dy \approx V(x)\int_2^\infty \frac{dy}{y^{1-\alpha/2}|y-z|^{1+\alpha}} \approx \frac{V(x)}{|z|^{1+\alpha/2}},$$

which yields

$$P_{(0,\infty)}(x,z) \approx \frac{V(x)}{|z|^{1+\alpha/2}} \approx \frac{V(x)}{V(|z|)}\,\frac{1}{x-z}.$$

Case 4: $\alpha \le 2$ and $-1 < z < 0$, $x \ge 1$. We split the integral defining the Poisson kernel into three parts,

$$\begin{aligned} P_{(0,\infty)}(x,z) = \int_0^{|z|/4} G_{(0,\infty)}(x,y)\nu(y-z)\,dy + \int_{|z|/4}^{1/2} G_{(0,\infty)}(x,y)\nu(y-z)\,dy \\ + \int_{1/2}^\infty G_{(0,\infty)}(x,y)\nu(y-z)\,dy. \end{aligned}$$

For $y \le 1/2$, by (4.8), we have $G_{(0,\infty)}(x,y) \approx V(x)V(y)/x$. Moreover $\nu(z) \approx 1/|z|$, hence

$$\int_0^{|z|/4} G_{(0,\infty)}(x,y)\nu(y-z)\,dy \approx \frac{V(x)}{x}\nu(z)\int_0^{|z|/4} V(y)\,dy \le c\frac{V(x)}{x}.$$

Next, applying (4.5), the second integral is estimated in the following way:

$$\begin{aligned} \int_{|z|/4}^{1/2} G_{(0,\infty)}(x,y)\nu(y-z)\,dy &\approx \frac{V(x)}{x}\int_{|z|/4}^{1/2} V(y)\nu(y)\,dy \\ &\approx \frac{V(x)}{x}\int_{|z|/4}^{1/2} \frac{V(y)}{y}\,dy \approx \frac{V(x)}{xV(|z|/4)}. \end{aligned}$$

Summing both estimates we infer that

$$\int_0^{1/2} G_{(0,\infty)}(x,y)\nu(y-z)\,dy \approx \frac{V(x)}{xV(|z|/4)} \approx \frac{V(x)}{V(|z|)}\,\frac{1}{|x-z|}.$$

For $y \le x/2$ or $y \ge 2x$, by (4.8), we have $G_{(0,\infty)}(x,y) \approx V(x)V(y)/(x+y)$, hence applying (3.6), we arrive at

$$\begin{aligned}\int_{1/2}^{\infty} & G_{(0,\infty)}(x,y)\nu(y-z)\,dy \\ &\le c\frac{V(x)}{x}\int_{1/2}^{\infty} V(y)\nu(y)\,dy + \int_{x/2}^{2x} G_{(0,\infty)}(x,y)\nu(z-y)\,dy \\ &\le c\frac{V(x)}{x} + \nu(x/2)\int_{0}^{2x} G_{(0,\infty)}(x,y)\,dy \\ &\le c\frac{V(x)}{x} + \nu(x/2)V(x)V(2x) \approx \frac{V(x)}{x}.\end{aligned}$$

Combining all the estimates of the integrals we obtain

$$P_{(0,\infty)}(x,z) \approx \frac{V(x)}{V(|z|)}\frac{1}{x-z} \quad \text{for } \alpha \le 2.$$

Noting that for $\alpha = 2$ we have $V(x)/(x-z) \approx 1$, we can rewrite the above as

$$P_{(0,\infty)}(x,z) \approx \frac{1}{V(|z|)} \quad \text{for } \alpha = 2. \ \blacksquare$$

THEOREM 4.7. *Let $z < 0 < x$ and $\alpha \in (0,2]$, then*

$$P_{(0,\infty)}(x,z) \approx \begin{cases} \dfrac{V(x\wedge 1)}{V(|z|)}\dfrac{1}{(x-z)\log\left(1+\frac{1}{x-z}\right)}e^{z}, & \alpha = 2,\\[2ex] \dfrac{V(x)}{V(|z|)}\dfrac{1}{(x-z)\log\left(2+\frac{1}{x-z}\right)}, & \alpha < 2.\end{cases}$$

Proof. By Lemma 4.6 it remains to consider the case $-1<z<0<x<1$. By Remark 4.5 we have

$$\begin{aligned}R(x,z) &= \int_0^2 G_{(0,\infty)}(x,y)\nu(y-z)\,dy \\ &\approx \int_0^2 \left(1\wedge\frac{V(x)V(y)}{V^2(|x-y|)}\right)\frac{1}{|x-y|\log^2\left(1+\frac{1}{|x-y|}\right)}\frac{dy}{y-z}.\end{aligned}$$

Let us denote

$$\begin{aligned}I_1 &= \int_0^{x/2} V(y)\frac{dy}{y-z},\\ I_2 &= \int_{x/2}^{3x/2} \frac{1}{|x-y|\log^2\left(1+\frac{1}{|x-y|}\right)}\,dy,\\ I_3 &= \int_{3x/2}^{2} \frac{V^3(y)}{y}\frac{dy}{y-z}.\end{aligned}$$

Note that

$$R(x,z) \approx \frac{V^3(x)}{x} I_1 + \frac{1}{x-z} I_2 + V(x) I_3.$$

We start with the estimate of I_2:

$$I_2 \approx \int_x^{3x/2} \frac{1}{(y-x)\log^2(y-x)}\, dy \approx \frac{1}{\log(1+1/x)} \approx V^2(x).$$

For $|z| < x$, by (4.5),

$$I_3 \approx \int_{3x/2}^{2} \frac{V^3(y)}{y^2}\, dy \approx \frac{V^3(x)}{x}.$$

