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ATOMICITY AND THE FIXED DIVISOR IN CERTAIN PULLBACK CONSTRUCTIONS

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Abstract. Let D be an integral domain with field of fractions K. In this article, we use a certain pullback construction in the spirit of $\mathrm{Int}(E, D)$ that furnishes many examples of domains between $D[x]$ and $K[x]$ in which there are elements that do not admit a finite factorization into irreducible elements. We also define the notion of a fixed divisor for this pullback construction to characterize all of its irreducible elements and those nonzero nonunits that do admit a finite factorization into irreducibles. En route to these characterizations, we show that this construction yields a domain with infinite restricted elasticity.

1. Introduction. Let D be any integral domain with field of fractions K and let $D^\bullet$ denote the set of nonzero nonunit elements of D. An element $d \in D^\bullet$ is called *irreducible* (or an *atom*) if $d = ab$ with $a, b \in D$ implies that either a or b is a unit of D. We will write $\mathcal{A}(D)$ to denote the set of all irreducible elements of D. An element $d \in D^\bullet$ is called *atomic* if it admits a finite factorization $d = \pi_1 \cdots \pi_t$ where each $\pi_i \in \mathcal{A}(D)$. Let $\mathcal{F}(D)$ be the set of all atomic elements of D and $\mathcal{N}(D) = D^\bullet - \mathcal{F}(D)$. That is, $\mathcal{N}(D)$ is the set of elements of $D^\bullet$ that do not admit a factorization into irreducibles.

The domain D is called *atomic* if every element of $D^\bullet$ is atomic. Some standard examples of atomic domains include UFD's (every factorization into irreducibles has the same length and is unique up to associates), HFD's (every factorization of α into irreducibles has the same length), and domains satisfying ACCP (ascending chain condition on principal ideals). It is well known that we have the chain of implications displayed below:

$$\text{UFD} \Rightarrow \text{HFD} \Rightarrow \text{ACCP} \Rightarrow \text{atomic}.$$

Recall that if $E = \{e_1, \ldots, e_r\}$ is a subset of D, then $\mathrm{Int}(E, D) = \{g \in K[x] : g(E) \in D\}$ is called the *ring of integer-valued polynomials on D determined by E*. The purpose of this article is to extend some results from [1], [9], [10] to a more general context. In particular, we consider a special type of conductor square introduced in [3] that defines a ring between $D[x]$

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and $K[x]$. Let $v(x) = v_1(x) \cdots v_r(x)$ where $v_1, \ldots, v_r$ are distinct irreducible polynomials over the field K. If $C = v(x)K[X]$, then we have the natural surjection $\eta : K[x] \twoheadrightarrow K[x]/C \simeq \prod_{i=1}^r K[\theta_i]$ where, for each index $i \leq r$, θ_i is a root of v_i. If D_i is any overring of $D[\theta_i]$, then we have the inclusion $\iota : \Pi_{i=1}^r D_i \hookrightarrow \prod_{i=1}^r K[\theta_i]$. Taking the pullback of the maps η and ι, we obtain the ring $R = \{g(x) \in K[x] : g(\theta_i) \in D_i \text{ for each } i \leq r\}$ between $D[x]$ and $K[x]$ with the nonzero conductor C from $K[x]$ into R. In this case we will say that R *is defined by a conductor square of the type* (⊠):

$$(\boxtimes) \qquad \begin{array}{ccc} R & \hookrightarrow & K[x] \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \eta \\ \prod_{i=1}^r D_i & \overset{\iota}{\hookrightarrow} & \prod_{i=1}^r K[\theta_i] \end{array}$$

It was first proved in [15] that $\mathrm{Int}(E, D) = f(x)K[x] + \sum_{i=1}^r D\varphi_i(x)$ where $f(x) = (x - e_1) \cdots (x - e_r)$ and, for each $i \leq r$, the polynomial φ_i is the LaGrange interpolation polynomial on the set E. This representation indicates that $\mathrm{Int}(E, D)$ is definable by a conductor square of the type (⊠). Indeed, it is noted in [3] that if we set $v_i(x) = (x - e_i)$ and $D_i = D$ for each $i \leq r$, then the resulting pullback ring is $R = \mathrm{Int}(E, D)$.

