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ON SMALL DEVIATIONS OF GAUSSIAN PROCESSES USING MAJORIZING MEASURES

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Abstract. We give two examples of periodic Gaussian processes, having entropy numbers of exactly the same order but radically different small deviations. Our construction is based on Knopp's classical result yielding existence of continuous nowhere differentiable functions, and more precisely on Loud's functions. We also obtain a general lower bound for small deviations using the majorizing measure method. We show by examples that our bound is sharp. We also apply it to Gaussian independent sequences and to a generic class of ultrametric Gaussian processes.

1. Introduction and preliminaries. The relatively recent small deviations theory of Gaussian processes and of more general processes is a very active and interactive domain of research, having connections with statistics and operator theory. It also completes the theory of large deviations, which was earlier extensively investigated.

Suppose that $X = \{X(t),\, t \in T\}$ is a sample bounded Gaussian process defined on a countable set, and let $M(X) = \sup_{t\in T} |X(t)|$. The case of an uncountable parameter set T can be easily reduced to the countable case by using the familiar notion of separable processes, a good reference being [7]. The *small deviation problem* it to find satisfactory estimates for the probability

$$\mathbb{P}\{M(X) \le \lambda\}, \qquad \lambda = o(\mu),$$

where $\mu(X)$ denotes the median of $M(X)$. Typically, one can take $\mu(X) = 2\mathbb{E}\, M(X)$. The *moderate deviation problem* concerns the range of values $\lambda \approx \mu$ and seems not to be much investigated. The case $\mu = o(\lambda)$ naturally corresponds to the study of the large deviations of X. The large deviations theory is essentially based on the Borel–Sudakov–Tsirelson isoperimetric inequality, regularity methods (metric entropy method, majorizing measure method) as well as Slepian's comparison lemma. Although some of these tools are relevant to the study of small deviations, this area also relies upon intrinsic devices: Laplace transform, Tauberian theorems, subadditive

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lemma, and most importantly, Kathri–Sidák's inequality implying for any centered Gaussian vector $(X_1, \dots, X_J)$ that

$$\mathbb{P}\Big\{\sup_{j=1}^{J} |X_j| \le z\Big\} \ge \mathbb{P}\{|X_1| \le z\}\mathbb{P}\Big\{\sup_{j=2}^{J} |X_j| \le z\Big\}. \tag{1.1}$$

Talagrand's well-known lower bound [9] is based on this device. Let $\{X(t), t \in T\}$ be a Gaussian process, and as is customary, let

$$d(s,t) = \|X(s) - X(t)\|_2, \qquad s,t \in T.$$

Recall that the *entropy number* $N(T, d, \varepsilon)$ is the minimal number (possibly infinite) of d-balls of radius $\varepsilon > 0$ that suffice to cover T. Assume there exists a nonnegative function ϕ on $\mathbb{R}^+$ such that $N(T, d, \varepsilon) \le \phi(\varepsilon)$, and moreover $c_1\phi(\varepsilon) \le \phi(\varepsilon/2) \le c_2\phi(\varepsilon)$ for some constants $1 < c_1 \le c_2 < \infty$. Then, for some $K > 0$ and every $\varepsilon > 0$,

$$\mathbb{P}\Big\{\sup_{s,t\in T} |X(s) - X(t)| \le \varepsilon\Big\} \ge e^{-K\phi(\varepsilon)}. \tag{1.2}$$

This estimate has been recently improved in [1] where a much larger set of size functions ϕ is permitted. The basic idea is the use of inequality (1.1) to control the Laplace transform of some standard approximating chaining sum, and next to apply de Bruijn's Tauberian result. As moreover some general links between Kolmogorov's entropy function $H(\varepsilon) = \log N(T, d, \varepsilon)$ (relative to the unit ball of the associated reproducing Hilbert space) and $-\log \mathbb{P}\{\sup_{t\in T} |X(t)| \le \varepsilon\}$ were earlier established by Kuelbs and Li (see [4] or [5]), there seems to be a kind of dual behavior between small deviations and entropy numbers of a Gaussian process.

However, this is not exactly so. The convenient estimate (1.2) is indeed known to not always provide sharp lower estimates, whereas in some cases it is quite sharp. See [4, §3.4] and [5, §2–3]. A typical instance is $X(t) = g|t|^\alpha$, $t \in [0,1]$, where $0 < \alpha \le 1$. We have $d(s,t) \le |t-s|^\alpha$, so that $N(T, d, \varepsilon) \le C\varepsilon^{1/\alpha}$. However $\mathbb{P}\{\sup_{0\le s,t\le 1} |X(s) - X(t)| \le \varepsilon\} \approx \varepsilon$. In fact, much more can be said. In Section 2, we show that there exist two sample continuous periodic Gaussian processes, with entropy numbers of *exactly* the same order, but having radically different small deviations. There also exist aperiodic sample continuous Gaussian processes for which this duality fails even more dramatically. In Section 3, we establish a new general lower bound for small deviations by using the majorizing measure method. We show by examples that our bound is sharp. We also apply it to Gaussian independent sequences and to a generic class of ultrametric Gaussian processes.

Notation and conventions. All Gaussian processes we consider are supposed to be *centered*. The letter g is used throughout to denote a standard Gaussian random variable. Further $g_1, g_2, \dots$ will always denote a sequence of i.i.d. standard Gaussian random variables. The notation $f(t) \preceq h(t)$ (resp.

$f(t) \succeq h(t)$) near $t_0 \in \overline{\mathbb{R}}$ means that for t in a neighborhood of t_0, $|f(t)| \le c|h(t)|$ (resp. $|f(t)| \ge c|h(t)|$) for some constant $0 < c < \infty$. We write $f(t) \approx h(t)$ when $f(t) \preceq h(t)$ and $f(t) \succeq h(t)$.

2. Examples failing the duality with entropy numbers. By considering two kinds of processes, one of type $X(t) = gf(t)$, where g is $\mathcal{N}(0,1)$, and the other as in the example given in [5, (2.5), (2.6)] (see also [4, Section 3.4]), we will prove the following striking result.

THEOREM 2.1. *Let $0 < \alpha < 1$. There exist two cyclic continuous Gaussian processes $X_1(t), X_2(t)$, $t \ge 0$, such that, as $\varepsilon \to 0$,*

$$N([0,1], d_{X_i}, \varepsilon) \approx \varepsilon^{1/\alpha}, \qquad i = 1, 2,$$

but

$$\mathbb{P}\Big\{ \sup_{0\le t\le 1} |X_1(t)| \le \varepsilon \Big\} \approx \varepsilon, \qquad \log \mathbb{P}\Big\{ \sup_{0\le t\le 1} |X_2(t)| \le \varepsilon \Big\} \approx -\Big(\log \frac{1}{\varepsilon}\Big)^2.$$

Therefore the sole information on the size of the entropy numbers of the process is in general insufficient to estimate its small deviations. The proof is essentially based on two lemmas.