Moreover, by (4.2),

$$\begin{aligned} I_1 &\approx \int_{|z|/4}^{x/2} \frac{1}{y \log^{1/2}(1+y^{-1})}\, dy + \frac{1}{|z|} \int_0^{|z|/4} V(y)\, dy \\ &\approx \log^{1/2}(1+4/|z|) - \log^{1/2}(1+x^{-1}) + V(|z|) \\ &\approx V(|z|) \log\left(1 + \frac{x}{|z|}\right). \end{aligned} \tag{4.14}$$

Hence, for $x > |z|$,

$$R(x,z) \approx \frac{V^2(x)}{x}\left(1 + V(|z|)V(x)\log\left(1+\frac{x}{|z|}\right)\right).$$

Assume that $2|z| < x < 1/2$, then

$$\begin{aligned} & 1 + V(|z|)V(x)\log\left(1+\frac{x}{|z|}\right) \\ &\approx 1 + \frac{1}{\log^{1/2}(1/|z|)} \frac{1}{\log^{1/2}(1/x)} \log\frac{x}{|z|} \\ &= 1 + \frac{\log^{1/2}(1/|z|)}{\log^{1/2}(1/x)} - \frac{\log^{1/2}(1/x)}{\log^{1/2}(1/|z|)} \approx \frac{\log^{1/2}(1/|z|)}{\log^{1/2}(1/x)} \approx \frac{V(x)}{V(|z|)}. \end{aligned} \tag{4.15}$$

If $|z| < x \le 2|z|$, then

$$1 + V(|z|)V(x)\log\left(1+\frac{x}{|z|}\right) \approx 1 \approx \frac{V(x)}{V(|z|)}. \tag{4.16}$$

For $x \ge 1/2$, we have

$$1 + V(|z|)V(x)\log\left(1+\frac{x}{|z|}\right) \approx V(z)\log\left(1+\frac{1}{|z|}\right) \approx \frac{V(x)}{V(|z|)}. \tag{4.17}$$

That is,

$$R(x,z) \approx \frac{V(x)}{V(|z|)}\,\frac{V^2(x)}{x} \approx \frac{V(x)}{V(|z|)}\,\frac{V^2(x-z)}{x-z}.$$

If $|z| \geq x$ we have, by (4.2),

$$I_1 \approx \frac{1}{|z|}\int_0^{x/2} V(y)\,dy \approx \frac{x}{|z|}V(x),$$

and by (4.5),

$$\begin{aligned}
I_3 &\approx \int_{7/4|z|}^{2} \frac{V^3(y)}{y^2}\,dy + \frac{1}{|z|}\int_{3/2x}^{7/4|z|} \frac{1}{y\log^{3/2}(1+y^{-1})}\,dy \\
&\approx \frac{V^3(|z|)}{|z|} + \frac{1}{|z|}\left(\frac{1}{\log^{1/2}(1+\frac{4}{7|z|})} - \frac{1}{\log^{1/2}(1+\frac{2}{3x})}\right) \\
&\approx \frac{V(|z|)}{|z|}\left(V^2(|z|) + V^2(x)\log\left(1+\frac{|z|}{x}\right)\right) \\
&\approx \frac{V^2(x)V(|z|)}{|z|}\log\left(1+\frac{|z|}{x}\right).
\end{aligned}$$

Combining the estimates of the integrals I_1, I_2 and I_3 we arrive, for $x \geq |z|$, at

$$R(x,z) \approx \frac{V^2(x)}{|z|}\left(1 + V(|z|)V(x)\log\left(1+\frac{|z|}{x}\right)\right).$$

By symmetry and (4.15)–(4.17) we infer that

$$R(x,z) \approx \frac{V^2(x)}{|z|}\,\frac{V(|z|)}{V(x)} \approx \frac{V(x)}{V(|z|)}\,\frac{V^2(x-z)}{x-z}.$$

Next, observe that $R(x,-1) \leq R(x,z)$, hence from the above established bound and Lemma 4.1 we infer that

$$R(x,z) \geq cV(x). \tag{4.18}$$

Since, by (4.8), $G_{(0,\infty)}(x,y) \approx V(x)V(y)/y$ for $y > 2$, we obtain

$$\int_2^{\infty} G_{(0,\infty)}(x,y)\nu(y-z)\,dy \leq cV(x)\int_2^{\infty} \frac{V(y)}{y}\nu(y)\,dy \leq cV(x),$$

which together with (4.18) implies that the Poisson kernel is comparable to $R(x,z)$. Hence, by Lemma 4.1,

$$P_{(0,\infty)}(x,z) \approx \frac{V(x)}{V(|z|)}\,\frac{1}{(x-z)\log\left(2+\frac{1}{x-z}\right)}. \blacksquare$$

REMARK 4.8. If $-1 < z < 0 < x < 1$, then

$$P_{(0,\infty)}(x,z) \approx \int_0^{7(x\vee|z|)/4} \left(1 \wedge \frac{V(x)V(y)}{V^2(|x-y|)}\right) \frac{1}{|x-y|\log^2\left(1+\frac{1}{|x-y|}\right)} \frac{dy}{y-z}.$$

5. Boundary Harnack principle. In this section we derive the Harnack inequality for nonnegative harmonic functions in intervals. The method we apply for this purpose is a regularization of the Poisson kernel of an interval or rather its upper bound provided by the Poisson kernel of a half-line. We follow the approach of [BSS], where it was used to deal with a class of symmetric stable processes not necessarily rotation invariant. As a consequence of the Harnack inequality we obtain the boundary Harnack principle.

We start with two elementary lemmas, which we leave without rigorous proofs, giving only some explanation how to derive them. The first lemma follows from the Ikeda–Watanabe formula and the fact that ν is radially decreasing.

LEMMA 5.1. *For any $r > 0$ and $|x| < r < |z|$,*

$$E^x\tau_{(-r,r)}\nu(|z|+2r) \le P_{(-r,r)}(x,z) \le E^x\tau_{(-r,r)}\nu(|z|-r).$$

From Proposition 3.5 we have $E^x\tau_{(-r,r)} \approx V(r)V(r-|x|)$. Combining this with the above lemma and the properties of the Lévy measure we easily obtain the following estimates.

