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FINITE GROUPS WITH FEW SELF-NORMALIZING SUBGROUPS

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Abstract. We describe finite groups which contain just one conjugate class of self-normalizing subgroups.

1. Introduction and preliminaries. Throughout this paper the term group always means a group of finite order.

A subgroup of a finite group is called a *Carter subgroup* if it is nilpotent and self-normalizing. By [C2], any finite solvable group contains exactly one conjugacy class of Carter subgroups. P. Hall proved that a subgroup H of a finite group G is abnormal in G if and only if every subgroup containing H is self-normalizing ([DH, p. 251]). It has been shown in [C1] that if $G = G_1G_2$ where G_1 and G_2 are solvable subgroups and G_1 commutes with every self-normalizing subgroup of G_2, and G_2 commutes with every self-normalizing subgroup of G_1, then G is solvable. So, self-normalizing subgroups have a strong influence on the structure of groups. In this paper, we study finite groups which contain just one conjugacy class of self-normalizing subgroups and obtain a characterization of such groups.

Our notation is standard. All unexplained notation can be found in [DH] and [R].

For the proof of our main theorem, we need the following definition.

DEFINITION 1.1. Let M be a normal subgroup of a finite group G. If the subgroup MP is nilpotent for any Sylow subgroup P of G, then M is called an *NS subgroup* of G.

Let M be a normal subgroup of G. Then it is easy to see that M is an NS subgroup of G if and only of P is normal in PM for any Sylow subgroup P of G. Let M and N be two distinct NS subgroups of G. Then $P \trianglelefteq PN$ and $P \trianglelefteq PM$. Hence $P \trianglelefteq PNM$, which implies that MN is an NS subgroup of G. Thus the subgroup generated by all NS subgroups is also an NS subgroup of G, which is the unique maximal NS subgroup of G.

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Let $1 = Z_0(G) \leq Z_1(G) \leq \cdots$ be the upper central series of G and $Z_\infty(G) = \bigcup_{i=0}^{\infty} Z_i(G)$. Then $Z_\infty(G)$ is called the *hypercenter* of G (see [B]). It is well known that for any nilpotent subgroup A of G, the subgroup $AZ_\infty(G)$ is still nilpotent (see [W, p. 6]). Hence to a certain extent, an NS subgroup is "close" to being the hypercenter. In fact, as the following lemma shows, the unique maximal NS subgroup of G is just equal to the hypercenter $Z_\infty(G)$.

LEMMA 1.2. *Let G be a finite group. Then $Z_\infty(G)$ is the unique maximal NS subgroup of G.*

Proof. Let M be an NS subgroup of G, $P \in \mathrm{Syl}_p(M)$ and $Q \in \mathrm{Syl}_q(G)$, where $q \neq p$. Then PQ is nilpotent since M is an NS subgroup. Hence $Q \leq C_G(P)$, which implies that $O^p(G) \leq C_G(P)$. Therefore $G/C_G(P)$ is a p-group and hence $G = C_G(P)P$. Let $P_1 = [P, G]$. Then $P_1 \trianglelefteq G$. Since $G = C_G(P)P$, we see that $P_1 = [P, G] = [P, C_G(P)P] = [P, P] < P$. Now let $P_2 = [P_1, G]$, $P_3 = [P_2, G]$, ... Then there must exist a natural number k such that $P_k = 1$ by what has been said above. Hence

$$[P, \underbrace{G, \ldots, G}_{k}] = 1.$$

This implies that $P \leq Z_k(G) \leq Z_\infty(G)$. Thus we obtain $M \leq Z_\infty(G)$. On the other hand, obviously $Z_\infty(G)$ itself is an NS subgroup (see [W, p. 6]). Hence the claim of the lemma follows. ∎

The following lemma gives an interesting property of NS subgroups, which will play an important role in the next part of our paper.

