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WEAKLY PRECOMPACT SUBSETS OF $L_1(\mu, X)$

BY

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Abstract. Let (Ω, Σ, μ) be a probability space, X a Banach space, and $L_1(\mu, X)$ the Banach space of Bochner integrable functions $f : \Omega \to X$. Let $W = \{f \in L_1(\mu, X) :$ for a.e. $\omega \in \Omega, \|f(\omega)\| \leq 1\}$. In this paper we characterize the weakly precompact subsets of $L_1(\mu, X)$. We prove that a bounded subset A of $L_1(\mu, X)$ is weakly precompact if and only if A is uniformly integrable and for any sequence (f_n) in A, there exists a sequence (g_n) with $g_n \in \mathrm{co}\{f_i : i \geq n\}$ for each n such that for a.e. $\omega \in \Omega$, the sequence $(g_n(\omega))$ is weakly Cauchy in X. We also prove that if A is a bounded subset of $L_1(\mu, X)$, then A is weakly precompact if and only if for every $\epsilon > 0$, there exist a positive integer N and a weakly precompact subset H of NW such that $A \subseteq H + \epsilon B(0)$, where $B(0)$ is the unit ball of $L_1(\mu, X)$.

1. Introduction. Throughout this paper, X and Y will denote Banach spaces. The unit ball of X will be denoted by B_X, and the closed linear span of a sequence (x_n) in X will be denoted by $[x_n]$. The unit basis of ℓ_1 will be denoted by (e_n^*), and a continuous linear transformation $T : X \to Y$ will be referred to as an *operator*.

A subset S of X is said to be *weakly precompact* provided that every bounded sequence from S has a weakly Cauchy subsequence. A series $\sum x_n$ in X is said to be *weakly unconditionally convergent* (wuc) if for every $x^* \in X^*$, the series $\sum |x^*(x_n)|$ is convergent. An operator $T : X \to Y$ is *weakly precompact* if $T(B_X)$ is weakly precompact, and *unconditionally converging* if it maps weakly unconditionally convergent series to unconditionally convergent ones. An operator T is *completely continuous* (or *Dunford–Pettis*) if T maps weakly Cauchy sequences to norm convergent sequences.

A bounded subset A of X (resp. A of X^*) is called a V^*-*subset* of X (resp. a V-*subset* of X^*) provided that

$$\lim_n(\sup\{|x_n^*(x)| : x \in A\}) = 0 \qquad (\text{resp.} \lim_n(\sup\{|x^*(x_n)| : x^* \in A\}) = 0)$$

for each wuc series $\sum x_n^*$ in X^* (resp. wuc series $\sum x_n$ in X).

In his fundamental paper [Pe], Pełczyński introduced property (V) and property (V^*). The Banach space X has *property* (V) (resp. (V^*)) if every

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V-subset of X^* (resp. V^*-subset of X) is relatively weakly compact. The following results were also established in [Pe]:

(a) $C(K)$ spaces have property (V).
(b) L^1-spaces have property (V^*).
(c) A Banach space X has property (V) if and only if every unconditionally converging operator T from X to any Banach space Y is weakly compact.
(d) Every closed subspace of a Banach space with property (V^*) has property (V^*).
(e) In the last portion of the proof of Proposition 6 on p. 646, Pełczyński noted that every weakly Cauchy sequence is a V^*-set. Consequently, every bounded weakly precompact set in X is a V^*-set.
(f) If X has property (V^*), then X is weakly sequentially complete.

A Banach space X has *property weak* (V^*) (wV^*) if every V^*-subset of X is weakly precompact [Bom]. If X does not contain a copy of ℓ_1, then X has property (wV^*), by Rosenthal's theorem ([Di1, Ch. XI]). In particular, c_0 has property (wV^*), but it does not have property (V^*). A Banach space X has property (wV^*) if and only if every sequence in X equivalent to (e_n^*) contains a subsequence (x_{n_i}) so that $[x_{n_i}]$ is complemented in X [Bom]. A Banach space X has property (wV^*) if and only if every operator $T : Y \to X$ with unconditionally converging adjoint is weakly precompact [GL]. Every order continuous Banach lattice has property (wV^*) ([Bom], [Tz]). A Banach lattice has property (V^*) if and only if it is weakly sequentially complete if and only if it does not contain a copy of c_0 ([Bom], [LT], [Em]).