Much is known about the ring $\mathrm{Int}(E, D)$ when E is finite (see [6] for a survey). Recall that a ring R is said to have the *strong n-generator property* if the following condition holds for every finitely generated ideal I: For each nonzero $b \in I$, there exist $b_1, \ldots, b_{n-1} \in I$ such that $I = (b, b_1, \ldots, b_{n-1})$. For example, [8] proves that $\mathrm{Int}(E, D)$ has the strong 2-generator property if and only if D is a Bézout domain. A similar result for $\mathrm{Int}(E, D)$ can be found in [4] for a larger number of generators. Also, [15] uses the representation above to show that $\mathrm{Int}(E, D)$ is a Prüfer domain if and only if D is a Prüfer domain. As the previous paragraph suggests, analogous results hold for a ring R defined by a conductor square of the type (⊠) (see [5] for a survey of all of these articles). The following problems may provide impetus to study the construction:

PROBLEM 1.1 ([7, Problem 50]). *Study the ring- and ideal-theoretic properties that transfer in a conductor square where the conductor ideal is not maximal* (*or even prime*) *in the extension ring.*

The construction (⊠) has conductor ideal that is generally a finite intersection of maximal ideals of $K[x]$. The results of [3] and [4] provide some investigation toward Problem 1.1.

PROBLEM 1.2 ([7, Problem 52]). *Does there exist a pullback diagram of the type* (⊠) *that defines a Prüfer domain containing an ideal requiring more than two generators?*

The results of [4] show that for $n \geq 2$, there exists a Prüfer domain D with the n-generator property but not the $(n-1)$-generator property. In addition, $\mathrm{Int}(\{0\}, D) = D + xK[x]$ shares the same property as D. It follows from [15] and [3] that the answer to Problem 1.2 is affirmative.

PROBLEM 1.3 ([12, Problem PD2]). *Does each Prüfer overring of* $\mathbb{Z}[x]$ *have the* 2*-generator property?*

A partial affirmative answer to this question is given in [3] where it is shown that any Prüfer domain R between $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ with a nonzero conductor from $\mathbb{Q}[x]$ into R has the 2-generator property.

Some authors have also considered factorization in various pullback constructions in the spirit of $\mathrm{Int}(E, D)$. For example, [11] finds necessary and sufficient conditions on the pullback diagram defining $S = A + xB[x]$ in order that S is an HFD (see [14] for similar examples). On the other hand, [1] proves that if D is not a field, then $\mathrm{Int}(E, D)$ is never atomic. What is more, a ring R defined by the construction (⊠) does not satisfy ACCP (see below). At this point it would be reasonable to guess that the ring R is not atomic.

Indeed, we do show that the diagram (⊠) is quite useful in producing numerous examples of nonatomic domains. In addition, we introduce the concept of a fixed divisor (as in [2], [9], [10]) in order to characterize the irreducible elements in certain pullbacks of the type (⊠). We use the fixed divisor to show that these pullbacks have infinite restricted elasticity. This investigation might give some insight into the possibility of atomicity in a ring R defined by (⊠). If it is the case that such a ring R is atomic, then we will have a method of producing atomic domains that do not satisfy ACCP.

2. Atomicity. In this section we show that examples of nonatomic domains are quite easily obtained using the construction (⊠). It will become evident that in most cases, the divisors of the conductor polynomial $v(x)$ do not admit a finite factorization into irreducible elements in R. In fact, the closing result of this section shows that under certain conditions, we need look no further than the conductor ideal $v(x)K[x]$ for the nonatomic elements.

DEFINITION 2.1. Suppose that R is a ring defined by the diagram of the type (⊠).

(1) As in [6], $\mathfrak{I}_0(R)$ denotes the ring of constants in R. That is, $\mathfrak{I}_0(R) = R \cap K$.
(2) A polynomial $f \in R$ is called *pseudo-irreducible in* R if $g, h \in R$ and $f = gh$ imply g or $h \in \mathfrak{I}_0(R)$.

