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STRONGLY INVARIANT MEANS ON COMMUTATIVE HYPERGROUPS

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Abstract. We introduce and study strongly invariant means m on commutative hypergroups, $m(T_x\varphi \cdot \psi) = m(\varphi \cdot T_{\tilde{x}}\psi)$, $x \in K$, $\varphi, \psi \in L^\infty(K)$. We show that the existence of such means is equivalent to a strong Reiter condition. For polynomial hypergroups we derive a growth condition for the Haar weights which is equivalent to the existence of strongly invariant means. We apply this characterization to show that there are commutative hypergroups which do not possess strongly invariant means.

1. Introduction. Hypergroups generalize locally compact groups. For the theory of hypergroups we refer to [1] and [3]. A *hypergroup* K is a locally compact Hausdorff space with a convolution, i.e. a map $K \times K \to M^1(K)$, $(x, y) \mapsto \delta_x * \delta_y$, ($M^1(K)$ is the space of probability measures on K) and an involution, i.e. $K \to K$, $x \mapsto \tilde{x}$, satisfying certain axioms (see [1] or [3]). The support of each probability measure $\delta_x * \delta_y$ is compact. Hence, for $y \in K$ the translation of a locally integrable function f on K is defined by

$$T_x f(y) = \int_K f(z)\, d\delta_x * \delta_y\,(z).$$

Spector [11] has proven that each commutative hypergroup possesses a Haar measure μ, which is characterized by

$$\int_K T_x f(y)\, d\mu(y) = \int_K f(y)\, d\mu(y)$$

for all $x \in K$ and $f \in C_c(K)$, where $C_c(K)$ is the space of all continuous complex valued functions on K with compact support. Throughout this paper, K will be a *commutative* hypergroup. The Banach spaces $L^p(K) = L^p(K, \mu)$, $1 \le p \le \infty$, are invariant under the translations T_x, $x \in K$, and we have $\|T_x f\|_p \le \|f\|_p$.

The convex subset $P^1(K)$ of $L^1(K)$ is defined by

$$P^1(K) = \Big\{ f \in L^1(K) : f \ge 0,\ \int_K f(x)\, d\mu(x) = 1 \Big\}.$$

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Let $C_b(K)$ denote the space of bounded continuous complex valued functions on K equipped with the sup-norm. $C_b(K)$ is also invariant under the translation T_x. The space of bounded uniformly continuous functions is given by

$$UC(K) = \{f \in C_b(K) : x \mapsto T_x f \text{ is continuous from } K \to C_b(K)\}.$$

Our main interest is in *strongly invariant means* on $L^\infty(K)$: these are means $m \in L^\infty(K)^*$ satisfying

$$m((T_x\varphi) \cdot \psi) = m(\varphi \cdot T_{\tilde{x}}\psi)$$

for all $x \in K$ and $\varphi, \psi \in L^\infty(K)$. Invariant means on hypergroups are studied in [10] and [6]. An *invariant mean* satisfies $m(T_x\varphi) = m(\varphi)$ for all $x \in K$ and $\varphi \in L^\infty(K)$. Since K is commutative, there exist invariant means on $L^\infty(K)$. We will show that this is not necessarily the case for strongly invariant means. We will derive an equivalent condition for the existence of such means, namely a strong Reiter condition. For polynomial hypergroups K on $\mathbb{N}_0$ we give an explicit characterization when a strongly invariant mean exists.

Polynomial hypergroups K on $\mathbb{N}_0$ are generated by orthogonal polynomial sequences $(R_n(x))_{n\in\mathbb{N}_0}$. $\mathbb{N}_0$ is equipped with the discrete topology. The convolution is determined by the linearization coefficients $g(m,n;k)$ of the product $R_m(x)R_n(x)$, i.e.

$$R_m(x)R_n(x) = \sum_{k=|n-m|}^{n+m} g(m,n;k)R_k(x). \tag{1.1}$$

If all $g(m,n;k)$ are nonnegative, and if the $R_n(x)$ are normalized so that $R_n(1) = 1$, then, putting

$$\delta_m * \delta_n = \sum_{k=|n-m|}^{n+m} g(m,n;k)\delta_k,$$

a convex combination of the point measures δ_k, we get a convolution on $\mathbb{N}_0$. Together with $\tilde{n} = n$ as involution and $n = 0$ as unit, this convolution defines a commutative hypergroup on $\mathbb{N}_0$. For more details and a lot of examples we refer to [1] or [5, 6]. The Haar measure on the polynomial hypergroup $\mathbb{N}_0$ is the counting measure with weights $h(n) = g(n,n;0)^{-1}$ of the points $n \in \mathbb{N}_0$. Taking $m = 1$ in (1.1) we get the three-term recurrence relation

$$R_1(x)R_n(x) = a_nR_{n+1}(x) + b_nR_n(x) + c_nR_{n-1}(x)$$

for all $n \in \mathbb{N}$, with initial values $R_0(x) = 1$, $R_1(x) = (1/\alpha_0)(x - \beta_0)$. Hence, $a_n = g(1,n;n+1) > 0$, $b_n = g(1,n;n) \geq 0$, $c_n = g(1,n;n-1) > 0$ for all $n \in \mathbb{N}$, and $a_n + b_n + c_n = 1$, whereas $\alpha_0 > 0$, $\beta_0 \in \mathbb{R}$ with $\alpha_0 + \beta_0 = 1$. The Haar weights satisfy $h(0) = 1$, $h(n+1) = \frac{a_n}{c_{n+1}}h(n)$.