LEMMA 5.2. *Suppose that h is a nonnegative function. Let $p > 1$ and $r > 0$. Let*

$$h_2(x) = E^x[h(X_{\tau_{(-r,r)}}),\, |X_{\tau_r}| > pr].$$

Then there is $C = C(p,\alpha) > 0$ such that for $|x| < r$,

$$\begin{aligned} C^{-1}V(r)V(r-|x|) \int_{|z|>pr} h(z)\nu(z)\,dz &\le h_2(x) \\ &\le CV(r)V(r-|x|) \int_{|z|>pr} h(z)\nu(z)\,dz, \qquad 0<\alpha<2, \end{aligned}$$

and

$$\begin{aligned} C^{-1}e^{-2r}V(r)V(r-|x|) \int_{|z|>pr} h(z)\nu(z)\,dz &\le h_2(x) \\ &\le Ce^{r}V(r)V(r-|x|) \int_{|z|>pr} h(z)\nu(z)\,dz, \qquad \alpha = 2. \end{aligned}$$

THEOREM 5.3 (Harnack inequality). *Let $1 < p \le 3/2$. There exists a constant $C = C(\alpha,p)$ such that for any $r > 0$ and any nonnegative function*

h, harmonic in $(-2r,2r)$, we have, for $0<\alpha<2$,

$$C^{-1}V^2(r)\int\limits_{|z|>pr} h(z)\nu(z)\,dz \le h(x) \le CV^2(r)\int\limits_{|z|>pr} h(z)\nu(z)\,dz, \qquad x\in(-r,r).$$

For $\alpha=2$ we have

$$C^{-1}e^{-5r/2}V^2(r)\int\limits_{|z|>pr} h(z)\nu(z)\,dz \le h(x) \le Ce^{2r}V^2(r)\int\limits_{|z|>pr} h(z)\nu(z)\,dz, \qquad x\in(-r,r).$$

Proof. In the proof below the constants $c_1,c_2,\dots$ will depend on p,α only.

For simplicity, we will write $\tau_{(-r,r)}$ as τ_r. We start with the upper bound. Define

$$\tilde{P}(x,z)=\int\limits_{pr}^{13r/8\wedge|z|} P_{(-t,t)}(x,z)\,dt, \qquad |z|>pr.$$

Since h is harmonic on $(-2r,2r)$, for all $t\in[pr,13r/8]$, we have

$$h(x)=\int\limits_{|z|>t} P_{(-t,t)}(x,z)h(z)\,dz.$$

Therefore

$$\begin{aligned}(13/8-p)rh(x)&=\int\limits_{pr}^{13r/8}\int\limits_{|z|>t} P_{(-t,t)}(x,z)h(z)\,dz\,dt\\ &=\int\limits_{pr<|z|<7r/4}\tilde{P}(x,z)h(z)\,dz+\int\limits_{|z|>7r/4}\tilde{P}(x,z)h(z)\,dz\\ &=I_1+I_2.\end{aligned}$$

By Lemma 5.2 we have

$$I_2\le c_1\begin{cases}(13/8-p)rV^2(r)\displaystyle\int\limits_{|z|>7r/4} h(z)\nu(z)\,dz, & \alpha<2,\\ e^{2r}(13/8-p)rV^2(r)\displaystyle\int\limits_{|z|>7r/4} h(z)\nu(z)\,dz, & \alpha=2.\end{cases}$$

In order to estimate I_1 we need an upper bound of $\tilde{P}(x,z)$. We claim that there is a constant c_2 such that for $pr<|z|<7r/4$, $|x|<r$,

$$\tilde{P}(x,z)\le c_2V^2(r\wedge 1). \tag{5.1}$$

By symmetry, we can assume that $z < -pr$. Then we have

$$\tilde{P}(x,z) \leq \int_{pr}^{13r/8\wedge|z|} P_{(-t,\infty)}(x,z)\,dt = \int_{pr}^{13r/8\wedge|z|} P_{(0,\infty)}(x+t,z+t)\,dt.$$

Since $|x| < r$, we have $(p-1)r < x+t < 3r$ and $x - z > (p-1)r$. First, assuming $r \leq 1$ for $\alpha = 2$, or arbitrary r for $0 < \alpha < 2$, by Theorem 4.7 we obtain

$$\begin{aligned}
\tilde{P}(x,z) &\leq c_3 \frac{1}{(x-z)\log\big(2+\frac{1}{x-z}\big)} \int_{pr}^{13r/8\wedge|z|} \frac{V(x+t)}{V(|z|-t|)}\,dt \\
&\leq c_4 \frac{1}{(x-z)\log\big(2+\frac{1}{x-z}\big)} V(3r) \int_0^{|z|} \frac{dt}{V(t)} \\
&\leq c_5 \frac{1}{r\log(2+1/r)} V(r) \int_0^{2r} \frac{dt}{V(t)}.
\end{aligned}$$

Noting that $\int_0^{2r} dt/V(t) \approx r/V(r)$ we obtain

$$\tilde{P}(x,z) \leq c_6 \frac{1}{\log(2+1/r)} \approx V^2(r\wedge 1).$$

Similarly, for $\alpha = 2$ and $r \geq 1$,

$$\tilde{P}(x,z) \leq c_7 \int_{pr}^{13r/8\wedge|z|} \frac{V(1)}{V(|z|-t)} e^{-|z|+t}\,dt \leq c_7 V(1) \int_0^{\infty} \frac{e^{-u}}{V(u)}\,du.$$