A bounded subset A of X is called a *Dunford–Pettis* (DP) *subset* of X if each weakly null sequence in X^* tends to 0 uniformly on A. Every DP set is weakly precompact; see e.g., see [Ro, p. 377], [An], [GL]. A Banach space X the *Dunford–Pettis property* (DPP) if every weakly compact operator T with domain X is completely continuous. Equivalently, X has the DPP if and only if $x_n^*(x_n) \to 0$ for all weakly null sequences (x_n) in X and (x_n^*) in X^* ([Di2]). Schur spaces, $C(K)$ spaces, and $L_1(\mu)$ spaces have the DPP ([BDS], [DP], [Gr]). The reader can check [Di1], [Di2], [DU], and [An] for a guide to the extensive classical literature dealing with the DPP, equivalent formulations of the preceding definitions, and undefined notation and terminology.

Let $L_1(\mu, X)$ be the Banach space of all X-valued Bochner integrable functions on a probability space (Ω, Σ, μ). In this paper we characterize weakly precompact subsets of $L_1(\mu, X)$. The problem was also studied by Bourgain [Bou] (when X does not contain a copy of ℓ_1) and Talagrand [Ta]. N. Randrianantoanina [Ra] proved that $L_1(\mu, X)$ has property (V^*) if and only if X has property (V^*). The proof of Theorem 2 in [Ra] shows that $L_1(\mu, X)$ has property (wV^*) if and only if X has property (wV^*).

2. Weak precompactness in $L_1(\mu, X)$. Let (Ω, Σ, μ) be a probability space, X be a Banach space, and let $L_1(\mu, X)$ be the Banach space of (equivalence classes of) μ-strongly measurable X-valued Bochner integrable functions $f : \Omega \to X$, equipped with the norm

$$\|f\|_1 = \int_\Omega \|f(\omega)\|\, d\mu.$$

For a subset A of X, let $\mathrm{co}(A)$ denote the convex hull of A. Let $B(0)$ denote the unit ball of $L_1(\mu, X)$. A subset A of $L_1(\mu, X)$ is called *uniformly integrable* if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $\mu(E) < \delta$, then $\int_E \|f(\omega)\|\, d\mu < \epsilon$ for all $f \in A$.

Following [Ul2], let $W = \{f \in L_1(\mu, X) : \text{for a.e. } \omega \in \Omega,\ \|f(\omega)\| \le 1\}$. For a positive integer N, let $W(N) = \{f \in L_1(\mu, X) : \text{for a.e. } \omega \in \Omega,$ $\|f(\omega)\| \le N\}$. Note that $W(N) = NW$ and $W(1) = W$. For a subset H of W and $\omega \in \Omega$, let $H(\omega) = \{f(\omega) : f \in H\}$. Strictly speaking, as noted in [Ul2], $H(\omega)$ is not well defined since the elements of H are not single functions but classes of functions. To make the definition of $H(\omega)$ precise, one can introduce a lifting ρ of $L_\infty(\mu)$, and define $\rho(f)$ as in [Din, p. 212], or [IT, p. 76] and set $H(\omega) = \{\rho(f)(\omega) : f \in H\}$. However, not to complicate the notations, we do not introduce a lifting but deal with the elements of W as if they were strongly measurable bounded single functions. For a subset A of $L_1(\mu, X)$ and $\omega \in \Omega$, let $A(\omega) = \{f(\omega) : f \in A\}$.

The following two lemmas will be useful in our study.

Lemma 2.1 ([Ul1, Lemma 2.2]). *Let K be a bounded subset of X. Then K is weakly precompact if and only if for each sequence (x_n) in K, there is a sequence (y_n) so that $y_n \in \mathrm{co}\{x_i : i \ge n\}$ for each n and (y_n) is weakly Cauchy.*

Lemma 2.2 ([DRS, Theorem 2.4]). *Assume that (f_n) is a bounded sequence in $L_1(\mu, X)$. Then there exist a sequence (g_n) with $g_n \in \mathrm{co}\{f_i : i \ge n\}$ for each n, and three measurable subsets C_1, C_2, and L of Ω with $\mu(C_1 \cup C_2 \cup L) = 1$, such that*

(a) *for $\omega \in C_1$, the sequence $(g_n(\omega))$ is norm convergent in X;*
(b) *for $\omega \in C_2$, the sequence $(g_n(\omega))$ is weakly Cauchy but not weakly convergent in X;*
(c) *for $\omega \in L$, there exists a positive integer k with $(g_n(\omega))_{n \ge k} \sim (e_n^*)$.*

The main result of this paper is the following theorem.