A growth condition for the Haar weights characterizes the existence of strongly invariant means. Applying this growth condition we test some examples. We will see that some polynomial hypergroups have a strongly invariant mean, while others do not.

2. Strongly invariant means. Let K be a commutative hypergroup. Let $\mathbb{M}(K)$ be the set of invariant means on K, i.e.

$$\mathbb{M}(K) = \{m \in L^\infty(K)^* : m(1) = 1,\ m \geq 0,\ m(T_x\varphi) = m(\varphi) \\ \text{for all } x \in K \text{ and } \varphi \in L^\infty(K)\}.$$

Then $\mathbb{M}(K)$ is a nonempty (see [10]) weak-$*$-compact, convex subset of $L^\infty(K)^*$. Our main interest is in the set $S\mathbb{M}(K)$ of strongly invariant means in K, i.e.

$$S\mathbb{M}(K) = \{m \in L^\infty(K)^* : m(1) = 1,\ m \geq 0,\ m((T_y\varphi)\cdot\psi) = m(\varphi\cdot(T_{\tilde{y}}\psi)) \\ \text{for all } y \in K \text{ and } \varphi, \psi \in L^\infty(K)\}.$$

Obviously, $S\mathbb{M}(K) \subseteq \mathbb{M}(K)$ and $S\mathbb{M}(K)$ is also a weak-$*$-compact, convex subset of $L^\infty(K)^*$. Elements $x \in K$ act on $L^\infty(K)^*$ by the map $F \mapsto T_xF$, where $T_xF(\varphi) = F(T_x\varphi)$ for all $\varphi \in L^\infty(K)$. Note that

$$\|T_xF\| = \sup_{\|\varphi\|_\infty \leq 1} |T_xF(\varphi)| = \sup_{\|\varphi\|_\infty \leq 1} |F(T_x\varphi)| \leq \|F\|.$$

We define

$$J(L^\infty(K)) = \{F \in L^\infty(K)^* : T_xF = F \text{ for all } x \in K\}.$$

Let $\mathrm{Bil}(L^\infty(K)) = \{B : L^\infty(K) \times L^\infty(K) \to \mathbb{C} : B \text{ bilinear}\}$. Each $x \in K$ also determines a map $F \mapsto B^1_{x,F}$, $L^\infty(K)^* \to \mathrm{Bil}(L^\infty(K))$, by

$$B^1_{x,F}(\varphi,\psi) = F((T_x\varphi)\cdot\psi) \quad \text{ for all } \varphi, \psi \in L^\infty(K),$$

and a second map $F \mapsto B^2_{x,F}$, $L^\infty(K)^* \to \mathrm{Bil}(L^\infty(K))$, by

$$B^2_{x,F}(\varphi,\psi) = F(\varphi\cdot(T_{\tilde{x}}\psi)) \quad \text{ for all } \varphi, \psi \in L^\infty(K).$$

We define

$$SJ(L^\infty(K)) = \{F \in L^\infty(K)^* : B^1_{x,F} = B^2_{x,F} \text{ for all } x \in K\}.$$

Obviously $J(L^\infty(K))$ and $SJ(L^\infty(K))$ are linear subspaces of $L^\infty(K)^*$ and $0 \in SJ(L^\infty(K)) \subseteq J(L^\infty(K))$.

Proposition 2.1.

(i) *The linear span*

$$L := \mathrm{span}\{T_x\varphi - \varphi : \varphi \in L^\infty(K),\ x \in K\}$$

is not dense in $L^\infty(K)$ *if and only if* $J(L^\infty(K)) \neq \{0\}$.

(ii) *The linear span*

$$SL := \operatorname{span}\{T_x\varphi \cdot \psi - \varphi \cdot T_{\tilde{x}}\psi : \varphi, \psi \in L^\infty(K),\ x \in K\}$$

is not dense in $L^\infty(K)$ *if and only if* $SJ(L^\infty(K)) \neq \{0\}$.

(iii) $SJ(L^\infty(K)) \subsetneq J(L^\infty(K))$ *if and only if* $\overline{SL} \supsetneq \overline{L}$.

Proof. The proof of (i) follows the lines of the proof of Proposition (2.1) of [7]. The proofs of (ii) and (iii) also use simple applications of the Hahn–Banach theorem. For the sake of completeness they are given here.

(ii) Assume $\overline{SL} = L^\infty(K)$. If $F \in SJ(L^\infty(K))$, then $B^1_{x,F} = B^2_{x,F}$, which means $F(T_x\varphi \cdot \psi - \varphi \cdot T_{\tilde{x}}\psi) = 0$ for all $\varphi, \psi \in L^\infty(K)$ and $x \in K$, and $\overline{SL} = L^\infty(K)$ implies $F = 0$. Conversely, if $\overline{SL} \subsetneq L^\infty(K)$, then there exists some $F \in L^\infty(K)^*$, $F \neq 0$, such that $F|\overline{SL} = 0$. In particular, $F \in SJ(L^\infty(K))$.