By (5.1) and since the density of the Lévy measure is radially decreasing we have

$$r^{-1} I_1 = r^{-1} \int_{pr<|z|<7r/4} \tilde{P}(x,z)h(z)\,dz \leq c_2 \frac{V^2(r\wedge 1)}{r\nu(r)} \int_{pr<|z|<7r/4} \nu(z)h(z).$$

Note that

$$\frac{V^2(r\wedge 1)}{r\nu(r)} \approx \begin{cases} V^2(r) & \text{for } \alpha < 2, \\ V^2(r)\dfrac{e^r}{1+r} & \text{for } \alpha = 2. \end{cases}$$

Combining this with the above estimates of I_1 and I_2 we obtain

$$h(x) \leq \begin{cases} c_8 V^2(r) \displaystyle\int_{|z|>pr} h(z)\nu(z)\,dz, & 0<\alpha<2, \\ c_8 e^{2r} V^2(r) \displaystyle\int_{|z|>pr} h(z)\nu(z)\,dz, & \alpha = 2. \end{cases}$$

Finally we find the lower bound for $h(x)$. Let $q = (1+p)/2$. Next, for

$$h_2(x) = E^x[h(X_{\tau_{qr}}),\, |X_{\tau_{qr}}| > pr],$$

by Lemma 5.2, for $0 < \alpha < 2$, we arrive at

$$h(x) \geq h_2(x) \geq c_9 V^2(r) \int_{|z|>pr} h(z)\nu(z)\,dz.$$

Similarly, for $\alpha = 2$, we have

$$h(x) \geq h_2(x) \geq c_9 e^{-5/2r} V^2(r) \int_{|z|>pr} h(z)\nu(z)\,dz. \blacksquare$$

REMARK 5.4. A weak form of the Harnack inequality for the geometric stable process was proved in $\mathbb{R}^d$ for $d > \alpha$ in [SV, Theorem 6.6]. It was shown there that there is a constant $C = C(r, \alpha, d)$ such that for any harmonic function h in a ball $B(0, r)$ we have

$$h(x) \leq Ch(y), \qquad x, y \in B(0, r/2). \tag{5.2}$$

As a function of $r > 0$ the constant C tends to ∞ as $r \searrow 0$. This form is not scale invariant. Recently in [KM] this result was improved and the constant in (5.2) is bounded provided $r < 1$, so the Harnack inequality is scale invariant. We mention here that the geometric stable case is only an example included in a large class of processes for which the scale invariant Harnack inequality holds [KM].

The constant from Theorem 5.3 does not depend on r for $0 < \alpha < 2$. In the last section we find two-sided estimates for the Poisson kernel of any interval which allow us to improve the Harnack inequality for $\alpha = 2$ (see Theorem 6.5).

THEOREM 5.5 (Boundary Harnack property). *There exists a constant $C = C(\alpha)$ such that for any $r > 0$ and any nonnegative function h regular harmonic in $(0, 2r)$ which vanishes in $(-2r, 0)$ we have, for $\alpha = 2$,*

$$C^{-1} e^{-2r} \frac{V(x)}{V(r)} \leq \frac{h(x)}{h(r)} \leq C e^{2r} \frac{V(x)}{V(r)}, \qquad 0 < x < r.$$

For $0 < \alpha < 2$,

$$C^{-1} \frac{V(x)}{V(r)} \leq \frac{h(x)}{h(r)} \leq C \frac{V(x)}{V(r)}, \qquad 0 < x < r.$$

Proof. We provide the proof for the case $0 < \alpha < 2$ only. Let

$$h_2(x) = E^x[h(X_{\tau_r}), |X_{\tau_r}| > 3r/2].$$

Note that by the Harnack inequality and Lemma 5.2 we have $h_2(r/2) \approx h(r/2) \approx h(r)$. Moreover, by Lemma 5.2,

$$\frac{h_2(x)}{h_2(r)} \approx \frac{V(x)}{V(r)}.$$

Hence,

$$h_2(x) \approx h(r)\frac{V(x)}{V(r)}.$$

Next, by the Harnack inequality,

$$\begin{aligned} h_1(x) &= E^x[h(X_{\tau_r}),\, r \le X_{\tau_r} < 3r/2] \\ &\le Ch(r)P^x(r < X_{\tau_r} < 3r/2) \le Ch(r)\frac{V(x)}{V(r)}. \end{aligned}$$

This implies that

$$h(x) = h_1(x) + h_2(x) \approx h(r)\frac{V(x)}{V(r)}, \qquad 0 < x < r. \ \blacksquare$$

6. Green function and Poisson kernel of the interval. This section is devoted to extension of the results of Section 4 to intervals. We show optimal estimates of the Green functions and Poisson kernels for intervals taking into account their size. Note that by letting the length of the intervals tend to infinity we recover the estimates from Section 4. This does not mean that the results of Section 4 can be obtained from the current section. In fact, we strongly use the estimates for half-lines, showing that for some choice of variables and interval lengths the Green functions and Poisson kernels are comparable for intervals and half-lines.

LEMMA 6.1.

(A) *There exists a constant* $a \le 1/2$ *such that, for* $0 < x, y \le aR$,

$$G_{(0,R)}(x,y) \ge G_{(0,\infty)}(x,y)/2.$$

(B) *For any* $0 < a < 1/2$ *there is a constant* $b < a/2$ *such that, for* $R \le 4$, *and* $aR/2 < x < y < (1 - a/2)R$,

$$G_{(0,R)}(x,y) \ge G_{(0,\infty)}(x,y)/2$$

if $|x - y| \le bR$.