To begin, recall a classical result from real analysis, the existence of continuous nowhere differentiable functions (see Knopp's construction in [3]). In [6], Loud has given an example of a function $f(t)$ which satisfies, for every real t, a Lipschitz condition of order precisely α $(0 < \alpha < 1)$. The proof is based on the method used in [3], as well as on van der Waerden's construction [11]. More precisely, if $0 < \alpha < 1$, there exists a continuous periodic function f and a pair of positive constants K_1, K_2 such that

(a) for any t and any h, $|f(t+h) - f(t)| \le K_1|h|^\alpha$
(b) for any t and infinitely many, arbitrarily small h,

$$|f(t+h) - f(t)| \ge K_2|h|^\alpha. \tag{2.1}$$

Let $\varphi(t,h)$ be the saw-tooth function equal to 0 for even multiples of h, to 1 for odd multiples of h, and linear otherwise. Loud's function is defined as follows: Let A be an integer such that $2^{2A(1-\alpha)} > 2$ and put

$$f(t) = \sum_{n=1}^{\infty} 2^{-2\alpha A n} \varphi(t, 2^{-2An}). \tag{2.2}$$

Then f satisfies (2.1). Notice that f is 2^{-2A}-periodic. The leading idea in Loud's proof is that for every pair of values of t and h, at most one or two terms of the series (2.2) make a significant contribution to the difference

$f(t+h)-f(t)$. Further, it is of interest to notice that property (b) is established for the values $h=2^{-2An}$, $n>1$. From this and by considering $X(\omega,t)=g(\omega)f(t)$, one easily deduces

LEMMA 2.2. *For any $0<\alpha\le 1$, there exists a cyclic Gaussian process $X(t)$, $t\ge 0$, with sample paths satisfying a Lipschitz condition of order precisely α. Moreover, as $\varepsilon\to 0$,*

$$N([0,1],d_X,\varepsilon)\asymp \varepsilon^{-1/\alpha}\quad \textit{whereas}\quad \mathbb{P}\Big\{\sup_{0\le t\le 1}|X(t)|\le\varepsilon\Big\}\asymp\varepsilon.$$

Now consider the following example. Let $0<\alpha<1$, let $p\ge 2$ be some integer, and let A be an integer such that $p^{2(1-\alpha)A}>2$. For each integer k, let $\varphi_k(t)=p^{-2\alpha Ak}\varphi(t,p^{-2Ak})$. Put

$$f(t)=\sum_{k=1}^{\infty}\varphi_k(t),\qquad X(t)=\sum_{k=1}^{\infty}g_k\varphi_k(t). \tag{2.3}$$

To prove Theorem 2.1 as well as Proposition 2.6, we will use the lemma below providing estimates of both the increments of f and of its random counterpart X.

LEMMA 2.3.

(a) *For all $0\le s,t\le 1$,*

$$c_1|s-t|^\alpha\le\|X(s)-X(t)\|_2\le c_2|s-t|^\alpha,$$

where

$$c_1=p^{-2A},\qquad c_2=\bigg(\frac{p^{4A\alpha}}{1-p^{-4(1-\alpha)A}}+\frac{1}{1-p^{-4\alpha A}}\bigg)^{1/2}.$$

(b) *For all $0\le s,t\le 1$ with $|s-t|=p^{-2A(m+1)}$ for some integer $m\ge 1$,*

$$|f(s)-f(t)|\ge\kappa_p|s-t|^\alpha.$$

Moreover, for all $0\le s,t\le 1$,

$$|f(s)-f(t)|\le\mathcal{K}_p|s-t|^\alpha.$$

Here

$$\kappa_p=p^{-2(1-\alpha)A}\,\frac{1-2p^{-2(1-\alpha)A}}{1-p^{-2(1-\alpha)A}},\qquad \mathcal{K}_p=\frac{p^{4A\alpha}}{1-p^{-4(1-\alpha)A}}+\frac{1}{1-p^{-4\alpha A}}.$$

Proof. This is just reproducing Loud's proof for $p\ne 2$, which we do because the way the constants depend on p and α matters in what follows. Given any function f, we write $\Delta f=f(t+\Delta t)-f(t)$ for any t and Δt. Let m be the integer such that $p^{-2A(m+1)}<\Delta t\le p^{-2Am}$. The slope of $\varphi_k(t)$ is $\pm p^{2(1-\alpha)Ak}$, so that

$$|\Delta\varphi_k|\le p^{2(1-\alpha)Ak}|\Delta(t)|\le p^{2(1-\alpha)Ak-2Am}=p^{-2(1-\alpha)A(m-k)-2A\alpha m}.$$

Moreover φ_k has maximal oscillation $p^{-2\alpha Ak}$. Therefore

$$(2.4)\qquad \begin{aligned}|\Delta f(t)| &\le \sum_{k=1}^{\infty}|\Delta\varphi_k(t)| \le \sum_{k=1}^{m} p^{-2(1-\alpha)A(m-k)-2A\alpha m} + \sum_{k=m+1}^{\infty} p^{-2\alpha Ak}\\ &\le \frac{p^{-2A\alpha m}}{1-p^{-2(1-\alpha)A}} + \frac{p^{-2\alpha A(m+1)}}{1-p^{-2\alpha A}}\\ &\le |\Delta t|^{\alpha}\left(\frac{p^{2A\alpha}}{1-p^{-2(1-\alpha)A}} + \frac{1}{1-p^{-2\alpha A}}\right).\end{aligned}$$

Furthermore,

$$\begin{aligned}\|\Delta X(t)\|_2^2 &= \sum_{k=1}^{\infty}[\Delta\varphi_k(t)]^2 \le 2\sum_{k=1}^{m} p^{-4(1-\alpha)A(m-k)-4A\alpha m} + 2\sum_{k=m+1}^{\infty} p^{-4\alpha Ak}\\ &\le 2\left\{\frac{p^{-4A\alpha m}}{1-p^{-4(1-\alpha)A}} + \frac{p^{-4\alpha A(m+1)}}{1-p^{-4\alpha A}}\right\}\\ &\le 2|\Delta t|^{2\alpha}\left(\frac{p^{4A\alpha}}{1-p^{-4(1-\alpha)A}} + \frac{1}{1-p^{-4\alpha A}}\right).\end{aligned}$$

In the other direction, fix t and let $\Delta t = p^{-2A(m+1)}$. By periodicity, $\Delta\varphi_k = 0$ if $k > m$, while $\Delta\varphi_k = \pm p^{2(1-\alpha)Ak-2Am}$ if $k \le m$. Thus

$$\begin{aligned}|\Delta f(t)| &= p^{-2A(m+1)}[\pm p^{2(1-\alpha)Am} \pm p^{2(1-\alpha)A(m-1)} \pm \cdots \pm p^{2(1-\alpha)A}]\\ &= p^{-2A-2\alpha Am}[\pm 1 \pm p^{-2(1-\alpha)A} \pm \cdots \pm p^{-2(1-\alpha)A(m-1)}].\end{aligned}$$

As $r := p^{-2(1-\alpha)A} < 1/2$, it follows that

$$g|\pm 1 \pm p^{-2(1-\alpha)A} \pm \cdots \pm p^{-2(1-\alpha)A(m-1)}| \ge 1 - \frac{r}{1-r} = \frac{1-2r}{1-r}.$$

As $|\Delta t|^{\alpha} = p^{-2\alpha A-2\alpha Am}$, we therefore get

(2.5)

$$|\Delta f(t)| \ge p^{-2A-2\alpha Am}\frac{1-2p^{-2(1-\alpha)A}}{1-p^{-2(1-\alpha)A}} = |\Delta t|^{\alpha}p^{-2(1-\alpha)A}\frac{1-2p^{-2(1-\alpha)A}}{1-p^{-2(1-\alpha)A}}.$$

The corresponding estimate for ΔX is very easy. Let m be such that $p^{-2A(m+1)} \le |\Delta t| < p^{-2Am}$. We have $\Delta\varphi_m(t) = \pm p^{-2A\alpha m}p^{2Am}\Delta t$. Thus

$$\begin{aligned}\|\Delta X(t)\|_2^2 &\ge [\Delta\varphi_m(t)]^2 = p^{-4A\alpha m}p^{4Am}p^{-4A(m+1)}\\ &= p^{-4A}p^{-4A\alpha m} \ge p^{-4A}|\Delta t|^{2\alpha}.\end{aligned}$$

This yields the lower bound with $c_1 = p^{-2A}$. ■

The following known estimate will be used. We give a proof because it is elementary and may be easily adapted (to some extent) to other non-geometric coeffcients.