(iii) Assume that there is some $F \in J(L^\infty(K)) \setminus SJ(L^\infty(K))$. Then $F|\overline{L} = 0$ and there are some $x \in K$ and $\varphi, \psi \in L^\infty(K)$ such that

$$F(T_x\varphi \cdot \psi - \varphi \cdot T_{\tilde{x}}\psi) \neq 0.$$

Hence, $\overline{L} \subsetneq SL$. The inverse implication follows immediately by the Hahn–Banach theorem. ∎

To show the existence or nonexistence of strongly invariant means on K we use another version of the Hahn–Banach theorem (cf. [8, Lemma 4.2.7]). Let $SL_{\mathbb{R}}$ be the $\mathbb{R}$-linear span of $\{T_x\varphi \cdot \psi - \varphi \cdot T_{\tilde{x}}\psi : \varphi, \psi \in L^\infty_{\mathbb{R}}(K),\ x \in K\}$, where $L^\infty_{\mathbb{R}}(K)$ is the space of real-valued elements of $L^\infty(K)$.

Proposition 2.2. *The following properties are equivalent:*

(i) *There exists* $m \in L^\infty_{\mathbb{R}}(K)^*$ *with* $m(1) = 1$, $m \geq 0$ *and* $m|SL_{\mathbb{R}} = 0$.

(ii) $\sup_{x\in K} \eta(x) \geq 0$ *for all* $\eta \in SL_{\mathbb{R}}$.

Proof. By the definition of means we have $m(\varphi) \leq \sup_{x\in K} \varphi(x)$ for all $\varphi \in L^\infty_{\mathbb{R}}(K)$. Supposing (i) we obtain $0 = m(\eta) \leq \sup_{x\in K} \eta(x)$ for all $\eta \in SL_{\mathbb{R}}$.

Conversely, assume that (ii) is valid. Let $N(\varphi) = \sup_{x\in K} \varphi(x)$ for all $\varphi \in L^\infty_{\mathbb{R}}(K)$. Then N is a sublinear functional on $L^\infty_{\mathbb{R}}(K)$ and it dominates the zero functional on $SL_{\mathbb{R}}$. By the Hahn–Banach theorem [9] the zero-functional on $SL_{\mathbb{R}}$ can be extended to a linear functional m on $L^\infty_{\mathbb{R}}(K)$ such that $m(\varphi) \leq \sup_{x\in K} \varphi(x)$ for all $\varphi \in L^\infty_{\mathbb{R}}(K)$. Then m is a mean on $L^\infty_{\mathbb{R}}(K)$ [8, Proposition 3.2] and $m(\eta) = 0$ for all $\eta \in SL_{\mathbb{R}}$. ∎

Writing $\varphi \in L^\infty(K)$ as $\varphi = \varphi_1 + i\varphi_2$, $\varphi_1, \varphi_2 \in L^\infty_{\mathbb{R}}(K)$, and putting $m(\varphi) = m_{\mathbb{R}}(\varphi_1) + im_{\mathbb{R}}(\varphi_2)$ for a mean $m_{\mathbb{R}}$ on $L^\infty_{\mathbb{R}}(K)$ we obtain a mean on $L^\infty(K)$. If $\varphi, \psi \in L^\infty(K)$ and $\varphi = \varphi_1 + i\varphi_2$, $\psi = \psi_1 + i\psi_2$ with

$\varphi_1, \varphi_2, \psi_1, \psi_2 \in L^\infty_{\mathbb{R}}(K)$, then

$$\begin{aligned} T_x\varphi \cdot \psi - \varphi \cdot T_{\tilde{x}}\psi = (T_x\varphi_1 \cdot \psi_1 - \varphi_1 \cdot T_{\tilde{x}}\psi_1) - (T_x\varphi_2 \cdot \psi_2 - \varphi_2 \cdot T_{\tilde{x}}\psi_2) \\ + i(T_x\varphi_1 \cdot \psi_2 - \varphi_1 \cdot T_{\tilde{x}}\psi_2) + i(T_x\varphi_2 \cdot \psi_1 - \varphi_2 \cdot T_{\tilde{x}}\psi_1). \end{aligned}$$

Therefore, if $m_{\mathbb{R}}$ is a mean on $L^\infty_{\mathbb{R}}(K)$ with $m_{\mathbb{R}}|SL_{\mathbb{R}} = 0$, then m is a mean on $L^\infty(K)$ such that $m|SL = 0$. So we have the following characterization of the existence of a strongly invariant mean on K.

THEOREM 2.3. *Let K be a commutative hypergroup. There exists a strongly invariant mean m on K if and only if* $\sup_{x\in K} \eta(x) \geq 0$ *for all* $\eta \in SL_{\mathbb{R}}$.

3. Reiter's conditions. Skantharajah [10] has shown for general hypergroups that an invariant mean exists if and only if the following *Reiter condition* (P_1) is satisfied: For every $\epsilon > 0$ and compact $C \subseteq K$ there exists some $f \in P^1(K)$ such that $\|T_x f - f\|_1 < \epsilon$ for all $x \in C$.

We now investigate a modification of the Reiter condition (P_1) which is related to the existence of strongly invariant means $m \in S\mathbb{M}(K)$.