Theorem 2.3. *Let A be a bounded subset of $L_1(\mu, X)$. Then A is weakly precompact if and only if A is uniformly integrable and for any sequence (f_n) in A, there exists a sequence (g_n) with $g_n \in \mathrm{co}\{f_i : i \ge n\}$ for each n such that for a.e. $\omega \in \Omega$, the sequence $(g_n(\omega))$ is weakly Cauchy in X.*

Proof. Suppose that A is weakly precompact. Then A is uniformly integrable ([DU, Theorem IV.2.4, p. 104]). Let (f_n) be a sequence in A. By Lemma 2.2, there exist a sequence (g_n) with $g_n \in \mathrm{co}\{f_i : i \geq n\}$ for each n, and three sets C_1, C_2, and L in Σ with $\mu(C_1 \cup C_2 \cup L) = 1$ and satisfying conditions (a)–(c) of that lemma.

If $\mu(L) > 0$, then by [Ta, Lemma 4], there exists a positive integer k such that $(g_n)_{n \geq k} \sim (e_n^*)$. Since $(g_n)_{n \geq k}$ lies in the set $\mathrm{co}(A)$, which is weakly precompact ([Ro, p. 377], [Sch, p. 27]), one obtains a contradiction. Hence $\mu(L) = 0$, and for a.e. $\omega \in \Omega$, the sequence $(g_n(\omega))$ is weakly Cauchy in X.

Conversely, let (f_n) be a sequence in A. Let (g_n) be a sequence with $g_n \in \mathrm{co}\{f_i : i \geq n\}$ such that for a.e. $\omega \in \Omega$, the sequence $(g_n(\omega))$ is weakly Cauchy in X. By [Ta, Lemma 8], (g_n) is weakly Cauchy in $L_1(\mu, X)$. By Lemma 2.1, A is weakly precompact. ∎

Talagrand showed that if A is a uniformly integrable subset of $L_1(\mu, X)$ and for each $\omega \in \Omega$, the set $A(\omega)$ is weakly precompact, then A is weakly precompact ([Ta, p. 704]). Theorem 2.3 enables an efficient proof of a stronger implication.

COROLLARY 2.4. *Let A be a bounded uniformly integrable subset of $L_1(\mu, X)$.*

- (i) *If the set $A(\omega)$ is weakly precompact for a.e. $\omega \in \Omega$, then A is weakly precompact.*
- (ii) *Suppose that X has property (wV^*). If $A(\omega)$ is a V^*-set for a.e. $\omega \in \Omega$, then A is a V^*-set.*

Proof. (i) Let (f_n) be a sequence in A. By Lemma 2.2, there exist a sequence (g_n) with $g_n \in \mathrm{co}\{f_i : i \geq n\}$ for each n, and three sets C_1, C_2, and L in Σ with $\mu(C_1 \cup C_2 \cup L) = 1$, such that conditions (a)–(c) of Lemma 2.2 are satisfied. Since for a.e. $\omega \in \Omega$, the set $\mathrm{co}(A(\omega))$ is weakly precompact ([Ro, p. 377], [Sch, p. 27]), and for $\omega \in L$ the sequence $(g_n(\omega))_{n \geq k}$ lies in this set, we have $\mu(L) = 0$. Then for a.e. $\omega \in \Omega$, the sequence $(g_n(\omega))$ is weakly Cauchy in X. Apply Theorem 2.3.

(ii) Suppose that X has property (wV^*). For a.e. $\omega \in \Omega$, the set $A(\omega)$ is a V^*-set, and thus weakly precompact (since X has property (wV^*)). Since A is bounded and weakly precompact (by (i)), A is a V^*-set ([Pe]). ∎

COROLLARY 2.5. *Let $g : \Omega \to \mathbb{R}$ be a positive integrable function and (f_n) be a sequence in $L_1(\mu, X)$ such that*

- (i) *for a.e. $\omega \in \Omega$ and all $n \in \mathbb{N}$, $\|f_n(\omega)\| \leq g(\omega)$;*
- (ii) *for a.e. $\omega \in \Omega$, the sequence $(f_n(\omega))$ is weakly precompact.*

Then the sequence (f_n) is weakly precompact.

Proof. Let $A = \{f_n : n \in \mathbb{N}\}$. Then A is bounded and uniformly integrable and for a.e. $\omega \in \Omega$, the set $A(\omega)$ is weakly precompact. Apply Corollary 2.4(i). ∎

The next result is motivated by [Bou, Corollary 9].