LEMMA 2.4. *Given any* $0 < \rho < 1$,

$$\log \mathbb{P}\Big\{\sum_{n=1}^{\infty} |g_n|\rho^n \le \varepsilon\Big\} \approx \Big(\log \frac{1}{\varepsilon}\Big)^2 \quad \text{as } \varepsilon \to 0.$$

Here, we recall (see end of Section 1) that $g_1, g_2, \ldots$ denotes a sequence of i.i.d. standard Gaussian random variables.

Proof. We begin with the lower bound. Let $H = \sqrt{\rho}/(1-\sqrt{\rho})$. Plainly,

$$\mathbb{P}\Big\{\sum_{n=1}^{\infty} |g_n|\rho^n \le \varepsilon_0\Big\} \ge \prod_{n=1}^{\infty} \mathbb{P}\Big\{|g| < \frac{\varepsilon_0}{H}\rho^{-n/2}\Big\}.$$

Thus it suffices to estimate the product $\prod_{n=1}^{\infty} \mathbb{P}\{|g| < \varepsilon\delta^n\}$ with $\varepsilon = \varepsilon_0/H$, $\delta = \rho^{-1/2}$, $\delta > 1$. Let a be such that $\mathbb{P}\{|g| \ge a\} \le 1/2$, and put $N = \sup\{n : \delta^n \le a/\varepsilon\}$. Then

$$\prod_{n=1}^{N} \mathbb{P}\{|g| < \varepsilon\delta^n\} \ge \mathbb{P}\{|g| < \varepsilon\}^N \ge \exp\Big\{-C_\delta\Big(\log\frac{1}{\varepsilon}\Big)^2\Big\}.$$

Now,

$$\begin{aligned}\sum_{n=N+1}^{\infty} \mathbb{P}\{|g| \ge \varepsilon\delta^n\} &= \int_a^{\infty}\Big\{\sum_{a \le \varepsilon\delta^n \le u} 1\Big\}e^{-u^2/2}\,du \\ &\le C_\delta \int_a^{\infty} \{1 \vee \log u\} e^{-u^2/2}\,du < \infty.\end{aligned}$$

As $\log(1-x) \ge -2x$ if $0 \le x \le 1/2$ and $\mathbb{P}\{|g| > \varepsilon\delta^n\} \le 1/2$ if $n > N$, we get

$$\prod_{n=N+1}^{\infty} \mathbb{P}\{|g| < \varepsilon\delta^n\} \ge \exp\Big\{-\sum_{n=N+1}^{\infty} \mathbb{P}\{|g| \ge \varepsilon\delta^n\}\Big\} \ge c_\delta > 0.$$

Thus $\prod_{n=1}^{\infty} \mathbb{P}\{|g| < \varepsilon\delta^n\} \ge c_\delta \exp\{-C_\delta(\log\frac{1}{\varepsilon})^2\}$. To get the upper bound is faster. Let $N' = \sup\{n : \delta^n \le 1/\sqrt{\varepsilon}\}$. Then

$$\begin{aligned}\prod_{n=1}^{\infty} \mathbb{P}\{|g| < \varepsilon\delta^n\} &\le \prod_{n=1}^{N'} \mathbb{P}\{|g| < \varepsilon\delta^n\} \le \mathbb{P}\{|g| < \sqrt{\varepsilon}\}^{N'} \\ &= \exp\Big\{-N'\log\frac{1}{\mathbb{P}\{|g| < \sqrt{\varepsilon}\}}\Big\} \le \exp\Big\{-C_\delta\Big(\log\frac{1}{\varepsilon}\Big)^2\Big\}. \quad \blacksquare\end{aligned} \tag{2.6}$$

We can now prove Theorem 2.1. Take X_1 as in Lemma 2.2. Let $p = 2$ in (2.3) and choose $X_2 = X$. The entropy numbers clearly satisfy

$N([0,1], d_{X_i}, \varepsilon) \approx \varepsilon^{1/\alpha}$, $i = 1, 2$. First, by using Lemma 2.4,

$$\mathbb{P}\Big\{\sup_{0\le t\le 1} |X_2(t)| \le \varepsilon\Big\} \ge \mathbb{P}\Big\{\sum_{k=1}^{\infty} 2^{-2\alpha A k}|g_k| \le \varepsilon\Big\} \ge e^{-C_\alpha \log^2 \frac{1}{\varepsilon}}.$$

Next we notice

$$\varphi_j(2^{-2Ak}) = \begin{cases} 2^{-2A\alpha j}2^{-2A(k-j)} & \text{if } j \le k, \\ 0 & \text{if } j > k. \end{cases}$$

Thus $X_2(2^{-2Ak}) = 2^{-2Ak}\sum_{j=1}^{k} g_j 2^{2A(1-\alpha)j}$. And as

$$2^{2Ak}X_2(2^{-2Ak}) - 2^{2A(k-1)}X_2(2^{-2A(k-1)}) = g_k 2^{2A(1-\alpha)k},$$

it follows from (2.6) that

$$\begin{aligned}\mathbb{P}\Big\{\sup_{0\le t\le 1} |X_2(t)| \le \varepsilon\Big\} &\le \mathbb{P}\Big\{\sup_{s,t\in T} |X(s) - X(t)| \le \varepsilon(1 + 2^{-2A})\Big\} \\ &\le \prod_{k=1}^{\infty} \mathbb{P}\{|g_k| \le 2^{2A\alpha k}(1 + 2^{-2A})\varepsilon\} \\ &\le \exp\Big\{-C_\alpha\Big(\log\frac{1}{\varepsilon}\Big)^2\Big\}.\end{aligned}$$

This completes the proof.

Remark 2.5. Let $\psi(t) = 1 - |2\{t\} - 1|$ where $\{t\}$ denotes the fractional part of t. Lifshits has considered the following example:

$$X(t) = g_0 t + \sum_{n=1}^{\infty} g_n 2^{-\alpha n/2}\psi(\{2^n t\}), \qquad t \in [0,1]. \tag{2.7}$$

It is observed in [5] that $\|X(s) - X(t)\|_2 \ge c|t-s|^{\alpha/2}$ whereas

$$\log \mathbb{P}\Big\{\sup_{t\in T} |X(t)| \le \varepsilon\Big\} \approx -\log^2\frac{1}{\varepsilon}.$$

As we said at the beginning, our second process is of the same type since $\psi(t) = \varphi(t, 1/2)$.