DEFINITION 3.1. We say that K satisfies the *strong Reiter condition* (SP_1^*) if for every $\epsilon > 0$ and finite subset $F \subseteq K$ and finite subset $\Phi \subseteq L^\infty(K)$ there is some $f \in P^1(K)$ such that

$$\|T_x(\varphi \cdot f) - f \cdot T_x\varphi\|_1 < \epsilon \tag{3.1}$$

for all $x \in F$ and $\varphi \in \Phi$.

THEOREM 3.2. *If there exists a strongly invariant mean m on K, then K satisfies (SP_1^*).*

Proof. The embedding of the unit ball $B \subseteq L^1(K)$ into $L^\infty(K)^*$ is dense in the unit ball $C \subseteq L^\infty(K)^*$ with respect to the weak-$*$-topology (see [2, p. 424]). Hence, the existence of a strongly invariant mean $m \in S\mathbb{M}(K)$ yields a net $(f_j)_{j\in I}$, $f_j \in B$, such that

$$\int_K f_j(x)\psi(x)\,d\mu(x) \to m(\psi) \quad \text{for all } \psi \in L^\infty(K).$$

Since $m(1) = 1$ and m is positive we can choose the f_j from $P^1(K)$. For any $y \in K$ and $\varphi, \psi \in L^\infty(K)$ we have

$$\int_K T_y(\varphi \cdot f_j)(x)\psi(x)\,d\mu(x) = \int_K f_j(x)\varphi(x)T_{\tilde{y}}\psi(x)\,d\mu(x) \to m(\varphi \cdot T_{\tilde{y}}\psi)$$

and

$$\int_K f_j(x)T_y\varphi(x)\psi(x)\,d\mu(x) \to m(T_y\varphi \cdot \psi).$$

Hence,

$$\int_K (T_y(\varphi f_j)(x) - (f_j T_y\varphi)(x))\psi(x)\,d\mu(x) \to 0$$

for all $\psi \in L^\infty(K)$.

For $F = \{y_1, \ldots, y_m\}$ and $\Phi = \{\varphi_1, \ldots, \varphi_n\}$ define

$$g_{k,l,j} = T_{y_k}(\varphi_l f_j) - f_j(T_{y_k}\varphi_l).$$

Then the net $g_j = (g_{1,1,j}, \ldots, g_{m,n,j}) \in L^1(K)^{mn} = L^1(K) \times \cdots \times L^1(K)$ converges to zero in the product space $L^1(K)^{mn}$ with respect to the weak topology. Since for convex subsets of $L^1(K)^{mn}$ the norm-closure coincides with the weak closure, a convex combination of the g_j converges to zero in the norm of $L^1(K)^{mn}$ (see [2, p. 422]). Hence, there exists some $f \in P^1(K)$, a convex combination of the functions g_j, such that

$$\|T_x(\varphi \cdot f) - f \cdot T_x\varphi\|_1 < \epsilon \quad \text{for all } x \in F,\ \varphi \in \Phi. \ \blacksquare$$

Now, we show that the converse of Theorem 3.2 is also true.

THEOREM 3.3. *If K satisfies (SP_1^*), then there exists a strongly invariant mean m on K.*

Proof. Let $\epsilon > 0$, $F \subseteq K$ finite and $\Phi \subseteq L^\infty(K)$ finite. Then there exists $f \in P^1(K)$ such that $\|T_x(\varphi \cdot f) - (T_x\varphi) \cdot f\|_1 < \epsilon$ for all $x \in F$ and $\varphi \in \Phi$. The function f determines a linear functional $m_{\epsilon,F,\Phi} \in L^\infty(K)^*$ by

$$m_{\epsilon,F,\Phi}(\psi) = \int_K \psi(y) f(y)\,d\mu(y) \quad \text{for all } \psi \in L^\infty(K).$$

This functional is positive and satisfies

$$\|m_{\epsilon,F,\Phi}\| = m_{\epsilon,F,\Phi}(1) = \int_K f(y)\,d\mu(y) = 1.$$

In particular all $m_{\epsilon,F,\Phi}$ are elements of a weak-$*$-compact subset of $L^\infty(K)^*$.

For every $\varphi, \psi \in L^\infty(K)$ and $x \in K$ we have

$$m_{\epsilon,F,\Phi}(\varphi \cdot T_{\tilde{x}}\psi) = \int_K \varphi(y) T_{\tilde{x}}\psi(y) f(y)\,d\mu(y) = \int_K T_x(\varphi f)(y)\psi(y)\,d\mu(y).$$

Therefore,

$$\begin{aligned} &|m_{\epsilon,F,\Phi}(\varphi \cdot T_{\tilde{x}}\psi) - m_{\epsilon,F,\Phi}(\psi \cdot T_x\varphi)| \\ &\qquad = \Big|\int_K (T_x(\varphi f)(y) - f(y)T_x\varphi(y))\psi(y)\,d\mu(y)\Big| \\ &\qquad \le \|\psi\|_\infty \|T_x(\varphi f) - (T_x\varphi) \cdot f\|_1 < \epsilon \end{aligned}$$

for all $\varphi \in \Phi$ with $x \in F$ and $\psi \in L^\infty(K)$ with $\|\psi\|_\infty \le 1$.