If $c \in \mathfrak{I}_0(R)$, then we get nothing new when evaluating at θ_i and it follows that $c \in \bigcap_{i=1}^{r} D_i$. In other words, $\mathfrak{I}_0(R) \subset \bigcap_{i=1}^{r} D_i$. It is also worth noting that if $f \in R$, then the irreducibility of f in $K[x]$ implies the pseudo-irreducibility of f in R. However, if $g(x) = x(x-1)/2$ and $R = \mathrm{Int}(\{0,1\}, \mathbb{Z})$, then g is a pseudo-irreducible element of R while it is not irreducible in $\mathbb{Q}[x]$.

LEMMA 2.2. *Suppose that R is a ring defined by a diagram of the type ($\boxtimes$). If $g(x) \in R$ has the property that $g(x)/c \in R$ for every nonzero nonunit $c \in \mathfrak{J}_0(R)$, then the following hold:*

(1) *The polynomial $g(x)/c$ is not an irreducible element of R.*
(2) *If $g(x)/c$ is a pseudo-irreducible polynomial then it is not atomic.*
(3) *There exists an infinite chain of principal ideals that properly ascends from $(g(x))$.*

Proof. (1) Choose any nonzero nonunit $d \in \mathfrak{J}_0(R)$. Then $g(x)/c = d \cdot g(x)/cd$ is a proper factorization.

(2) Suppose that $g(x)/d = p_1(x) \cdots p_s(x)$ where each p_k is in $\mathcal{A}(R)$. By assumption, all but one of the p_k are constant in R. That is, after a suitable reordering, we have $\deg(g) = \deg(p_1)$ and $p_2, \ldots, p_s \in \mathfrak{J}_0(R)^\bullet$. But then $g(x)/(p_2(x) \cdots p_s(x)) = p_1(x) \in \mathcal{A}(R)$, contradicting (1) above.

(3) Choose any nonzero nonunit $d \in \mathfrak{J}_0(R)$. Then $(g(x)) \subset (g(x)/d) \subset (g(x)/d^2) \subset \cdots$ is a properly ascending chain of principal ideals that does not terminate. ∎

REMARK 2.3. Notice that if R is a ring defined by ($\boxtimes$), then the conductor polynomial $v(x)$ has the property that $v(x)/c \in R$ for every $c \in \mathfrak{J}_0(R)$. Indeed, we have $v(\theta_i)/c = 0 \in D_i$ for each $i \le r$ and it is evident that R never satisfies the ascending chain condition on its principal ideals. In addition, usually one need not look much further than $v(x)$ (or its irreducible factors $v_i(x)$) in order to find an element of R that is not atomic.

The proof of the next result essentially uses the same argument as [1].

THEOREM 2.4. *Suppose that R is a ring defined by a diagram of the type ($\boxtimes$) with the following properties:*

(1) *There exists a nonunit $d \in D$ that remains a nonunit in D_iD_j for each $i \neq j$.*
(2) *The Vandermonde determinant Δ of the full set of roots in a splitting field of v is nonzero.*

Then the polynomial $v(x)/\Delta^2 d$ is a pseudo-irreducible element of R so that R is not atomic.

Proof. Suppose that we can write $v(x)/\Delta^2 d = g(x)h(x)$ with $g, h \in R$ and such that $\deg(g), \deg(h) \ge 1$. Lifting this equation up to the UFD $K[x]$, after a suitable reordering of the v_i, we find that $g = \alpha v_1 \cdots v_k$ and $h = \beta v_{k+1} \cdots v_r$ where $\alpha, \beta \in K$. It follows that $\alpha\beta\Delta^2 d = 1$ so that $(\alpha\Delta)(\beta\Delta)d$