A class of examples. If τ is a piecewise C^2 expanding map on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, by the Lasota–Yorke theorem there exists a τ-invariant probability measure μ which is absolutely continuous with respect to Lebesgue measure. So is the case for ψ. This leads us to introduce the following family of processes: let $\{a_n, n \ge 1\} \in \ell_1$, $f \in L^1(\mathbb{T}, \mu)$ and put

$$X(t) = \sum_{n=1}^{\infty} a_n g_n f(\psi^n(t)).$$

We have just considered the case $f(t) = t$. It would certainly be very informative to describe the small deviations of this class of Gaussian processes.

By Birkhoff's theorem,

$$\frac{1}{n}\sum_{k=0}^{n-1} f(\psi^n(t)) \to \int_{\mathbb{T}} f\,d\mu \quad \text{almost everywhere.}$$

The rate of this convergence, which for specific f only may be given explicitly, certainly plays a role since by using Abel summation we formally have

$$X(t) = \sum_{n=1}^{\infty} n(a_n - a_{n+1}) g_n \left[\frac{1}{n}\sum_{k=0}^{n-1} f(\psi^n(t))\right] \sim \Big(\int_{\mathbb{T}} f\,d\mu\Big) \sum_{n=1}^{\infty} n(a_n - a_{n+1}) g_n.$$

Shao and Li [4] argued from example (2.7) that stationarity (in fact, periodicity) should play a big role in upper estimates for small deviations.

We show that Loud's functions can be used to build aperiodic examples failing the duality even more dramatically. The intuitive idea behind the construction is that adding infinitely many functions with periods q_n^{-1}, where q_n are mutually coprime integers, produces aperiodic functions.

PROPOSITION 2.6. *There exists an aperiodic sample continuous Gaussian process* $\{X(t), 0 \le t \le 1\}$ *such that*

$$\lim_{\varepsilon\to 0} \frac{\log N([0,1], d_X, \varepsilon)}{\log\frac{1}{\varepsilon}} = \infty \quad \textit{while} \quad \liminf_{\varepsilon\to 0} \frac{\log \mathbb{P}\Big\{\sup_{0\le t\le 1} |X(t)| \le \varepsilon\Big\}}{\big(\log\frac{1}{\varepsilon}\big)^2} > -\infty.$$

Proof. Let $\mathcal{P}$ be an infinite set of mutually coprime integers larger than 2. Let $0 < \alpha_p < 1/2$, $\alpha_p \downarrow 0$. Take $A = 1$; then the condition $p^{2(1-\alpha_p)A} > 2$ holds. We further assume that

$$\lim_{p\to\infty} \alpha_p \log p = 0, \qquad 2^{hp}\alpha_p \log p \uparrow \infty \quad (\forall h > 0). \tag{2.8}$$

Let $\varphi_{p,k}(t) = p^{-2\alpha_p k}\varphi(t, p^{-2k})$, $k = 1, 2, \ldots$, and put $f_p = \sum_{k=1}^{\infty} \varphi_{p,k}$. Then f_p is p^{-2}-periodic. Now let $\{a_p, p \in \mathcal{P}\}$ be a sequence of reals such that $\sum_p a_p^2 < \infty$, and consider the Gaussian process

$$X(t) = \sum_p g_p a_p f_p(t). \tag{2.9}$$

Since $\mathcal{P}$ is a set of mutually coprime integers, periodicity is destroyed and so by considering its covariance, X is no longer periodic.

By Lemma 2.3, $|f_p(s) - f_p(t)| \ge \kappa_p |s-t|^{\alpha_p}$ whenever $|s-t| = p^{-2(m+1)}$, m an integer. By assumption (2.8), $p^{\alpha_p} \sim 1$ as $p \to \infty$, so that

$$\kappa_p = p^{-2(1-\alpha_p)} \frac{1 - 2p^{-2(1-\alpha_p)}}{1 - p^{-2(1-\alpha_p)}} \sim p^{-2}.$$

Moreover,

$$\|f_p\|_\infty \le \sum_{k=1}^{\infty} p^{-2\alpha_p k} = \frac{p^{-2\alpha_p}}{1-p^{-2\alpha_p}} \le \frac{1}{1-e^{-2\alpha_p \log p}} \le \frac{C}{\alpha_p \log p}.$$

Let $0 \le s,t \le 1$ be such that $|s-t| = p^{-2(m+1)}$. Then

$$\|X(s)-X(t)\|_2^2 = \sum_q a_q^2 |f_q(s)-f_q(t)|^2 \ge a_p^2 |f_p(s)-f_p(t)|^2 \ge a_p^2 \kappa_p^2 |s-t|^{2\alpha_p}.$$

Thus $\|X(s)-X(t)\|_2 \ge C a_p p^{-2} |s-t|^{\alpha_p}$. Now let $\alpha > 0$. We choose an integer m so that

$$m+1 \sim \frac{p\alpha \log 2}{2\alpha_p \log p}.$$

Then $|s-t|^{\alpha_p} = p^{-2\alpha_p(m+1)} \sim 2^{-p\alpha}$. Let β, γ be such that $0 < \beta < \alpha < \alpha+\beta < \gamma$, and choose $a_p = 2^{-\beta p}$. Then, for all p large enough,

$$\|X(s)-X(t)\|_2 \ge C 2^{-(\alpha+\beta)p} p^{-2} \ge 2^{-\gamma p}.$$

Put $\varepsilon = 2^{-\gamma p}$. Then

$$N([0,1], d_X, \varepsilon) \ge p^{2(m+1)} = e^{2(m+1)\log p} \ge 2^{c\,p\alpha/\alpha_p},$$

and so

$$\frac{\log N([0,1], d_X, \varepsilon)}{\log \frac{1}{\varepsilon}} \ge \frac{c\,p\alpha/\alpha_p}{\gamma p} = \frac{c\alpha}{\gamma \alpha_p},$$

which implies our first claim since $\alpha_p \downarrow 0$ as $p \to \infty$.

Now let $0 < \beta' < \beta$. As $2^{hp}\alpha_p \log p \uparrow \infty$ for any $h > 0$, it follows that

$$|X(t)| \le \sum_p |g_p| 2^{-\beta p} \|f_p\|_\infty \le C \sum_p |g_p| \frac{2^{-\beta p}}{\alpha_p \log p} \le C \sum_p |g_p| 2^{-\beta' p}.$$

Therefore, by using Lemma 2.4,

$$\mathbb{P}\Big\{ \sup_{0\le t\le 1} |X(t)| \le \varepsilon \Big\} \ge \mathbb{P}\Big\{ \sum_p |g_p| 2^{-\beta' p} \le \varepsilon / C \Big\} \ge e^{-C(\log \frac{1}{\varepsilon})^2}. \quad \blacksquare$$

3. A general lower bound using majorizing measures. The results from the previous section suggest the search of lower bounds for small deviations by using the majorizing measure method. It is known from the general theory of Gaussian processes that this is the paramount method for studying the regularity of Gaussian processes. And also that in general, entropy numbers are not a sufficiently precise tool. A classical example is provided by independent Gaussian sequences. See [8], [10], [12]. Generally speaking, once having Kathri–Sidák's inequality in hand, the argument leading to lower bounds is relatively direct. Familiarity with the chaining technique is however necessary. In [13], we obtained a general lower estimate for small deviations by using the majorizing measure method. Since the result is relevant there and in the next section, we present a slightly updated formulation of it and provide a proof.