Defining

$$(\epsilon_1, F_1, \Phi_1) < (\epsilon_2, F_2, \Phi_2) \quad \text{whenever} \quad \epsilon_2 \le \epsilon_1, F_1 \subseteq F_2, \Phi_1 \subseteq \Phi_2$$

we get a partial order. With respect to this partial order, the functionals $m_{\epsilon,F,\Phi}$ form a net. This net has an accumulation point $m \in L^\infty(K)^*$ satisfying $\|m\| = m(1) = 1$ and $m(\varphi(T_{\tilde{x}}\psi)) = m((T_x\varphi)\psi)$ for all $\varphi, \psi \in L^\infty(K)$ and $x \in K$. ∎

The function $f \in P^1(K)$ in the (SP_1^*) condition can even be chosen to be continuous with compact support.

Lemma 3.4. *If K satisfies (SP_1^*), then the function f in* (3.1) *can be chosen from $f \in P^1(K) \cap C_c(K)$.*

Proof. Let $\epsilon > 0$, $F \subseteq K$ finite, $\Phi \subseteq L^\infty(K)$ finite, and denote $M = \max\{\|\varphi\|_\infty : \varphi \in \Phi\}$. Then there exists $g \in P^1(K)$ such that $\|T_x(\varphi \cdot g) - (T_x\varphi) \cdot g\|_1 < \epsilon/3$ for all $x \in F$, $\varphi \in \Phi$. Furthermore, there is some $f \in P^1(K) \cap C_c(K)$ such that $\|g - f\|_1 < \epsilon/(3M)$. It follows that

$$\begin{aligned}\|T_x(\varphi \cdot f) - (T_x\varphi) \cdot f\|_1 \le \|&T_x(\varphi \cdot f) - T_x(\varphi \cdot g)\|_1 \\ &+ \|T_x(\varphi \cdot g) - (T_x\varphi) \cdot g\|_1 \\ &+ \|(T_x\varphi) \cdot g - (T_x\varphi) \cdot f\|_1 < \epsilon.\end{aligned}$$ ∎

Restricting from $L^\infty(K)$ to $UC(K)$ we may replace the finiteness of $F \subseteq K$ in (SP_1^*) by compactness. In fact we can show:

Theorem 3.5. *Assume that (SP_1^*) is satisfied. For $\epsilon > 0$, $C \subseteq K$ compact and $\Phi \subseteq UC(K)$ finite there exists some $f \in P^1(K) \cap C_c(K)$ such that*

$$\|T_x(\varphi f) - T_x\varphi \cdot f\|_1 < \epsilon \qquad \textit{for all } x \in C, \varphi \in \Phi.$$

Proof. Let $M = \max\{\|\varphi\|_\infty : \varphi \in \Phi\}$. Choose $g \in P^1(K) \cap C_c(K)$. There exist $x_1, \ldots, x_n \in K$ and open neighbourhoods U_{x_i} of x_i, $i = 1, \ldots, n$, such that

$$C \subseteq \bigcup_{i=1}^n U_{x_i} \text{ and } \|T_x g - T_y g\|_1 < \frac{\epsilon}{3M} \quad \text{and} \quad \|T_x\varphi - T_y\varphi\|_\infty < \frac{\epsilon}{3}$$

for all $x, y \in U_{x_i}$ and every $\varphi \in \Phi$. Condition (SP_1^*) yields $f \in P^1(K) \cap C_c(K)$ such that

$$\|T_{x_i}(\varphi \cdot f) - (T_{x_i}\varphi) \cdot f\|_1 < \frac{\epsilon}{3} \quad \text{for all } \varphi \in \Phi,$$

$i = 1, \ldots, n$.

Now, let $x \in C$. Then $x \in U_{x_i}$ for some $i = 1, \ldots, n$, and we obtain

$$\begin{aligned}\|T_x(\varphi \cdot f) * g - T_{x_i}(\varphi \cdot f) * g\|_1 &= \|(\varphi \cdot f) * T_x g - (\varphi \cdot f) * T_{x_i} g\|_1 \\ &\le \|\varphi \cdot f\|_1 \|T_x g - T_{x_i} g\|_1 < \frac{\epsilon}{3}.\end{aligned}$$

Hence,

$$\|T_x(\varphi\cdot f)*g-((T_x\varphi)\cdot f)*g\|_1\le\|T_x(\varphi\cdot f)*g-T_{x_i}(\varphi\cdot f)*g\|_1$$
$$+\|T_{x_i}(\varphi\cdot f)*g-((T_{x_i}\varphi)\cdot f)*g\|_1+\|((T_{x_i}\varphi)\cdot f)*g-((T_x\varphi)\cdot f)*g\|_1<\epsilon.$$

Choosing $g\in P^1(K)\cap C_c(K)$ from an approximate identity in $L^1(K)$ it follows that $\|T_x(\varphi\cdot f)-T_x\varphi\cdot f\|_1\le\epsilon$ for all $x\in C$ and $\varphi\in\Phi$. ∎

4. Strongly invariant means on polynomial hypergroups. Now, we study the special case of polynomial hypergroups K on $\mathbb{N}_0$. We begin by deriving a more accessible description of the linear space

$$SL=\operatorname{span}\{T_n\varphi\cdot\psi-\varphi\cdot T_n\psi:n\in\mathbb{N}_0,\ \varphi,\psi\in\ell^\infty\}.$$

We shall use the following relation for the translation operators T_n:

$$T_{n+1}=\frac{1}{a_n}T_1\circ T_n-\frac{b_n}{a_n}T_n-\frac{c_n}{a_n}T_{n-1},\qquad n\ge1. \tag{4.1}$$

Applying induction and (4.1) we will prove the following result.