$= 1$. Now $\alpha\Delta = g(\theta_i) \cdot \alpha\Delta/g(\theta_i)$ for any index $k+1 \leq i \leq r$, and since $g \in R$, we have $g(\theta_i) \in D_i$. If $\{\theta_1, \ldots, \theta_n\}$ is the full set of roots in a splitting field of $v(x)$ where $n \geq r$, then $\alpha\Delta/g(\theta_i) \in D[\theta_1, \ldots, \theta_n]$. It follows that $\alpha z \in D_i[\theta_1, \ldots, \theta_n]$ and similarly that $\beta z \in D_j[\theta_1, \ldots, \theta_n]$ for any index $1 \leq j \leq k$. This means that d is a unit in the ring $D_iD_j[\theta_1, \ldots, \theta_n]$. But d is a nonunit of D_iD_j, and since we can assume that $\theta_1, \ldots, \theta_n$ are all integral over D (see [3] for a justification), it must be the case that d is a nonunit in $D_iD_j[\theta_1, \ldots, \theta_n]$. It follows that the polynomial $v(x)/\Delta^2 d$ cannot be factored into two polynomials of smaller degree and is hence pseudo-irreducible. Lemma 2.2 and Remark 2.3 imply that R is not atomic. ∎

The previous results together with the next examples suggest that nonatomic domains between $D[x]$ and $K[x]$ defined by the diagram (⊠) are quite numerous.

Examples 2.5. Suppose that R is a ring defined by a diagram of the type (⊠).

(1) If $r = 1$, then the ring R defined by (⊠) is never atomic. Since $v(x) = v_1(x)$ is irreducible in $K[x]$, it is pseudo-irreducible and Remark 2.3 implies that R is not atomic. In particular, the ring $\operatorname{Int}(\{0\}, D) = D + xK[x]$ is nonatomic whenever D is not a field.

(2) (See [1].) If D is not a field, then the ring $\operatorname{Int}(E, D)$ of integer-valued polynomials on D determined by the finite subset $E = \{e_1, \ldots, e_r\} \subset D$ is never atomic. As noted in the introduction, $\operatorname{Int}(E, D)$ is defined by a conductor square of the type (⊠) where $C = (x-e_1)\cdots(x-e_r)K[x]$ and $A = \prod_{i=1}^{r} D$. That is, each D_i equals D so that $D_iD_j = D \subsetneq K$. Nonatomicity of R follows from Theorem 2.4.

(3) Suppose that $r = 2$ and $D_2 = K(\theta_2)$ and consider the polynomial $v_1(x) \in D[x]$. If c is any nonzero element of D, then $v_1(\theta_1)/c = 0 \in D_1$ and $v_1(\theta_2)/c \in K(\theta_2)$ putting $v_1(x)/c \in R$. Lemma 2.2 implies that R is not atomic.

(4) If $C = x(x-1)\mathbb{Q}[x]$ and $D_1 \times D_2 = \mathbb{Z}_{(2)} \times \mathbb{Z}_{(3)}$, then the polynomial $x/2 \in R$ is not atomic. First note that $\mathfrak{J}_0(R) = \mathbb{Z}_{(2)} \cap \mathbb{Z}_{(3)}$ and if 3 divides $x/2$, then $x/6 \in R$. But this is impossible since $1/6 \notin \mathbb{Z}_{(3)}$. It now follows that $x/2$ cannot be factored into irreducibles. To see this, if $x/2 = c_1 \cdots c_n g(x)$ is a factorization into irreducibles, then each c_i equals 2 and $g(x)$ is irreducible in R. But then $g(x) = x/2^k$ is irreducible, which is false.

We close this section with a result that gives conditions in which the badly behaved elements with respect to factorization are confined to the conductor ideal $C = v(x)K[x]$. This theorem is a slight strengthening of Proposition 7 in [10]. First, we make a relevant definition.

DEFINITION 2.6. A D-module M *satisfies D-ACC* if every ascending chain of cyclic submodules of M stabilizes.