Let $X = \{X(t),\, t \in T\}$ be a centered Gaussian process, with basic probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and let

$$d(s,t) = \|X(s) - X(t)\|_2, \qquad D = \operatorname{diam}(T,d).$$

We assume that $\sigma = \sup_{t\in T} \|X(t)\|_2 < \infty$ and that X is d-separable. Let $\Pi_0 \preceq \Pi_1 \preceq \cdots$ be a sequence of finite measurable ordered partitions of T (Π_{n+1} is a refinement of Π_n) such that

$$\max_{\pi\in\Pi_n} \max_{u,v\in\pi} d(u,v) \le 2^{-n} D, \qquad n = 0,1,\ldots. \tag{3.1}$$

Let $N_n = \#\{\Pi_n\}$. For any $\pi \in \Pi_n$, let $\bar\pi$ be such that $\pi \subset \bar\pi \in \Pi_{n-1}$. If $t \in T$, we also define $\pi_n(t)$ by the relations $t \in \pi_n(t) \in \Pi_n$. We now introduce a majorizing measure condition.

- *There exists a probability measure μ on T such that*

$$\lim_{n\to\infty} \sup_{t\in T} \sum_{m>n} 2^{-m} \left(\log\left(1 + \frac{1}{\mu(\pi_m(t))}\right)\right)^{1/2} = 0. \tag{3.2}$$

Put

$$H(n) = \sup_{t\in T} \sum_{m>n} (2^{-m} D) \left(\log\left(1 + \frac{1}{\mu(\pi_m(t))}\right)\right)^{1/2}.$$

Then $H(n)$ is finite and $H(n) \to 0$ as $n \to \infty$.

Theorem 3.1. *For $0 < \varepsilon\sigma < H(0)$, let $n(\varepsilon)$ be such that $H(n(\varepsilon)) \le 24\varepsilon\sigma$. Then*

$$\mathbb{P}\Big\{\sup_{t\in T} |X(t)| \le 25\varepsilon\sigma\Big\} \ge C e^{-N_{n(\varepsilon)} \log \frac{1}{\varepsilon}}.$$

Proof. Since X is d-separable, it suffices to produce a proof for a countable d-dense subset of T, which we will call again T. Put

$$X_\pi = \int_\pi X(u)\, \frac{\mu(du)}{\mu(\pi)}, \qquad X_n(t) = \int_{\pi_n(t)} X(u)\, \frac{\mu(du)}{\mu(\pi_n(t))}.$$

These Gaussian random variables are the bricks of the majorizing measure method. By (3.1), we have $\|X(t) - X_n(t)\|_2 \le 2^{-n}$. Elementary considerations then yield $X(t) \overset{\text{a.s.}}{=} \lim_{n\to\infty} X_n(t)$. It follows that $X(t) - X_n(t) \overset{\text{a.s.}}{=} \sum_{m=n+1}^{\infty} (X_m(t) - X_{m-1}(t))$ and we have the bound

$$|X(t)| \le \sup_{\pi\in\Pi_n} |X_\pi| + \sum_{m=n+1}^{\infty} |X_m(t) - X_{m-1}(t)|. \tag{3.3}$$

Let $n = n(\varepsilon)$. Then by using (3.3), and by applying Kathri–Sidák's inequality (1.1) repeatedly, we get

$$\mathbb{P}\Big\{\sup_{t\in T}|X(t) \le 25\varepsilon\sigma\Big\}$$

$$\ge \mathbb{P}\Big\{\sup_{\pi\in\Pi_n}|X_\pi| \le \varepsilon\sigma,\ \sup_{t\in T}|X(t)-X_n(t)| \le 24\varepsilon\sigma\Big\}$$

$$\ge \mathbb{P}\Big\{\sup_{\pi\in\Pi_n}|X_\pi| \le \varepsilon\sigma,\ \sup_{m\ge n}\frac{1}{D2^{-(m+1)}}\,\frac{|X_m(t)-X_{m+1}(t)|}{\big(\log_2\big(1+\frac{1}{\mu(\pi_m(t))}\big)\big)^{1/2}} \le 24\Big\}$$

$$\ge \prod_{\pi\in\Pi_n}\mathbb{P}\{|X_\pi| \le \varepsilon\sigma\}\,\mathbb{P}\Big\{\sup_{m\ge n}\frac{1}{D2^{-(m+1)}}\,\frac{|X_m(t)-X_{m+1}(t)|}{\big(\log_2\big(1+\frac{1}{\mu(\pi_{m+1}(t))}\big)\big)^{1/2}} \le 24\Big\}.$$

Now we apply the following inequality. Let $\varphi(x) = 2^{x^2}-1$. Then

$$\frac{x}{y} \le 1+\frac{\varphi(x)}{\varphi(y)}, \qquad x\ge 0,\ y>0.$$

We have (with $c = \sqrt{8(\log 2)/3}$)

$$\frac{1}{D2^{-(m+1)}}\,\frac{|X_m(t)-X_{m+1}(t)|}{\big(\log_2\big(1+\frac{1}{\mu(\pi_{m+1}(t))}\big)\big)^{1/2}}$$

$$\le \frac{4c}{\big(\log_2\big(1+\frac{1}{\mu(\pi_{m+1}(t))}\big)\big)^{1/2}}\int_{\pi_m(t)}\int_{\pi_{m+1}(t)}\frac{|X(u)-X(v)|}{cd(u,v)}\,\frac{\mu(du)}{\mu(\pi_m(t))}\,\frac{\mu(dv)}{\mu(\pi_{m+1}(t))}$$

$$\le 4\sqrt{2}\,c\int_{\pi_m(t)}\int_{\pi_{m+1}(t)}\frac{\frac{|X(u)-X(v)|}{cd(u,v)}}{\big(\log_2\big(1+\frac{1}{\mu(\pi_m(t))\mu(\pi_{m+1}(t))}\big)\big)^{1/2}}\,\frac{\mu(du)}{\mu(\pi_m(t))}\,\frac{\mu(dv)}{\mu(\pi_{m+1}(t))}$$

$$\le 4\sqrt{2}\,c\bigg(1+\int_{\pi_m(t)}\int_{\pi_{m+1}(t)}\varphi\bigg(\frac{|X(u)-X(v)|}{cd(u,v)}\bigg)\mu(du)\,\mu(dv)\bigg) \le 8(1+Z),$$

where

$$Z = \int_T\int_T\varphi\bigg(\frac{|X(u)-X(v)|}{d(u,v)}\bigg)\mu(du)\,\mu(dv).$$

We have $\mathbb{E}\,Z \le 1$. Thus

$$\mathbb{P}\Big\{\sup_{m\ge n}\frac{1}{D2^{-(m+1)}}\,\frac{|X_m(t)-X_{m+1}(t)|}{\big(\log_2\big(1+\frac{1}{\mu(\pi_{m+1}(t))}\big)\big)^{1/2}} \le 24\Big\} \ge \mathbb{P}\{Z\le 2\}$$

$$\ge 1-\frac{1}{2}\mathbb{E}\,Z \ge \frac{1}{2}.$$

Since $\|X_\pi\|_2 \le \sigma$ for all $\pi \in \Pi_n$ and n, we finally obtain

$$\mathbb{P}\Big\{\sup_{t\in T}|X(t) \le 25\varepsilon\sigma\Big\} \ge \frac{1}{2}\prod_{\pi\in\Pi_n}\mathbb{P}\{|X_\pi| \le \varepsilon\sigma\} \ge \frac{1}{2}\mathbb{P}\{|g| \le \varepsilon\}^{N_n}$$
$$\ge C\exp\Big\{-N_n\log\frac{1}{\varepsilon}\Big\}. \blacksquare$$