Lemma 4.1. $SL=\operatorname{span}\{T_1\varphi\cdot\psi-\varphi\cdot T_1\psi:\varphi,\psi\in\ell^\infty\}$.

Proof. By induction on n we show

$$T_n\varphi_1\cdot\psi_1-\varphi_1\cdot T_n\psi_1\in\operatorname{span}\{T_1\varphi_2\cdot\psi_2-\varphi_2\cdot T_1\psi_2:\varphi_2,\psi_2\in\ell^\infty\}$$

for all $\varphi_1,\psi_1\in\ell^\infty$. Assume that $T_k\varphi_1\cdot\psi_1-\varphi_1\cdot T_k\psi_1$ are elements of

$$SL_1:=\operatorname{span}\{T_1\varphi_2\cdot\psi_2-\varphi_2\cdot T_1\psi_2:\varphi_2,\psi_2\in\ell^\infty\}$$

for $k=n-1,n$ and for all $\varphi_1,\psi_1\in\ell^\infty$. From (4.1) and $T_n\circ T_1=T_1\circ T_n$ we obtain, for all $\varphi_1,\psi_1\in\ell^\infty$,

$$\begin{aligned}T_{n+1}\varphi_1\cdot\psi_1-\varphi_1\cdot T_{n+1}\psi_1=&\left[T_1\left(\frac{T_n\varphi_1}{a_n}\right)-\frac{b_nT_n\varphi_1}{a_n}-\frac{c_nT_{n-1}\varphi_1}{a_n}\right]\psi_1\\&-\varphi_1\left[T_n\left(\frac{T_1\psi_1}{a_n}\right)-\frac{b_nT_n\psi_1}{a_n}-\frac{c_nT_{n-1}\psi_1}{a_n}\right].\end{aligned}$$

By the assumption we have

$$T_{n-1}\varphi_1\cdot\psi_1-\varphi_1\cdot T_{n-1}\psi_1\in SL_1,\qquad T_n\varphi_1\cdot\psi_1-\varphi_1\cdot T_n\psi_1\in SL_1.$$

Moreover,

$$T_1\left(\frac{T_n\varphi_1}{a_n}\right)\psi_1-\frac{T_n\varphi_1}{a_n}T_1\psi_1\in SL_1$$

and by the assumption

$$\varphi_1T_n\left(\frac{T_1\psi_1}{a_n}\right)-T_n\varphi_1\cdot\frac{T_1\psi_1}{a_n}\in SL_1.$$

Hence, we have

$$T_1\left(\frac{T_n\varphi_1}{a_n}\right)\psi_1 - \varphi_1 T_n\left(\frac{T_1\psi_1}{a_n}\right) = \left(T_1\left(\frac{T_n\varphi_1}{a_n}\right)\psi_1 - \frac{T_n\varphi_1}{a_n}T_1\psi_1\right) - \left(\varphi_1 T_n\left(\frac{T_1\psi_1}{a_n}\right) - T_n\varphi_1\cdot\frac{T_1\psi_1}{a_n}\right) \in SL_1,$$

and Lemma 4.1 is shown. ∎

Let $\varphi, \psi \in \ell^\infty$. Then

$$\begin{aligned} T_1\varphi(n)\,\psi(n) - \varphi(n)\,T_1\psi(n) &= a_n(\varphi(n+1)\psi(n) - \psi(n+1)\varphi(n)) \\ &\quad - c_n(\varphi(n)\psi(n-1) - \psi(n)\varphi(n-1)) \\ &= a_n\omega(n) - c_n\omega(n-1) \end{aligned} \tag{4.2}$$

with

$$\omega(n) = \varphi(n+1)\psi(n) - \psi(n+1)\varphi(n) \quad \text{for all } n \in \mathbb{N}_0.$$

Note that we set $a_0 = 1$ and $c_0 = 0$.

It is important to point out that for any sequence $\omega \in \ell^\infty$ there are $\varphi, \psi \in \ell^\infty$ such that $\omega(n) = \varphi(n+1)\psi(n) - \psi(n+1)\varphi(n)$. This can be easily proved by setting

$$\varphi(2k) = \begin{cases} 0 & \text{for } k \text{ even}, \\ 1 & \text{for } k \text{ odd}, \end{cases} \qquad \varphi(2k+1) = \begin{cases} \omega(2k) & \text{for } k \text{ even}, \\ -\omega(2k+1) & \text{for } k \text{ odd} \end{cases} \tag{4.3}$$

and

$$\psi(2k) = \begin{cases} 1 & \text{for } k \text{ even}, \\ 0 & \text{for } k \text{ odd}, \end{cases} \qquad \psi(2k+1) = \begin{cases} \omega(2k+1) & \text{for } k \text{ even}, \\ -\omega(2k) & \text{for } k \text{ odd}. \end{cases} \tag{4.4}$$

Therefore, there is again a simplification.