Proof. Throughout the proof we assume that $0 < x < y$ and $a < 1/2$. Denote $\tau_R = \tau_{(0,R)}$ and observe that

$$G_{(0,R)}(x,y) = G_{(0,\infty)}(x,y) - E^x G_{(0,\infty)}(X_{\tau_R}, y).$$

Note that $G_{(0,\infty)}(z,y)$ is decreasing on (y,∞) as a function of z, which together with Lemma 3.4 implies

$$E^x G_{(0,\infty)}(X_{\tau_R}, y) \le G_{(0,\infty)}(R,y)\frac{V(x)}{V(R)}. \tag{6.1}$$

Observe that for $x < y$ we have

$$\frac{V(x)V(y)}{1 \wedge \frac{V(x)V(y)}{V^2(|y-x|)}} \le V^2(y). \tag{6.2}$$

Suppose that $x, y \le 2 \wedge aR$. Then by Remark 4.5,

$$G_{(0,\infty)}(x,y) \approx \left(1 \wedge \frac{V(x)V(y)}{V^2(|y-x|)}\right)|y-x|^{-1}\log^{-2}(1+|y-x|^{-1}).$$

By (4.8), we have $G_{(0,\infty)}(R,y) \approx \frac{V(y)V(R)}{R\log(2+R^{-1})}$. Applying (6.2) we obtain

$$\begin{aligned}\frac{\frac{V(x)}{V(R)}G_{(0,\infty)}(R,y)}{G_{(0,\infty)}(x,y)} &\le c\frac{V^2(y)|y-x|\log^2\left(2+\frac{1}{|y-x|}\right)}{R\log(2+R^{-1})}\\ &\le c\frac{V^2(aR\wedge 1)(aR\wedge 1)\log^2\left(2+\frac{1}{aR\wedge 1}\right)}{R\log(2+R^{-1})}.\end{aligned}$$

Next, by Lemma 4.1, we infer that

$$V^2(aR\wedge 1) \approx \log^{-1}\left(2+\frac{1}{aR\wedge 1}\right),$$

which proves that

$$\frac{\frac{V(x)}{V(R)}G_{(0,\infty)}(R,y)}{G_{(0,\infty)}(x,y)} \le c\frac{(aR\wedge 1)\log\left(2+\frac{1}{aR\wedge 1}\right)}{R\log(2+R^{-1})} \le ca\log\left(2+\frac{1}{a}\right). \tag{6.3}$$

Assume now that $x<1<2<y<aR$ or $1<x<y<aR$. If $x<1<2<y<aR$, due to (4.8), $G_{(0,\infty)}(x,y) \approx V(x)V(y)/y$. If $1<x<y<aR$ then

$$G_{(0,\infty)}(x,y) \ge c\hat{G}^{(\alpha)}_{(0,\infty)}(x,y) \ge c\frac{V(x)V(y)}{y}.$$

By (4.8) we have

$$G_{(0,\infty)}(y,R) \approx \frac{V(y)V(R)}{R},$$

which together with $G_{(0,\infty)}(x,y) \ge cV(x)V(y)/y$ implies

$$\frac{G_{(0,\infty)}(R,y)\frac{V(x)}{V(R)}}{G_{(0,\infty)}(x,y)} \le c\frac{y}{R} \le ca. \tag{6.4}$$

Combining (6.1), (6.3) and (6.4) we infer that

$$E^x G_{(0,\infty)}(X_{\tau_R},y) \le ca\log\left(2+\frac{1}{a}\right)G_{(0,\infty)}(x,y) \le G_{(0,\infty)}(x,y)/2$$

for sufficiently small a, which completes the proof of the first part of the lemma.

Now we proceed to the proof of part (B). Let $R \le 4$ and $aR/2 < x < y < (1-a/2)R$. Assume that $|x-y| \le bR$. Let us observe that $V^2(y-x) \le$

$V^2(bR) \le V^2(aR/2) \le V(x)V(y)$. Then by Remark 4.5 we have

$$\begin{aligned}\frac{\frac{V(x)}{V(R)}G_{(0,\infty)}(R,y)}{G_{(0,\infty)}(x,y)} &\approx \frac{\frac{V(x)}{V(R)}\left(1\wedge \frac{V(R)V(y)}{V^2(R-y)}\right)\frac{1}{(R-y)\log^2(1+(R-y)^{-1})}}{\left(1\wedge\frac{V(x)V(y)}{V^2(y-x)}\right)\frac{1}{(y-x)\log^2(1+(y-x)^{-1})}}\\ &\le \frac{\frac{1}{(R-y)\log^2(1+(R-y)^{-1})}}{\frac{1}{(y-x)\log^2(1+(y-x)^{-1})}} \le c\frac{bR\log^2\left(1+\frac{1}{bR}\right)}{aR\log^2\left(1+\frac{1}{aR}\right)}\\ &\le c\frac{b}{a}\left(1+\log\frac{a}{b}\right)^2.\end{aligned}$$

Hence $E^xG_{(0,\infty)}(X_{\tau_R},y) \le G_{(0,\infty)}(x,y)/2$ for sufficiently small b. ∎

Standard arguments imply estimates of the Green function of the interval $(0,R)$, $R>0$, if R is bounded by a fixed positive number R_0. In the theorem below we choose $R_0=4$ as an upper bound for R, but we could choose any positive number at the expense of the comparability constant.

THEOREM 6.2. *Let $R<4$. Then*

$$G_{(0,R)}(x,y) \approx \left(1\wedge \frac{V(\delta_R(x))V(\delta_R(y))}{V^2(|y-x|)}\right)|y-x|^{-1}\log^{-2}(1+|y-x|^{-1}).$$

Proof. If $x,y<aR$ then, by Theorem 4.4 and Lemma 6.1 we get

$$G_{(0,R)}(x,y) \approx G_{(0,\infty)}(x,y) \approx \left(1\wedge\frac{V(x)V(y)}{V^2(|y-x|)}\right)|y-x|^{-1}\log^{-2}(1+|y-x|^{-1}).$$

By symmetry we have, for $x,y>(1-a)R$,

$$G_{(0,R)}(x,y) \approx \left(1\wedge \frac{V(R-x)V(R-y)}{V^2(|y-x|)}\right)|y-x|^{-1}\log^{-2}(1+|y-x|^{-1}).$$