LEMMA 1.3. *Let M be an NS subgroup of G, and H be a proper self-normalizing subgroup of G. Then M is contained in H.*

Proof. Suppose that M is not contained in H. Then there exists a prime r such that any Sylow r-subgroup R of M is not contained in H. Since M is an NS subgroup of G, R is characteristic in M. Hence for any Sylow p-subgroup P of H, we have $R \leq N_G(P)$, where p is a prime which is not equal to r. Let R_1 be a Sylow r-subgroup of H. Then there is an element $x \in R \setminus R_1$ such that $x \in N_G(R_1)$. It follows that there is an $x \notin H$ such that $x \in N_G(H)$, a contradiction. ∎

By the above lemma, we can immediately get the following corollary.

COROLLARY 1.4. *Let M be an NS subgroup of G. Then a subgroup H is self-normalizing in G if and only if H/M is self-normalizing in G/M, and two self-normalizing subgroups H_1 and H_2 are conjugate in G if and only if H_1/M and H_2/M are conjugate in G/M.*

2. Theorems. In this section, we give a characterization of finite groups with only one conjugacy class of proper self-normalizing subgroups.

MAIN THEOREM 2.1. *Let G be a finite group. Then the number of conjugacy classes of proper self-normalizing subgroups of G is 1 if and only if there exists an NS subgroup M of G such that $G/M = UW$, where U is an elementary abelian normal Sylow p-subgroup of G/M, W is a nilpotent Hall p'-subgroup of G/M, W acts irreducibly on U, and $N_U(N_W(U_1)) \nleq U_1$ for any proper subgroup U_1 of U.*

Proof. We first assume that the number of conjugacy classes of proper self-normalizing subgroups of G is 1. Since nilpotent groups have no proper self-normalizing subgroups, it can be easily seen that there exists at least one maximal subgroup H of G such that H is not normal in G. Similarly, G has a Sylow q-subgroup Q which is not normal. Hence $N_G(Q)$ is a proper self-normalizing subgroup of G. Without loss of generality, we may assume that $H = N_G(Q)$.

Assume that $p \mid |G : H|$ for some prime p. Then the Sylow p-subgroup P is normal in G. Indeed, otherwise, $N_G(P)$ must be a self-normalizing subgroup of G. Thus $N_G(P)$ and H are conjugate in G by hypothesis and then $p \mid |G : H| = |G : N_G(P)|$, a contradiction.

Now we claim that $|G : H|$ is a power of p. Indeed, let r be a prime which is not equal to p such that $r \mid |G : H|$. Then we can easily seen that both the Sylow r-subgroup R and the Sylow p-subgroup P are normal in G. Hence RH is a proper subgroup of G by hypothesis, a contradiction. This proves that $|G : H|$ is a power of p.

By the Schur–Zassenhaus Theorem, P has a complement K in G and all such complements are conjugate in G. Without loss of generality, we may assume that $K \leq H$.

We now claim that K is nilpotent. Let T be a Sylow t-subgroup of K. Then either $N_G(T) = G$, or $N_G(T)$ and H are conjugate in G by hypothesis. Therefore $|N_G(T)P/P| = |K|$. On the other hand, since $N_{G/P}(TP/P) = N_G(T)P/P$, we have $|N_{G/P}(TP/P)| = |K| = |G/P|$, hence it follows that $TP/P \trianglelefteq G/P$. Thus we find that G/P is nilpotent. Now as $K/K \cap P = K \cong KP/P = G/P$, it follows that K is nilpotent.

Let $M = P \cap H$. Then $H = H \cap G = H \cap KP = K(H \cap P) = KM$ and $M \trianglelefteq H$. Hence $\Phi(P) \leq M$ since $\Phi(P) \leq H$. Thus we have that $M \trianglelefteq P$ since $M/\Phi(P) \trianglelefteq P/\Phi(P)$. Let T be a Sylow subgroup of K. By hypothesis $N_G(T)$ and H are G-conjugate. Hence $M \leq N_G(T)$, which implies that H is nilpotent. Moreover, it is easily seen that M is an NS subgroup of G.

Let $U = PM/M$ and $W = H/M$. Then U is a normal elementary abelian p-subgroup of G/M and W is a nilpotent Hall p'-subgroup of G/M. Since W is a maximal subgroup of G/M, it follows that W acts irreducibly on U.