Lemma 4.2. *We have*

$$\operatorname{span}\{T_1\varphi\cdot\psi - \varphi\cdot T_1\psi : \varphi,\psi \in \ell^\infty\} = \{T_1\varphi\cdot\psi - \varphi\cdot T_1\psi : \varphi,\psi \in \ell^\infty\}.$$

Proof. Let $\varphi_1, \psi_1, \varphi_2, \psi_2 \in \ell^\infty$ and $n \in \mathbb{N}_0$. Then due to (4.2),

$$\begin{aligned} &(T_1\varphi_1\cdot\psi_1 - \varphi_1\cdot T_1\psi_1 + T_1\varphi_2\cdot\psi_2 - \varphi_2\cdot T_1\psi_2)(n) \\ &\qquad = a_n\omega_1(n) - c_n\omega_1(n-1) + a_n\omega_2(n) - c_n\omega_2(n-1) \\ &\qquad = a_n\omega(n) - c_n\omega(n-1) \end{aligned}$$

with $\omega_1, \omega_2, \omega \in \ell^\infty$. Finally, in view of (4.2)–(4.4) there exist $\varphi, \psi \in \ell^\infty$ with

$$T_1\varphi_1\cdot\psi_1 - \varphi_1\cdot T_1\psi_1 + T_1\varphi_2\cdot\psi_2 - \varphi_2\cdot T_1\psi_2 = T_1\varphi\cdot\psi - \varphi\cdot T_1\psi. \ ∎$$

Hence, in the case of polynomial hypergroups we are able to characterize the existence of strongly invariant means as follows.

PROPOSITION 4.3. *There exists $m \in S\mathbb{M}(\mathbb{N}_0)$ exactly when*

$$\sup_{n\in\mathbb{N}_0} (T_1\varphi(n)\psi(n) - \varphi(n)T_1\psi(n)) \geq 0 \qquad \text{for all } \varphi, \psi \in \ell^\infty_\mathbb{R}.$$

Proof. By Lemmas 4.1 and 4.2,

$$\sup_{n\in\mathbb{N}_0} (T_1\varphi(n)\psi(n) - \varphi(n)T_1\psi(n)) \geq 0 \quad \text{for all } \varphi, \psi \in \ell^\infty_\mathbb{R}$$

is equivalent to

$$\sup_{n\in\mathbb{N}_0} \eta(n) \geq 0 \quad \text{for all } \eta \in SL_\mathbb{R},$$

and Theorem 2.3 yields the statement. ∎

Consequently, the nonexistence of a strongly invariant mean in the case of polynomial hypergroups is characterized as follows.

PROPOSITION 4.4. *There does not exist $m \in S\mathbb{M}(\mathbb{N}_0)$ if and only if there exist $\delta > 0$ and $\omega \in \ell^\infty_\mathbb{R}$ such that*

$$a_n\omega(n) - c_n\omega(n-1) \leq -\delta \qquad \text{for all } n \in \mathbb{N}_0.$$

Proof. By the equivalence relation of Proposition 4.3 one only has to keep in mind (4.2)–(4.4). ∎

We give another sufficient and necessary condition for the existence of $m \in S\mathbb{M}(\mathbb{N}_0)$ starting with the sufficiency.

PROPOSITION 4.5. *Denote $H(n) = \sum_{k=0}^n h(k)$. If the set $\left\{\frac{H(n)}{h(n)a_n} : n \in \mathbb{N}_0\right\}$ is unbounded, then there exists $m \in S\mathbb{M}(\mathbb{N}_0)$.*

Proof. Suppose that $m \in S\mathbb{M}(\mathbb{N}_0)$ does not exist. By Proposition 4.4 there are $\omega \in \ell^\infty_\mathbb{R}$ and $\delta > 0$ such that

$$a_n\omega(n) + \delta \leq c_n\omega(n-1) \quad \text{for all } n \in \mathbb{N}_0.$$

Since $c_n = a_{n-1}h(n-1)/h(n)$ it follows that

$$h(n)a_n\omega(n) + h(n)\delta \leq a_{n-1}h(n-1)\omega(n-1) \quad \text{for all } n \in \mathbb{N}.$$

Adding $h(n-1)\delta$ to both sides of the inequality, we obtain, for all $n \geq 2$,

$$\begin{aligned} h(n)a_n\omega(n) + h(n)\delta + h(n-1)\delta &\leq a_{n-1}h(n-1)\omega(n-1) + h(n-1)\delta \\ &\leq a_{n-2}h(n-2)\omega(n-2). \end{aligned}$$

Iterating this step it follows that

$$h(n)a_n\omega(n) + \delta\sum_{k=1}^n h(k) \leq a_1h(1)\omega(1) + \delta h(1) \leq \omega(0) \leq -\delta$$

for all $n \in \mathbb{N}$. So for all $n \in \mathbb{N}_0$ we have

$$\frac{h(n)a_n}{H(n)}\omega(n) + \delta \leq 0.$$

But there exists a subsequence such that $h(n_k)a_{n_k}/H(n_k) \to 0$ as $k \to \infty$, contradicting $\omega \in \ell^\infty_{\mathbb{R}}$, $\delta > 0$. Hence, $m \in S\mathbb{M}(\mathbb{N}_0)$ exists. ∎