THEOREM 2.7. *Suppose that R is a ring defined by a diagram of the type ($\boxtimes$) with the following properties:*

(1) *Each nonunit of $\mathfrak{J}_0(R)$ remains a nonunit in D_i for each $i \leq r$.*
(2) *Each D_i satisfies ACC on its cyclic $\mathfrak{J}_0(R)$-submodules.*

Then $\mathcal{N}(R)$ is contained in the conductor ideal $C = v(x)K[x]$.

Proof. Since every polynomial in R can be factored into a product of pseudo-irreducible elements, it suffices to check the result for these polynomials. If $g \in \mathcal{N}(R)$ is pseudo-irreducible, then there exists in R an infinite chain of cyclic $\mathfrak{J}_0(R)$-submodules $\mathfrak{J}_0(R)g \subset \mathfrak{J}_0(R)g_1 \subset \mathfrak{J}_0(R)g_2 \subset \cdots$ that properly ascends from g. Evaluating at any θ_i gives a chain of cyclic $\mathfrak{J}_0(R)$-submodules $\mathfrak{J}_0(R)g(\theta_i) \subset \mathfrak{J}_0(R)g_1(\theta_i) \subset \mathfrak{J}_0(R)g_2(\theta_i) \subset \cdots$ in D_i that properly ascends from $g(\theta_i)$. If $g(\theta_i) \neq 0$, then condition (1) ensures that the chain remains infinite in D_i and condition (2) is violated. It follows that $g(\theta_i) = 0$ for all $i \leq r$ so that $g \in v(x)K[x]$. ∎

REMARK 2.8. If $v(x) = x - e$ is a linear polynomial, then it is enough to assume that $D = D_1$ is an atomic domain (without the full strength of ACCP) to conclude that $\mathcal{N}(R) \subset (x-e)K[x]$. Indeed, if $d + (x-e)q(x) \in R = D + (x-e)K[x]$ where $d \neq 0$, then $d(1 + (x-e)q(x)/d)$ can be factored into a finite product of irreducible elements. If $\deg(v) \geq 2$, it is not clear that we can replace condition (2) with "each D_i is $\mathfrak{J}_0(R)$-atomic" (see below for the definition).

3. The fixed divisor. In this section, we define the notion of a fixed divisor for a ring R defined by a special case of the diagram ($\boxtimes$) (see [2], [9], [10]). This tool will greatly facilitate the characterization of the irreducible and atomic elements of R. Using some of the ideas in [13] we can better understand the factorial behavior in ($\boxtimes$). Let D be any integral domain and let M be any torsion free D-module. The nonzero element $m \in M$ is said to be *D-irreducible in M* if whenever $m = dm'$ for some $d \in D$ and $m' \in M$, then d is a unit of D. The set of all D-irreducible elements of M is denoted by $\mathcal{A}_D(M)$. A nonzero element $m \in M$ is called *D-atomic* if there exists a finite factorization $m = c_1 \cdots c_t m'$ such that $c_i \in \mathcal{A}(D)$ for each $i \leq t$ and $m' \in \mathcal{A}_D(M)$. We will call M a *D-UFM* (or a *factorial module*) if M is D-atomic and if $c_1 \cdots c_t m' = d_1 \cdots d_s m''$ where $c_i, d_i \in \mathcal{A}(D)$ and $m', m'' \in \mathcal{A}_D(M)$ implies $t = s$, $c_i = u_i d_i$, and $m' = um''$ for some units $u, u_i \in D$. It is pointed out in [13] that if M is a D-UFM, then D is necessarily a UFD.

DEFINITION 3.1. Suppose that R is a ring defined by a conductor square of the type (⊠) and let $E = \{\theta_1, \ldots, \theta_r\}$. Assume further that each D_i is a $\mathfrak{I}_0(R)$-UFM and that every nonunit of $\mathfrak{I}_0(R)$ remains a nonunit in each D_i.

(1) We will say that R is a *ring defined by a conductor square of the type* $\mathfrak{F}$(⊠).
(2) Let $g(x) \in R$. Write each $g(\theta_i) = c_i g'(\theta_i)$ where $c_i \in \mathfrak{I}_0(R)$ and $g'(\theta_i)$ is an irreducible element in the $\mathfrak{I}_0(R)$-module D_i. Now define the *fixed divisor* of g to be $d(E, g) = \gcd(c_1, \ldots, c_n)$ in $\mathfrak{I}_0(R)$.