Let U_1 be any proper subgroup of U, and $W_1 = N_W(U_1)$, $G_1 = U_1W_1$. Then G_1 is a proper subgroup of G/M, and G_1 and W are not conjugate in G/M. By Corollary 1.4 and the hypothesis, G_1 is not self-normalizing in G/M. Therefore there exist $u \in U$ and $w \in W$ such that $uw \notin G_1$ but $G_1^{uw} = G_1$. Since $W_1 = N_W(U_1)$, we see that U_1 is the normal Sylow subgroup of G_1 and hence $U_1^{uw} = U_1$. Therefore $U_1^w = U_1$ since U_1 is normal in U. It follows that $w \in W_1$. Thus we get $u \notin U_1$. On the other hand, since $(U_1W_1)^u = U_1W_1$, there is an $u_1 \in U_1$ such that $W_1^u = W_1^{u_1}$, that is, $W_1^{uu_1^{-1}} = W_1$. Hence $uu_1^{-1} \in N_U(W_1)$, but $uu_1^{-1} \notin U_1$. This implies that $N_U(N_W(U_1)) \nleq U_1$.

Conversely, since W acts irreducibly on U, it follows that W is a maximal subgroup of G/M which is not normal in G/M. Hence, W is self-normalizing in G/M. Thus to complete the proof, we only need to show that G_1 is not self-normalizing in G/M as soon as $|G_1| \neq |W|$, for any proper subgroup G_1 of G/M. Let $G_1 = U_1W_1$, where U_1 is a Sylow p-subgroup of G_1, and W_1 is a Hall p'-subgroup of G_1. Then by the Schur–Zassenhaus Theorem, some conjugates of W_1 are contained in W. Hence we may assume that $U_1 \leq U$ and $W_1 \leq W$. If $U_1 = 1$, then we can easily see that our claim holds since W is nilpotent. If $U_1 = U$, then the claim follows from the nilpotency of W. For the case $1 < U_1 < U$, let $W_2 = N_W(U_1)$. If $W_1 < W_2$, then $G_1/U_1 \leq U_1W_2/U_1 \cong W_2 \leq W$. Thus G_1 is not self-normalizing in G/M, as expected. If $W_1 = W_2$, then there exists $u \in U \setminus U_1$ satisfying $W_1^u = W_1$. Thus $u \in N_{G/M}(G_1)$, which means G_1 is not self-normalizing in G/M. ∎

Let G be a group as in Theorem 2.1. By Lemma 1.2 and Theorem 2.1, we have $C_{G/Z_\infty(G)}(U) = U$ and hence $W \cong N_{G/Z_\infty(G)}(U)/C_{G/Z_\infty(G)}(U) \lesssim \mathrm{Aut}(U)$. Thus we get the following corollary.

COROLLARY 2.2. *Let G be a finite group. Then the number of conjugacy classes of proper self-normalizing subgroups of G is 1 if and only if $G/Z_\infty(G) = UW$, where U is an elementary abelian normal Sylow p-subgroup of $G/Z_\infty(G)$, W is a nilpotent Hall p'-subgroup of $G/Z_\infty(G)$, W acts irreducibly on U, $W \lesssim \mathrm{Aut}(U)$, and $N_U(N_W(U_1)) \nleq U_1$ for any proper subgroup U_1 of U. In addition, in this case, the number of proper self-normalizing subgroups of G is equal to the order of U.*

It is easy to see that the number of subgroups conjugate to any proper self-normalizing subgroup cannot be two. Hence if G contains just three or four or five proper self-normalizing subgroups, then those subgroups must be conjugate in G. Now Corollary 2.2 implies the following corollaries.

COROLLARY 2.3. *There is no group which contains exactly two proper self-normalizing subgroups.*

COROLLARY 2.4. *Let G be a finite group. Then G contains just three proper self-normalizing subgroups if and only if $G/Z_\infty(G) \cong S_3$.*

COROLLARY 2.5. *Let G be a finite group. Then G contains just four proper self-normalizing subgroups if and only if $G/Z_\infty(G) \cong A_4$.*

COROLLARY 2.6. *Let G be a finite group. Then G contains just five proper self-normalizing subgroups if and only if either $G/Z_\infty(G) \cong C_5 \rtimes C_4$ and $Z(C_5 \rtimes C_4) = 1$, or $G/Z_\infty(G) \cong C_5 \rtimes C_2$.*

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