Let $\delta : [0,1] \to \mathbb{R}^+$ be increasing, $\delta(0) = 0$, and satisfying the integral condition

$$\int_0^D \bigg(\log\Big(1+\frac{1}{\delta^{-1}(u)}\Big)\bigg)^{1/2} du < \infty.$$

Corollary 3.2. *Assume there exists a family* $\{\Pi_n,\ n \ge 0\}$ *of finite measurable ordered partitions of* T *satisfying* (3.1) *and a probability measure* μ *on* T *such that*

$$\min\{\mu(\pi) : \pi \in \Pi_m\} \ge \delta(2^{-m}D)/2 \qquad (\forall m \ge 0).$$

Let

$$n(\varepsilon) = \sup\bigg\{n : \int_0^{2^{-n}D} \bigg(\log\Big(1+\frac{2}{\delta^{-1}(u)}\Big)\bigg)^{1/2} du \le 12\varepsilon\sigma\bigg\}.$$

Then (*recalling that* $N_n = \#\{\Pi_n\}$)

$$\mathbb{P}\Big\{\sup_{t\in T}|X(t)| \le 25\varepsilon\sigma\Big\} \ge C\exp\Big\{-N_{n(\varepsilon)}\log\frac{1}{\varepsilon}\Big\}.$$

Proof. We have

$$\sum_{m>n}(2^{-m}D)\bigg(\log\Big(1+\frac{1}{\mu(\pi_m(t))}\Big)\bigg)^{1/2}$$
$$\le \sum_{m>n}(2^{-m}D)\bigg(\log\Big(1+\frac{2}{\delta^{-1}(2^{-m}D)}\Big)\bigg)^{1/2}$$
$$\le 2\int_0^{\varepsilon_n}\bigg(\log\Big(1+\frac{2}{\delta^{-1}(u)}\Big)\bigg)^{1/2} du.$$

Therefore

$$\mathbb{P}\Big\{\sup_{t\in T}|X(t)| \le 25\varepsilon\sigma\Big\} \ge C\exp\Big\{-N_{n(\varepsilon)}\log\frac{1}{\varepsilon}\Big\}. \blacksquare$$

Example 3.3. Consider a Gaussian processes $X(t)$, $t \in [0,1]$, which satisfy the increment condition

$$\|X(s) - X(t)\|_2 \le \delta(|s-t|) \qquad (\forall s,t \in [0,1]).$$

For $m = 0,1\ldots,$ let Π_m be a partition of $[0,1]$ into consecutive intervals of length less than or equal to $\varepsilon_m = \delta^{-1}(2^{-m}D)$, $D = \delta(1)$. One can arrange it so that each interval has length greater than $\varepsilon_m/2$. Let μ be the Lebesgue measure. Then $\mu(\pi) \geq \delta^{-1}(2^{-m}D)/2$ if $\pi \in \Pi_m$. Thus Corollary 3.2 applies. In the particular case $\delta(u) = (\log(2/u))^{-\beta}$ with $\beta > 1/2$, this gives

$$\log\Big|\log \mathbb{P}\Big\{\sup_{t\in T}|X(t)| \leq 2\varepsilon\sigma\Big\}\Big| = \mathcal{O}(\varepsilon^{-2/(2\beta-1)}). \tag{3.4}$$

This estimate can also be deduced from the very recent work [1, Theorem 3 with $\gamma = \beta^{-1}$], where a growth condition on entropy numbers (namely on the induced Gaussian metric) is given.

4. Gaussian independent sequences. Let $\varphi(n) \uparrow \infty$ with n and consider the Gaussian sequence $G(\varphi) = \{G_n,\, n \in \overline{\mathbb{N}}\}$ defined by

$$G_n = \frac{g_n}{\varphi(n)}, \qquad G_\infty = 0.$$

It is known ([8, p. 102]) that already in these elementary examples, the metric entropy approach fails to describe their regularity. As

$$\limsup_{n\to\infty} \frac{|g_n|}{\sqrt{2\log n}} \overset{\text{a.s.}}{=} 1, \tag{4.1}$$

$G(\varphi)$ is sample bounded if $\varphi(n) = \mathcal{O}(\sqrt{\log n})$, and is sample continuous on $\overline{\mathbb{N}}$ if and only if

$$\sqrt{\log n} = o(\varphi(n)). \tag{4.2}$$

We begin with a general remark. From Talagrand's representation of bounded or continuous Gaussian processes ([8, Theorems 2–3]), we know that a Gaussian process $\{X(t),\, t \in T\}$ is sample bounded if and only if there exists a (not necessarily independent) Gaussian sequence $\{\xi_n,\, n \geq 1\}$ with $\|\xi_n\|_2 \leq Ka(\log n + a^2/b^2)^{-1/2}$, and that for each $t \in T$ one can write

$$X(t) = \sum_{n=1}^{\infty} \alpha_n(t)\xi_n$$

where $\alpha_n(t) \geq 0$, $\sum_{n=1}^{\infty} \alpha_n(t) \leq 1$ and the series converges a.s. and in L^2. And if T is a compact metric space, $\{X(t),\, t \in T\}$ is sample continuous if and only if its covariance function is continuous, and the same representation holds with $\|\xi_n\|_2 = o(\sqrt{\log n})$. Thus by Kathri–Sidák's inequality,

$$\mathbb{P}\Big\{\sup_{t\in T}|X(t)| \leq \varepsilon\Big\} \geq \mathbb{P}\Big\{\sup_{n=1}^{\infty}|\xi_n| \leq \varepsilon\Big\} \geq \prod_{n=1}^{\infty} \mathbb{P}\{|\xi_n| \leq \varepsilon\}.$$

This makes the study of small deviations of sequences $G(\varphi)$ of particular interest in this general context. We shall show that Theorem 3.1 allows one to get sharp lower bounds. The sequence of ordered partitions associated to

φ is based on an intrinsic sieve of $\mathbb{N}$, and the majorizing measure we will construct turns up to be very simple.

We notice that

$$\|G_n - G_m\|_2 = \left(\frac{1}{\varphi(n)^2} + \frac{1}{\varphi(m)^2}\right)^{1/2}$$

and

$$D = \sup_{n,m\ge 1} \|G_n - G_m\|_2 = \left(\frac{1}{\varphi(1)^2} + \frac{1}{\varphi(2)^2}\right)^{1/2}, \qquad \sigma = \sup_{n\ge 1} = \|G_n\|_2 = \frac{1}{\varphi(1)}.$$

THEOREM 4.1. *Assume that* (4.2) *holds and*

$$\int_0^D \left(\log \varphi^{-1}\left(\frac{1}{u}\right)\right)^{1/2} du < \infty. \tag{4.3}$$

Let $\varepsilon_n = 2^{-n} D$ *and put* $H(n) = \int_0^{\varepsilon_n} (\log \varphi^{-1}(1/u))^{1/2}\, du$, $n \ge 0$. *For* $0 < \varepsilon < \varphi(1)H(1)/24$, *let* $n(\varepsilon)$ *be such that* $H(n(\varepsilon)) \le 24\varepsilon/\varphi(1)$. *There exists an absolute constant* C *such that*

$$\mathbb{P}\left\{\sup_{n\ge 1} |G_n| \le \frac{25}{\varphi(1)}\varepsilon\right\} \ge C e^{-\varphi^{-1}(1/\varepsilon_{n(\varepsilon)}) \log \frac{1}{\varepsilon}}.$$

Proof. Let $F_n = \varphi^{-1}(1/\varepsilon_n)$, $n \ge 0$. We notice that $F_1 = \varphi^{-1}(\varphi(1)) = 1$. For $u \ge 1$, let $\nu(u)$ denote the unique integer such that $F_{\nu(u)} \le u < F_{\nu(u)+1}$.