The following theorem collects some basic properties of the fixed divisor. Similar statements can be found in [9] and [10] for $\mathrm{Int}(E, D)$.

THEOREM 3.2. *Suppose that R is a ring defined by a conductor square of the type $\mathfrak{F}$(⊠). The following hold for a nonzero polynomial $g \in R$:*

(1) $g(x)/d(E, g)$ *belongs to the ring R.*
(2) $d(E, g) = 0$ *implies g is not irreducible.*
(3) $g = cg_1$, *where $g_1 \in R$ and $c \in \mathfrak{I}_0(R)$, implies $d(E, g) = cd(E, g_1)$.*
(4) $g = g_1 \cdots g_k$, *where each g_s is in R, implies $d(E, g_1) \cdots d(E, g_k) \mid d(E, g)$.*
(5) $g = g_1^k$, *where $g_1 \in R$, implies $d(E, g) = d(E, g_1)^k$.*

Proof. (1) Follows immediately from the fact that $d(E, g) \mid g(\theta_i)$ for each $i \leq r$.

(2) If $d(E, g) = 0$, then $g(\theta_i) = 0$ for all $i \leq r$. That is, $g \in v(x)K[x]$ and we can write $g(x) = v(x)q(x)$ for some $q(x) \in K[x]$. There is a nonzero nonunit $d \in D$ such that $dq(x) \in D[x]$ and we have a factorization $g(x) = \frac{v(x)q(x)}{d} \cdot d$.

(3) Follows from the identity $\gcd(cd_1, \ldots, cd_r) = c\gcd(d_1, \ldots, d_n)$.

(4) Suppose that p is any prime element of $\mathfrak{I}_0(R)$ that divides $d(E, g_1) \cdots d(E, g_k)$. We will assume that $p \mid d(E, g_1)$ so that $p \mid g_1(\theta_i)$ in the $\mathfrak{I}_0(R)$-module D_i for each $i \leq r$. Using the fact that D_i is a $\mathfrak{I}_0(R)$-UFM, we can write $g_1(\theta_i) = \gamma_{1,i} g_{1,i}(\theta_i)$ where $g_{1,i}(\theta_i)$ is $\mathfrak{I}_0(R)$-irreducible and $\gamma_{1,i} \in \mathfrak{I}_0(R)$. Since p is prime (irreducible) in the UFD $\mathfrak{I}_0(R)$, it is prime in the $\mathfrak{I}_0(R)$-module D_i. That is, $p \mid \gamma_{1,i}$ and it is evident from the equation $g(\theta_i) = \gamma_{1,i} \cdots \gamma_{k,i} g_{1,i}(\theta_i) \cdots g_{k,i}(\theta_i)$ that $p \mid \gamma_{1,i} \cdots \gamma_{k,i}$ for all $i \leq r$. Therefore, $p \mid d(E, g_1 \cdots g_k)$ and we have $d(E, g_1) \cdots d(E, g_k) \mid d(E, g)$ as desired.

(5) From (4), we have $d(E, g_1)^k \mid d(E, g)$. Now suppose that p is any prime element of $\mathfrak{I}_0(R)$ that divides $d(E, g)$. If we write $g(\theta_i) = (\gamma_{1,i} g_{1,i}(\theta_i))^k = (\gamma_{1,i})^k g_{1,i}^k(\theta_i)$, then $p \mid (\gamma_{1,i})^k$ in the ring $\mathfrak{I}_0(R)$. Hence, $p \mid \gamma_{1,i}$ for each $i \leq r$ so that $p \mid d(E, g_1)^k$. ∎

We now define a notion similar to that of "primitive" for a polynomial ring over a UFD. Theorem 3.4 below collects some results similar to those found in [9] and [10].

Definition 3.3. Suppose that R is a ring defined by a conductor square of the type $\mathfrak{F}(\boxtimes)$. An element $g \in R$ is called *image primitive* if $d(E, g) = 1$.