Let $aR/2<x<y<(1-a/2)R$. If $|x-y|\le bR$ then again, by Lemma 6.1 and Theorem 4.4,

$$G_{(0,R)}(x,y) \approx |y-x|^{-1}\log^{-2}(1+|y-x|^{-1}).$$

If $R>|x-y|>bR$, the Harnack inequality implies

$$G_{(0,R)}(x,y) \approx G_{(0,R)}(y-bR,y) \approx |y-x|^{-1}\log^{-2}(1+|y-x|^{-1}).$$

For $x<aR/2$ and $y>aR$ we use the boundary Harnack principle to get

$$G_{(0,R)}(x,y) \approx \frac{V(x)}{V(aR/2)}G_{(0,R)}(aR/2,y).$$

If $y>(1-a/2)R$ we again use the boundary Harnack principle

$$G_{(0,R)}(x,y) \approx \frac{V(x)}{V(aR/2)}G_{(0,R)}(aR/2,(1-a/2)R)\frac{V(y)}{V((1-a/2)R)}.$$

Hence

$$G_{(0,R)}(x,y) \approx \frac{V(\delta_R(x))V(\delta_R(y))}{V^2(|y-x|)}|y-x|^{-1}\log^{-2}(1+|y-x|^{-1}). \ \blacksquare$$

To extend the above uniform bound to large intervals we define a function $\hat{G}^{(\alpha)}_{(0,R)}(x,y)$, $x, y \in (0,R)$, such that

$$\hat{G}^{(\alpha)}_{(0,R)}(x,y) = \begin{cases} \min\left\{\frac{1}{|x-y|^{1-\alpha}}, \frac{V(\delta_R(x))V(\delta_R(y))}{|x-y|}\right\}, & \alpha < 1, \\ \log\left(1 + \frac{V(\delta_R(x))V(\delta_R(y))}{|x-y|}\right), & \alpha = 1, \\ \min\left\{\frac{V(\delta_R(x))V(\delta_R(y))}{(\delta_R(x)\delta_R(y))^{1/2}}, \frac{V(\delta_R(x))V(\delta_R(y))}{|x-y|}\right\}, & 1<\alpha<2, \\ \frac{(V(x)V(R-y)) \wedge (V(R-x)V(y))}{R}, & \alpha = 2. \end{cases}$$

THEOREM 6.3. *Let $R \geq 4$ and $x \leq y$. Then for $|x-y| \leq 1$,*

$$G_{(0,R)}(x,y) \approx \min\{G_{(0,\infty)}(x,y), G_{(0,\infty)}(R-x,R-y)\},$$

and for $|x-y| > 1$,

$$G_{(0,R)}(x,y) \approx \hat{G}^{(\alpha)}_{(0,R)}(x,y).$$

Proof. For $\alpha = 2$ we can use similar methods to the proof of Theorem 6.4 in [GR]. Therefore we assume that $\alpha < 2$. By symmetry we have $G_{(0,R)}(x,y) = G_{(0,R)}(R-x,R-y)$ and we can assume that $x \leq y$. Hence $G_{(0,R)}(x,y) \leq \min\{G_{(0,\infty)}(x,y), G_{(0,\infty)}(R-x,R-y)\}$. Let $|x-y| \leq 1$, and $x \leq R/2$. Then $\delta_R(y) \geq y/2$ and by Theorem 6.2 we infer

$$\begin{aligned} G_{(0,R)}(x,y) &\geq G_{((x-2)\vee 0,(x+2)\vee R)}(x,y) \\ &\approx \left(1 \wedge \frac{V(x\wedge 2)V(y \wedge 2)}{V^2(|x-y|)}\right)\frac{1}{|x-y|\log^2\left(1+\frac{1}{|x-y|}\right)}. \end{aligned}$$

Hence, Remark 4.5, for $x \leq 1$, and Theorem 4.4, Lemma 2.1 and (4.6), for $x > 1$, imply

$$G_{(0,R)}(x,y) \geq cG_{(0,\infty)}(x,y).$$

For $x > R/2$, we use symmetry to get

$$G_{(0,R)}(x,y) \geq cG_{(0,\infty)}(R-x,R-y),$$

which proves, for $|x-y| \leq 1$,

$$G_{(0,R)}(x,y) \approx \min\{G_{(0,\infty)}(x,y), G_{(0,\infty)}(R-x,R-y).$$

Assume that $|x-y| > 1$. Let us observe that, for $x, y \leq 3R/4$, we have $\hat{G}^{(\alpha)}_{(0,R)}(x,y) \approx \hat{G}^{(\alpha)}_{(0,\infty)}(x,y)$. Hence, by Remark 4.5,

$$G_{(0,R)}(x,y) \leq c\hat{G}^{(\alpha)}_{(0,R)}(x,y).$$

Lemma 2.1 implies, for $x, y \geq 1/2$,

$$G_{(0,R)}(x,y) \approx \hat{G}^{(\alpha)}_{(0,R)}(x,y). \tag{6.5}$$

If $x < 1/2$ we use the boundary Harnack principle to get the above estimate. By symmetry, (6.5) is true for $x, y \geq R/4$ as well. For $x < R/4$ and $y > 3R/4$ the boundary Harnack principle implies

$$G_{(0,R)}(x,y) \approx \frac{V(x)}{V(R/4)} G_{(0,R)}(R/4, 3R/4) \frac{V(R-y)}{V(3R/4)} \approx \hat{G}^{(\alpha)}_{(0,R)}(x,y). \quad \blacksquare$$

Now, we prove estimates for the Poisson kernel of the interval $(0, R)$. By symmetry, $P_{(0,R)}(x, z) = P_{(0,R)}(R-x, R-z)$. Therefore it is enough to prove estimates for $z < 0$ and $x \in (0, R)$.