LEMMA 4.2. *Let* $B(u,\varepsilon) = \{v \ge 1 : \|G_u - G_v\|_2 \le \varepsilon\}$. *Then*

$$\begin{aligned} B(u,\varepsilon_n) &= \{u\} && (\forall n > \nu(u)),\\ B(u,\varepsilon_n) &\supseteq [F_{n+1},\infty) && (\forall n < \nu(u)). \end{aligned}$$

Proof. Plainly $\varepsilon_{\nu(u)+1} < 1/\varphi(u) \le \varepsilon_{\nu(u)}$. If $n > \nu(u)$, then for any $v \ne u$,

$$\|G_u - G_v\|_2 > \frac{1}{\varphi(u)} > \varepsilon_{\nu(u)+1} \ge \varepsilon_n.$$

Hence $B(u,\varepsilon_n) = \{u\}$. Now notice that if $m \le \nu(u)$, then $v \ge F_m = \varphi^{-1}(1/\varepsilon_m)$ implies that $1/\varphi(v) \le \varepsilon_m$, and so

$$\|G_u - G_v\|_2 \le (\varepsilon_{\nu(u)}^2 + \varepsilon_m^2)^{1/2} \le \sqrt{2}\,\varepsilon_m < \varepsilon_{m-1}.$$

Hence with $n = m-1$ the second assertion follows. ∎

Let $\Pi_0 = \mathbb{N}$. For $\nu \ge 1$, let Π_ν be the finite partition of $\mathbb{N}$ defined by

$$\pi \in \Pi_\nu \iff \pi = \{u\},\ u < F_\nu \text{ or } \pi = [F_\nu,\infty).$$

Then $\#\{\Pi_\nu\} = F_\nu$ and $\Pi_{\nu+1}$ is a refinement of Π_ν. Further, assumption (3.1) is satisfied since by Lemma 4.2,

$$\max_{\pi\in\Pi_\nu} \max_{u,v\in\pi} d(u,v) \le \varepsilon_\nu.$$

Let μ be the probability measure on $\mathbb{N}$ defined by $\mu\{t\} = ct^{-2}$, $c = (\sum_{t=1}^{\infty} t^{-2})^{-1}$. For $t \geq 1$, we set $\pi_m(t) = \{t\}$ if $t < F_m$ and $\pi_m(t) = [F_m, \infty)$ otherwise. It follows that

$$\mu(\pi_m(t)) \geq \begin{cases} Ct^{-2} & \text{if } m > \nu(t), \\ CF_m^{-1} & \text{if } m \leq \nu(t). \end{cases} \tag{4.4}$$

Fix some integer n and let $t \geq 1$. If $n > \nu(t)$, then $t < F_n = \varphi^{-1}(1/\varepsilon_n)$ and

$$\begin{aligned}\sum_{m=n}^{\infty} \varepsilon_m \left(\log \frac{1}{\mu(\pi_m(t))}\right)^{1/2} &\leq C\Big(\sum_{m=n}^{\infty} \varepsilon_m\Big)(\log t)^{1/2} \\ &\leq C\varepsilon_n \left(\log \varphi^{-1}\left(\frac{1}{\varepsilon_n}\right)\right)^{1/2}.\end{aligned}$$

Now let $n \leq \nu(t)$. If $\nu(t) \geq m \geq n$, then $\mu(\pi_m(t)) \geq CF_m^{-1} \geq CF_{\nu(t)}^{-1}$ and as $t < F_{\nu(t)+1}$, we may write

$$\begin{aligned}\sum_{m=n}^{\infty} \varepsilon_m \left(\log \frac{1}{\mu(\pi_m(t))}\right)^{1/2} &\leq C \sum_{m=n}^{\nu(t)} \varepsilon_m \left(\log \varphi^{-1}\left(\frac{1}{\varepsilon_m}\right)\right)^{1/2} \\ &\quad + \Big(\sum_{m>\nu(t)} \varepsilon_m\Big)(\log t)^{1/2} \\ &\leq C \sum_{m=n}^{\nu(t)} \varepsilon_m \left(\log \varphi^{-1}\left(\frac{1}{\varepsilon_m}\right)\right)^{1/2} + C\varepsilon_{\nu(t)}(\log t)^{1/2} \\ &\leq C \sum_{m=n}^{\nu(t)+1} \varepsilon_m \left(\log \varphi^{-1}\left(\frac{1}{\varepsilon_m}\right)\right)^{1/2} \\ &\leq C \int_{\varepsilon_{\nu(t)+2}}^{\varepsilon_n} \left(\log \varphi^{-1}\left(\frac{1}{u}\right)\right)^{1/2} du.\end{aligned}$$

Therefore

$$\sup_{t\geq 1} \sum_{m=n}^{\infty} \varepsilon_m \left(\log \frac{1}{\mu(\pi_m(t))}\right)^{1/2} \leq C \int_0^{\varepsilon_n} \left(\log \varphi^{-1}\left(\frac{1}{u}\right)\right)^{1/2} du \to 0$$

as $n \to \infty$, by assumption. Condition (3.2) is thus realized. Let $n(\varepsilon)$ be such that $H(n(\varepsilon)) \leq 24\varepsilon/\varphi(1)$. By applying Theorem 3.1, it follows that

$$\mathbb{P}\Big\{\sup_{t\geq 1} |G_t| \leq \frac{25}{\varphi(1)}\varepsilon\Big\} \geq Ce^{-N_{n(\varepsilon)} \log \frac{1}{\varepsilon}}. \ \blacksquare$$

The following corollary easily follows.

COROLLARY 4.3.
(a) *Let* $\varphi(t) = (\log t)^\beta$, $\beta > 1/2$. *Then*

$$\log \left| \log \mathbb{P}\left\{ \sup_{n\ge 1} \frac{|g_n|}{\varphi(n)} \le \varepsilon \right\} \right| \preceq \varepsilon^{-2/(2\beta-1)}.$$

(b) *Let* $\varphi(t) = (\log t)^{1/2}(\log\log t)^{1+h}$, $h > 0$. *Then*

$$\log\log \left| \log \mathbb{P}\left\{ \sup_{n\ge 1} \frac{|g_n|}{\varphi(n)} \le \varepsilon \right\} \right| \preceq \varepsilon^{-1/h}.$$

5. Ultrametric Gaussian processes. For ultrametric Gaussian processes, a general upper bound of small deviations can be established. And by using Theorem 3.1, this is completed with a sharp lower bound. A metric space (T,d) is called *ultrametric* when d satisfies the strong triangle inequality

$$d(s,t) \le \max(d(s,u), d(u,t)) \quad (\forall s,t,u \in T).$$

Thus two balls of the same radius are either disjoint or identical. Let $B(t,u) = \{s \in T : d(s,t) \le u\}$, and let $v \le u$. It also follows that $s \in B(t,u) \Rightarrow B(s,v) \subset B(t,u)$. When (T,d) is separable, it is easy to show that (T,d) embeds continuously into a projective limit of sets, itself endowed with an ultrametric structure. Since we need this construction, we briefly recall it.