Theorem 3.4. *Suppose that R is a ring defined by a conductor square of the type $\mathfrak{F}(\boxtimes)$. The following hold for a nonzero polynomial $g \in R$:*

1. *$d(E, g) = 1$ and $g = g_1 \cdots g_k$, where each $g_i \in R$, implies $d(E, g_i) = 1$ for each $i \leq k$.*
2. *$g \in \mathcal{A}(R)$ implies $d(E, g) = 1$.*
3. *$d(E, g) = 1$ implies that g is not divisible by any element of $\mathfrak{J}_0(R)^{\bullet}$.*
4. *If $g \in \mathfrak{J}_0(R)[x]$ and g is primitive, then g irreducible in R if and only if g is irreducible in $\mathfrak{J}_0(R)[x]$ and $d(E, g) = 1$.*

Proof. (1) Follows immediately from (4) in the previous theorem.

(2) If $d(E, g) = d$ is an element of $\mathfrak{J}_0(R)^{\bullet}$, then $g(x)/d \in R$ by (1) of the previous theorem and $g(x) = d \cdot (g(x)/d)$ is a proper factorization of $g(x)$ in R.

(3) Suppose that $g(x) = ch(x)$ for some $c \in \mathfrak{J}_0(R)^{\bullet}$ and some $h \in R$. Since D_i is a $\mathfrak{J}_0(R)$-UFM, and since $h(\theta_i) \in D_i$, we have the unique factorization $h(\theta_i) = \gamma_i h_i(\theta_i)$ where $h_i(\theta_i)$ is irreducible. It now follows that $g(\theta_i) = c\gamma_i h_i(\theta_i)$ and that $1 = d(E, g) = \gcd(c\gamma_1, \ldots, c\gamma_r) = c \gcd(\gamma_1, \ldots, \gamma_r)$, making c a unit in the ring $\mathfrak{J}_0(R)$.

(4) ($\Rightarrow$) If $g(x)$ is irreducible in R, then $g(x)$ is image primitive by (2). Suppose $g(x) = g_1(x)g_2(x)$ is a proper factorization of $g(x)$ in $\mathfrak{J}_0(R)[x]$. If g_1, g_2 are both nonconstant, then this factorization of g is proper. If $g(x) = dg_2(x)$ is a proper factorization in $\mathfrak{J}_0(R)[x]$, then d is a nonunit in $\mathfrak{J}_0(R)$, making it a nonunit of R.

($\Leftarrow$) If $g(x)$ is irreducible in $\mathfrak{J}_0(R)[x]$ and image primitive, then $g(x)$ is irreducible in $K[x]$. Suppose $g(x) = g_1(x)g_2(x)$ is a proper factorization of $g(x)$ in R. Since $g(x)$ is image primitive, both $g_1(x)$ and $g_2(x)$ are nonconstant, which contradicts the irreducibility of $g(x)$ over $K[x]$. ∎

The results below are extensions of some results [9] and [10]. They all follow from the properties of the generalized fixed divisor given in Theorems 3.2 and 3.4. The proofs are essentially the same as the ones provided in [9] and [10] but we replace $\mathrm{Int}(E, D)$ and $D[x]$ with the more general constructions R defined by $\mathfrak{F}(\boxtimes)$ and $\mathfrak{J}_0(R)[x]$. We therefore omit the details of the proofs and refer the reader to the previously cited articles.

The definition below is introduced in [10] and is given in order to retain the notion of elasticity in an integral domain that is not atomic.

DEFINITION 3.5. Let D be any integral domain with $\mathcal{A}(D) \neq \emptyset$. We define the *restricted elasticity* to be

$$\rho_r(D) = \sup\Big\{\frac{m}{n} \in \mathbb{Q} : \alpha \in \mathcal{F}(D) \text{ and } \prod_{i=1}^{n} p_n = \alpha = \prod_{i=1}^{m} q_i \text{ where } p_i, q_i \in \mathcal{A}(D)\Big\}.$$

It is shown in [10, Corollary 6] that for every real number $t \geq 1$, there exists a domain D with a finite subset E such that $\rho_r(\mathrm{Int}(E, D)) = t$. That is, there exists a ring R defined by (⊠) such that $\rho_r(R) = t$. In addition, we have the following result analogous to [10, Proposition 12].