THEOREM 6.4. *Assume that $z < 0 < x < R$. For $0 < \alpha \leq 2$ and $x, |z| \leq 2 \wedge R$ we have*

$$P_{(0,R)}(x,z) \approx \frac{V(x)}{V(|z|)} \frac{V(R-x)}{V(R+|z|)} \frac{1}{(x-z)\log\left(2+\frac{1}{x-z}\right)}.$$

For $0 < \alpha < 2$, when $x > 2$ or $|z| > 2 \wedge R$, we have

$$P_{(0,R)}(x,z) \approx \frac{V(x)V(R-x)}{V(|z|)V(|z|+R)} \frac{1}{x-z}.$$

For $\alpha = 2$,

$$P_{(0,R)}(x,z) \approx \begin{cases} e^{-|z|}\dfrac{V(x \wedge 1)V(R-x)}{RV(|z|)}, & R \geq 4,\ x > 2 \text{ or } |z| > 2, \\ e^{-|z|}\dfrac{V(x)V(R-x)}{|z|}, & R \leq 4,\ |z| \geq R. \end{cases}$$

Proof. We present arguments for $\alpha = 2$ only, since the proof is similar to the proof of Theorem 4.7. Moreover, for intervals of length not greater than $R_0 = 4$ (the upper bound 4 can be replaced by any R_0 at the expense of the comparability constant), the proof below is suitable for all α's, provided that $|z| \leq R$.

We will use Theorems 6.2 and 6.3, therefore we will first prove estimates for $R \leq 4$ and next for $R > 4$.

Assume that $R \leq 4$ and additionally that $-R/2 < z < 0$. Clearly

$$P_{(0,R)}(x,z) \leq P_{(0,\infty)}(x,z).$$

Note that $V(R-y) \approx V(y)$ for $R/2 \leq y \leq 7R/8$. Therefore by Theorem 6.2

and Remark 4.8, for $x \le R/2$, we have

(6.6)
$$P_{(0,R)}(x,z) \ge c \int_0^{7R/8} \left(1 \wedge \frac{V(x)V(y)}{V^2(|y-x|)}\right) \frac{1}{|y-x|\log^2(1+|y-x|^{-1})} \frac{dy}{y-z} \ge cP_{(0,\infty)}(x,z),$$

yielding

$$P_{(0,R)}(x,z) \approx P_{(0,\infty)}(x,z). \tag{6.7}$$

For $x > R/2$, we have $x - z \approx R \approx R/2 - z$ and $V(x) \approx V(R/2)$, hence by the boundary Harnack principle, (6.7) and Theorem 4.7,

$$\begin{aligned} P_{(0,R)}(x,z) &\approx P_{(0,R)}(R/2,z)\frac{V(R-x)}{V(R/2)} \\ &\approx \frac{V(R)}{(x-z)\log\left(2+\frac{1}{x-z}\right)} \frac{V(R-x)}{V(R/2)} \\ &\approx \frac{V(R-x)}{(x-z)\log\left(2+\frac{1}{x-z}\right)}. \end{aligned}$$

The last relation, (6.7) and Theorem 4.7 imply that

$$P_{(0,R)}(x,z) \approx \frac{V(x)}{V(|z|)} \frac{V(R-x)}{V(R+|z|)} \frac{1}{(x-z)\log\left(2+\frac{1}{x-z}\right)}, \qquad -R/2 < z < 0 < x < R \le 4.$$

For $z < -R/2$, we have $\nu(|z|) \approx \nu(z + 3R/2)$. Hence, by Lemma 5.1 and Proposition 3.5 we get, for $z \le -R/2$,

$$P_{(0,R)}(x,z) \approx E^x\tau_{(0,R)}\nu(z) \approx V(x)V(R-x)\nu(z).$$

This ends the proof for $R \le 4$.

Assume that $R \ge 4$. If $-1 < z < 0 < x < 1$ then by (6.7),

$$CP_{(0,\infty)}(x,z) \le P_{(0,3)}(x,z) \le P_{(0,R)}(x,z) \le P_{(0,\infty)}(x,z),$$

which, by Theorem 4.7, yields

$$P_{(0,R)}(x,z) \approx \frac{V(x)}{V(|z|)} \frac{V(R-x)}{V(R+|z|)} \frac{1}{(x-z)\log\left(2+\frac{1}{x-z}\right)}, \qquad -1 < z < 0 < x < 1.$$

For $x \vee |z| \ge 1$ and $x \le R/2$ we use the same arguments as in the proof Lemma 4.6 to get

$$P_{(0,R)}(x,z) \ge ce^z \frac{V(x \wedge 1)}{V(|z|)} \approx P_{(0,\infty)}(x,z).$$

Hence,

$$P_{(0,R)}(x,z) \approx P_{(0,\infty)}(x,z) \approx e^{-|z|}\frac{V(x\wedge 1)V(R-x)}{RV(|z|)}.$$

Now, assume that $x > R/2$. Denote

$$W(x,z) = \int_0^R \hat{G}^{(2)}(x,y)\nu(y-z)\,dy.$$

Then by (4.4),

$$\begin{aligned} W(x,z) &= \int_0^x \frac{V(R-x)V(y)}{R}\nu(y-z)\,dy + \int_x^R \frac{V(x)V(R-y)}{R}\nu(y-z)\,dy \\ &= \frac{e^z}{R}\left(V(R-x)\int_0^x \frac{V(y)e^{-y}}{y-z}\,dy + V(x)\int_0^{R-x}\frac{V(y)e^{-R+y}}{R-y-z}\,dy\right) \\ &\approx \frac{e^z}{R}\left(V(R-x)\frac{1}{V(|z|)} + V(x)e^{-x}\frac{(1\wedge(R-x))V(R-x)}{R-z}\right) \\ &\approx e^z\frac{V(R-x)}{RV(|z|)}. \end{aligned}$$

Moreover

$$\begin{aligned} &\int_{x-1}^{R\wedge(x+1)}\left(1\wedge\frac{V(R-x)V(R-y)}{V^2(|x-y|)}\right)\frac{\nu(y-z)\,dy}{|x-y|\log^2(1+|x-y|^{-1})} \\ &\qquad\qquad \le cP_{(0,\infty)}(R-x,R-z) \le cW(x,z). \end{aligned}$$

Hence, by Theorem 6.3,

$$P_{(0,R)}(x,z) \approx e^z\frac{V(R-x)}{RV(|z|)} \approx e^{-|z|}\frac{V(x\wedge 1)V(R-x)}{RV(|z|)}. \ \blacksquare$$

The next result is an improvement of the Harnack inequality for $\alpha = 2$, which was proved in the previous section.