Let $D = \operatorname{diam}(T,d)$. Let S_n be the set of centers of balls forming a minimal covering of (T,d) with closed balls of radius $\varepsilon_n = 2^{-n}D$, $n = 0,1,\ldots.$ Notice that each ball $B(t,\varepsilon_n)$ contains at least one element of S_{n+1}, hence a ball $B(s,\varepsilon_{n+1})$ for some $s \in S_{n+1}$. Otherwise, there is one ball $B(t_0,\varepsilon_n)$, say, such that $\min\{d(t_0,s) : s \in S_{n+1}\} > \varepsilon_n > \varepsilon_{n+1}$, which contradicts the fact that S_{n+1} yields a covering of T of order ε_{n+1}. For $n = 0,1,\ldots$ consider the mappings $\theta_n : T \to S_n$ and $\Pi_{n,n-1} : S_n \to S_{n-1}$ defined by $d(s,\theta_n(s)) \le \varepsilon_n$ and $d(t,\Pi_{n,n-1}(t)) \le \varepsilon_{n-1}$. Next define $\Pi_{n,k} : S_n \to S_k$ for $n \ge k$ as follows: $\Pi_{n,n} = \mathrm{Id}(S_n)$ and

$$\Pi_{n,k} = \Pi_{n,n-1} \circ \cdots \circ \Pi_{k+1,k}.$$

The following elementary lemma arises from the construction itself, so we omit the proof.

LEMMA 5.1. *The pair* $\big((S_n),(\Pi_{n,k})\big)$ *defines a projective system of sets and we have the relations*

$$\theta_k = \Pi_{n,k} \circ \theta_n \quad (\forall n \ge k \ge 0).$$

Let $L = \varprojlim((S_n),(\Pi_{n,k}))$ *denote its projective limit, and* $G = \prod_{k=0}^\infty S_k$. *Let* Π_k *be the restriction to* L *of the projection of* G *onto* S_k, $k = 0,1,\ldots.$ *For* *any two elements s, t of L put*

$$\delta(s,t) = \varepsilon_{n(s,t)},$$

where $n(s,t) = \sup\{k \geq 0 : \Pi_k(s) = \Pi_k(t)\}$. Then (L, δ) is a compact ultrametric space. Moreover, the mapping $\ell : (T, d) \to (L, \delta)$ defined by $\ell(t) = \{\theta_k(t),\, k \geq 0\}$ is a continuous embedding from (T, d) to (L, δ), and

$$\frac{1}{2}\delta(\ell(s), \ell(t)) \leq d(s,t) \leq \delta(\ell(s), \ell(t)) \qquad (\forall s, t \in T).$$

The projective limit L, and thereby T, is easily visualized as a tree with branches in G, any two of them separating at offshoots of high $n(s,t)$. One can attach to any such tree an ultrametric Gaussian process. These processes have been much investigated by Fernique [2]. Let $\{g_n, n \in \coprod S_k\}$ be a sequence of independent Gaussian standard random variables. We put

$$Z(t) = \sum_{n=0}^{\infty} \varepsilon_n g_{\Pi_n(t)} \qquad (\forall t \in T).$$

THEOREM 5.2.

(a) *For some absolute constant $\gamma > 0$, we have for all $\varepsilon \leq D$,*

$$\mathbb{P}\Big\{\sup_{s,t\in L} |Z(s) - Z(t)| \leq \varepsilon\Big\} \leq e^{-\gamma N(T,\delta,2\varepsilon)}.$$

(b) *Assume that condition (3.2) is fulfilled. Then, with the notation of Theorem 3.1, letting $\sigma = 2D/\sqrt{3}$,*

$$\mathbb{P}\Big\{\sup_{t\in T} |Z(t)| \leq 2\varepsilon\sigma\Big\} \geq C e^{-N_{n(\varepsilon)} \log \frac{1}{\varepsilon}}.$$

Proof. (a) The assumption implies that from each offshoot of S_n grows at least one new branch. A plain calculation yields $d_Z(s,t) := \|Z(s) - Z(t)\|_2 = \varepsilon_{n(s,t)}(3/2)^{1/2}$, $s, t \in T$. Further, we notice that

$$Z(t) - Z(s) = \sum_{n > n(s,t)}^{\infty} \varepsilon_n (g_{\Pi_n(t)} - g_{\Pi_n(s)}).$$

Write $S_n = \{s_{n,j},\, 1 \leq j \leq N_n\}$, where we set $N_n = N(T, \varepsilon_n)$. Let $L_n \subset L$, $L_n = \{t_{n,j},\, 1 \leq j \leq N_n\}$, be such that $\Pi_n(t_{n,j}) = s_{n,j}$ for each j. Then $\mathbb{E}\,(Z(t_{n,i}) - Z(t_{n,i-1}))^2 = (3/2)\varepsilon_n^2$, and since the random variables g_n are independent, we observe that

(5.1)
$$\mathbb{E}\,(Z(t_{n,2i}) - Z(t_{n,2i-1}))(Z(t_{n,2j}) - Z(t_{n,2j-1})) = 0 \qquad (\forall 1 \leq j < i \leq N_n/2).$$

Hence the covariance matrix of $\{Z(t_{n,2i}) - Z(t_{n,2i-1}),\, 1 \leq i \leq N_n/2\}$ is diagonal with all diagonal entries equal to $(3/2)\varepsilon_n^2$. Consequently,

$$\begin{aligned}\mathbb{P}\Big\{\sup_{s,t\in L}|Z(s)-Z(t)|\le\varepsilon_n\Big\} &\le \mathbb{P}\Big\{\sup_{1\le i\le N_n/2}|Z(t_{n,2i})-Z(t_{n,2i-1})|\le\varepsilon_n\Big\}\\ &= \mathbb{P}\Big\{\sup_{1\le i\le N_n/2}\frac{|Z(t_{n,2i})-Z(t_{n,2i-1})|}{\|Z(t_{n,2i})-Z(t_{n,2i-1})\|_2}\le c\Big\}\\ &\le e^{-\gamma N(T,\varepsilon_n)},\end{aligned}$$

c, γ being absolute constants. Let $0<\varepsilon\le \operatorname{diam}(T,d)$, and let n be such that $\varepsilon_{n+1}<\varepsilon\le\varepsilon_n$. Then

$$\begin{aligned}\mathbb{P}\Big\{\sup_{s,t\in L}|Z(s)-Z(t)|\le\varepsilon\Big\} &\le \mathbb{P}\Big\{\sup_{s,t\in L}|Z(s)-Z(t)|\le\varepsilon_n\Big\}\le e^{-\gamma N(T,\varepsilon_n)}\\ &\le e^{-\gamma N(T,2\varepsilon)}.\end{aligned}$$

(b) This is a direct consequence of Theorem 3.1. ■

Notice to conclude that when $\mathbb{E}\,X^2(t)=1$ for all $t\in T$, it is easy to modify Z so that more precise comparison relations holds: for all s and t in T,

$$\mathbb{E}\,Z^2(t)=\mathbb{E}\,X^2(t),\qquad \mathbb{E}\,Z(s)Z(t)\ge\mathbb{E}\,X(s)X(t). \tag{5.2}$$

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