THEOREM 3.6. *Suppose that R is a ring defined by a conductor square of the type $\mathfrak{F}$(⊠). Then $\rho_r(R) = \infty$.*

The next theorem is critical in determining the irreducible elements of a domain R defined by a diagram of the type $\mathfrak{F}$(⊠). It is analogous to a familiar representation for elements in a polynomial ring over a UFD.

THEOREM 3.7. *Suppose that R is a ring defined by a conductor square of the type $\mathfrak{F}$(⊠). If $g \in R$ is image primitive, then there exists a unique (up to associates) primitive polynomial $g_1 \in \mathfrak{I}_0(R)[x]$ and $d \in \mathfrak{I}_0(R)$ such that*

$$(*) \qquad g(x) = g_1(x)/d.$$

In the result that follows, we lay more of the ground work needed in characterizing the irreducible and atomic elements of R defined by $\mathfrak{F}$(⊠).

LEMMA 3.8. *Suppose that R is a ring defined by a conductor square of the type $\mathfrak{F}$(⊠). The following statements are equivalent for a nonconstant primitive polynomial $f \in \mathfrak{I}_0(R)[x]$:*

(1) *$f(x)/d(E, f)$ is irreducible in R.*
(2) *Either f is irreducible in $\mathfrak{I}_0(R)[x]$, or for every pair of nonconstant polynomials $g, h \in R$ such that $f = gh$, one has $d(E, f) \nmid d(E, g)d(E, h)$.*

With Theorem 3.7 and Lemma 3.8 at hand, we are able to characterize the irreducible elements of a ring defined by $\mathfrak{F}$(⊠) and those elements of the conductor ideal that do admit a finite factorization into irreducibles. These results will shed some light on the factorization properties of the construction $\mathfrak{F}$(⊠) and may provide the necessary tools to find other examples of atomic domains that do not satisfy ACCP.

THEOREM 3.9. *Suppose that R is a ring defined by a conductor square of the type $\mathfrak{F}$(⊠). Let $g(x)$ be a nonunit in R and write $g(x) = g_1(x)/d$ as in (∗). Then $g(x)$ is irreducible in R if and only if*

(1) $\deg(g) = 0$ *and* g *is irreducible in* $\mathfrak{I}_0(R)$, *or*
(2) $\deg(g) > 0$, $g(x)$ *is image primitive in* R, *and* $d(E, g_1) = d$ *where either*
   (a) $g_1(x)$ *is irreducible in* $\mathfrak{I}_0(R)[x]$, *or*
   (b) *if* $g_1(x) = f(x)h(x)$ *is any proper factorization* $\mathfrak{I}_0(R)$, *then* $d \nmid d(E, f)d(E, h)$.

THEOREM 3.10. *Suppose that* R *is a ring defined by a conductor square of the type* $\mathfrak{F}(\boxtimes)$. *Let* $g \in v(x)K[x]$ *and write* $g(x) = g_1(x)/d$ *as in* $(*)$. *In order for* g *to admit a finite factorization into irreducibles, it is necessary and sufficient that there exists a proper factorization* $g_1(x) = p_1(x) \cdots p_r(x)$ *in* $\mathfrak{I}_0(R)[x]$ *and* $d = c_1 \cdots c_r$ *in* $\mathfrak{I}_0(R)$ *such that:*

(1) $d(E, p_i) \neq 0$ *whenever* $1 \leq i \leq r$, *and*
(2) $c_i \mid d(E, p_i)$ *whenever* $1 \leq i \leq r$.

We conclude this article with a natural question: Can we find necessary and sufficient conditions on the diagram $(\boxtimes)$ in order that the resulting pullback is atomic? Also, if such conditions are found, can we find examples that satisfy them?

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