Proposition 4.6. *If $m \in S\mathbb{M}(\mathbb{N}_0)$ exists, then $\left\{\frac{H(n)}{h(n)a_n} : n \in \mathbb{N}_0\right\}$ is unbounded.*

Proof. Suppose that $\left\{\frac{H(n)}{h(n)a_n} : n \in \mathbb{N}_0\right\}$ is bounded. Putting $\omega(n) = \frac{H(n)}{h(n)a_n}$ for all $n \in \mathbb{N}_0$ we get a sequence $\omega \in \ell^\infty_{\mathbb{R}}$ with

$$\begin{aligned} a_n\omega(n) - c_n\omega(n-1) &= \frac{H(n)}{h(n)} - \frac{H(n-1)c_n}{h(n-1)a_{n-1}} \\ &= \frac{1}{h(n)}(H(n) - H(n-1)) = 1. \end{aligned}$$

Now, for this ω choose $\varphi, \psi \in \ell^\infty_{\mathbb{R}}$ according to (4.3) and (4.4). Then we obtain $1 = m(1) = m(T_1\varphi \cdot \psi - \varphi \cdot T_1\psi) = 0$, which is a contradiction. ∎

Combining Propositions 4.5 and 4.6 we have a characterization of the existence of strongly invariant means on polynomial hypergroups.

Theorem 4.7. *There exists a strongly invariant mean on a polynomial hypergroup $\mathbb{N}_0$ if and only if the set*

$$\left\{\frac{H(n)}{h(n)a_n} : n \in \mathbb{N}_0\right\}$$

is unbounded.

We present some examples of polynomial hypergroups for which the set $\left\{\frac{H(n)}{h(n)a_n} : n \in \mathbb{N}_0\right\}$ is bounded or unbounded, respectively.

(i) *Orthogonal polynomials defined by homogeneous trees.* The polynomials $R_n(x;\alpha)$ with $\alpha \geq 2$ are determined by the recurrence coefficients

$$a_n = \frac{\alpha - 1}{\alpha}, \quad b_n = 0, \quad c_n = \frac{1}{\alpha} \qquad \text{for all } n \in \mathbb{N}$$

and $\alpha_0 = 1$, $\beta_0 = 0$ (see Section 1). They generate a polynomial hypergroup on $\mathbb{N}_0$ for each $\alpha \geq 2$ (see [4]). The Haar weights are

$$h(0) = 1, \quad h(n) = \alpha(\alpha - 1)^{n-1}, \qquad n \in \mathbb{N}.$$

For $\alpha > 2$ we get

$$\begin{aligned} H(n) &= \frac{\alpha(\alpha-1)^n - 2}{\alpha - 2}, \\ \frac{H(n)}{h(n)a_n} &= \frac{\alpha}{\alpha - 2}\left(1 - \frac{2}{\alpha(\alpha-1)^n}\right). \end{aligned}$$

Hence, $\left\{\frac{H(n)}{h(n)a_n} : n \in \mathbb{N}_0\right\}$ is bounded and so there does not exist a strongly invariant mean on these hypergroups.

(ii) *Little q-Legendre polynomials.* We now consider a class of orthogonal polynomials $P_n(x;q)$, $0<q<1$, having Haar weights which grow exponentially, but with an unbounded sequence $\left(\frac{H(n)}{h(n)a_n}\right)_{n\in\mathbb{N}_0}$. Given $0<q<1$ set

$$\alpha_0 = \frac{1}{q+1}, \qquad \beta_0 = \frac{q}{q+1}$$

and for all $n \in \mathbb{N}$,

$$\begin{aligned} a_n &= q^n \frac{(1+q)(1-q^{n+1})}{(1-q^{2n+1})(1+q^{n+1})}, \\ b_n &= \frac{(1-q^n)(1-q^{n+1})}{(1+q^n)(1+q^{n+1})}, \\ c_n &= q^n \frac{(1+q)(1-q^n)}{(1-q^{2n+1})(1+q^n)}. \end{aligned}$$

These polynomials define a polynomial hypergroup on $\mathbb{N}_0$ (see [1]). The Haar weights are

$$h(n) = \frac{1}{1-q}(q^{-n} - q^{n+1}),$$

so they are exponentially growing. But

$$a_n h(n) = \frac{1+q}{1-q}\,\frac{1-q^{n+1}}{1+q^{n+1}},$$

and hence $\left\{\frac{H(n)}{h(n)a_n} : n \in \mathbb{N}_0\right\}$ is unbounded, and strongly invariant means exist.

(iii) *Jacobi polynomials, generalized Chebyshev polynomials, q-ultraspherical polynomials, Pollaczek polynomials and others.* These classes of orthogonal polynomials generate polynomial hypergroups and their Haar weights have polynomial growth and the recurrence coefficients a_n converge towards $1/2$ (see [1] or [4, 5]). Hence, in all these cases $\left\{\frac{H(n)}{h(n)a_n} : n \in \mathbb{N}_0\right\}$ is unbounded. Therefore, strongly invariant means exist for each of these hypergroups.

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