THEOREM 6.5. *There exists a constant $C = C(\alpha)$ such that for any $r > 0$ and any nonnegative function h, harmonic in $(-2r, 2r)$, we have, for $0 < \alpha \le 2$,*

$$h(x) \le Ch(y), \qquad x, y \in (-r, r).$$

Proof. By Theorem 5.3 it is enough to prove the scale invariant Harnack inequality for $\alpha = 2$ when $r > 4$. We use Theorem 6.4 to get

$$\begin{aligned} P_{(-3r/2,3r/2)}(x,z) &\approx e^{-(|z|-3r/2)}\frac{1}{V(|z|-3r/2)} \\ &\approx P_{(-3r/2,3r/2)}(0,z), \qquad |x| < r,\ |z| > 3r/2, \end{aligned}$$

which yields

$$h(x) = \int_{|z|>3r/2} P_{(-3r/2,3r/2)}(x,z)h(z)\,dz \approx \int_{|z|>3r/2} P_{(0,R)}(0,z)h(z)\,dz = h(0).$$

Hence, $h(x) \approx h(y)$ for any $x, y \in (-r, r)$. ■

REFERENCES

[B] R. F. Bass, *Probabilistic Techniques in Analysis*, Springer, New York, 1995.

[Be] J. Bertoin, *Lévy Processes*, Cambridge Univ. Press, Cambridge, 1996.

[BG] R. M. Blumenthal and R. K. Getoor, *Markov Processes and Potential Theory*, Springer, New York, 1968.

[BGR] R. M. Blumenthal, R. K. Getoor and D. B. Ray, *On the distribution of first hits for the symmetric stable processes*, Trans. Amer. Math. Soc. 99 (1961), 540–554.

[BSS] K. Bogdan, A. Stós and P. Sztonyk, *Harnack inequality for stable processes on d-sets*, Studia Math. 158 (2003), 163–198.

[B–V] K. Bogdan, T. Byczkowski, T. Kulczycki, M. Ryznar, R. Song and Z. Vondraček, *Potential Analysis of Stable Processes and its Extensions*, Lecture Notes in Math. 1980, Springer, 2009.

[BB] H. Byczkowska and T. Byczkowski, *One-dimensional symmetric stable Feynman–Kac semigroups*, Probab. Math. Statist. 21 (2001), 381–404.

[CZ] K. L. Chung and Z. Zhao, *From Brownian Motion to Schrödinger's Equation*, Springer, New York, 1995.

[D] R. A. Doney, *Fluctuation Theory for Lévy Processes*, Lecture Notes in Math. 1897, Springer, Berlin, 2007.

[F] B. Fristedt, *Sample functions of stochastic processes with stationary, independent increments*, in: Advances in Probability and Related Topics, Vol. 3, Dekker, New York, 1974, 241–396.

[GR] T. Grzywny and M. Ryznar, *Two-sided optimal bounds for green functions of half-spaces for relativistic α-stable process*, Potential Anal. 28 (2008), 201–239.

[IW] N. Ikeda and S. Watanabe, *On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes*, J. Math. Kyoto Univ. 2 (1962), 79–95.

[KM] P. Kim and A. Mimica, *Harnack inequalities for subordinate Brownian motions*, Electron. J. Probab. 17 (2012), no. 37, 23 pp.

[KSV1] P. Kim, R. Song and Z. Vondraček, *Boundary Harnack principle for subordinate Brownian motions*, Stoch. Process. Appl. 119 (2009), 1601–1631.

[KSV2] P. Kim, R. Song and Z. Vondraček, *On the potential theory of one-dimensional subordinate Brownian motions with continuous components*, Potential Anal. 33 (2010), 153–173.

[KSV3] P. Kim, R. Song and Z. Vondraček, *Potential theory of subordinate Brownian motions revisited*, in: Stochastic Analysis and Applications to Finance, Interdisciplinary Math. Sci. 13, World Sci., 2012, 243–290.

[KKR] A. Kuznetsov, A. Kyprianou and V. Rivero, *The theory of scale functions for spectrally negative Lévy processes*, in: Lévy Matters II, Lecture Notes in Math. 2061, Springer, 2012, 97–186.

[KMR] M. Kwaśnicki, J. Małecki and M. Ryznar, *Suprema of Lévy processes*, Ann. Probab., to appear.

[MR] S. Mittnik and S. T. Rachev, *Modeling asset returns with alternative stable distributions*, Econometric Rev. 12 (1993), 261–330.

[RSV] M. Rao, R. Song and Z. Vondraček, *Green function estimates and Harnack inequality for subordinate Brownian motions*, Potential Anal. 25 (2006), 1–27.

[SV] H. Sikic, R. Song and Z. Vondraček, *Potential theory of geometric stable processes*, Probab. Theory Related Fields 135 (2006), 547–575.

Tomasz Grzywny, Michał Ryznar
Institute of Mathematics and Computer Sciences
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
E-mail: tomasz.grzywny@pwr.wroc.pl
michal.ryznar@pwr.wroc.pl

Received 2 April 2012 (5660)