COROLLARY 2.6. *Suppose that X contains no copy of ℓ_1, and let A be a bounded subset of $L_1(\mu, X)$. Then the following are equivalent:*

(i) *A is uniformly integrable.*
(ii) *A is weakly precompact.*
(iii) *A is a V^*-set.*

Proof. (i)⇒(ii). Suppose that A is uniformly integrable. Let (f_n) be a sequence in A. By Lemma 2.2, there exist a sequence (g_n) with $g_n \in \mathrm{co}\{f_i : i \geq n\}$ for each n, and three sets C_1, C_2, and L in Σ with $\mu(C_1 \cup C_2 \cup L) = 1$, such that conditions (a)–(c) of Lemma 2.2 are satisfied. However, since X contains no copy of ℓ_1, condition (c) is not possible. Therefore $\mu(L) = 0$, and for a.e. $\omega \in \Omega$, the sequence $(g_n(\omega))$ is weakly Cauchy. By Theorem 2.3, A is weakly precompact.

(ii)⇒(iii). If A is weakly precompact, then A is a V^*-set [Pe].

(iii)⇒(i). If A is a V^*-set, then A is uniformly integrable, by [Bom, Proposition 3.1]. ∎

LEMMA 2.7 ([Bom, Theorem 1.1 and Proposition 1.1]). *Let A be a bounded subset of a Banach space X. Then A is a V^*-set if and only if $T(A)$ is relatively compact for each operator $T : X \to \ell_1$.*

THEOREM 2.8. *If A is a V^*-set in $L_1(\mu, X)$, then the set*

$$\{\|f(\cdot)\|_X : f \in A\}$$

is weakly precompact in $L_1(\mu)$.

Proof. Suppose that $\{\|f(\cdot)\|_X : f \in A\}$ is not weakly precompact in $L_1(\mu)$. By [AK, Theorem 5.2.9], there is a sequence (A_n) of pairwise disjoint sets in Ω, a sequence (f_n) in A, and an $\epsilon > 0$ such that

$$\int_{A_n} \|f_n(\omega)\| \, d\mu > \epsilon$$

for all $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, choose $g_n \in L_\infty(\mu, X^*)$ such that $\|g_n\|_\infty \leq 1$, g_n vanishes off A_n, and

$$\int_{A_n} \langle f_n(\omega), g_n(\omega) \rangle \, d\mu > \epsilon.$$

Define $T : L_1(\mu, X) \to \ell_1$ by

$$T(f) = \sum_i \Big(\int_{A_i} \langle f(\omega), g_i(\omega) \rangle \, d\mu \Big) e_i^*$$

for $f \in L_1(\mu, X)$. Note that T is a well-defined operator, $\|T\| \leq 1$, and

$$\langle T(f_n), e_n\rangle = \int_{A_n} \langle f_n(\omega), g_n(\omega)\rangle \, d\mu > \epsilon$$

for all n. Then $\{T(f_n) : n \geq 1\}$, and thus $T(A)$ is not relatively compact in ℓ_1. By Lemma 2.7, A is not a V^*-set, a contradiction. ■

COROLLARY 2.9. *Let A be a bounded set in $L_1(\mu, X)$. Then the following are equivalent:*

(i) *A is weakly precompact.*
(ii) *The set $\{\|f(\cdot)\|_X : f \in A\}$ is relatively weakly compact in $L_1(\mu)$, A is uniformly integrable, and for any sequence (f_n) in A, there is a sequence (g_n) with $g_n \in \mathrm{co}\{f_i : i \geq n\}$ for each n such that for a.e. $\omega \in \Omega$, the sequence $(g_n(\omega))$ is weakly Cauchy in X.*

Proof. (i)⇒(ii). If A is weakly precompact, then A is uniformly integrable ([DU, Theorem IV.2.4]) and a V^*-set [Pe]. By the previous theorem, the set $\{\|f(\cdot)\|_X : f \in A\}$ is weakly precompact, and thus relatively weakly compact (and uniformly integrable) in $L_1(\mu)$ ([AK, Theorem 5.2.9]). The third assertion of (ii) follows from Theorem 2.3.

(ii)⇒(i). Apply Theorem 2.3. ■

In order to prove a result similar to Lemma 2.1 for V^*-sets, we need the following two lemmas.

LEMMA 2.10 ([BL, Lemma 3.3]). *Let (x_n^*, x_n) be a sequence in $X^* \times X$ such that (x_n^*) is bounded and (x_n) is weakly null. If (ϵ_j) is a sequence of positive numbers, then there exists a subsequence $(x_{n_j}^*, x_{n_j})$ of (x_n^*, x_n) such that $|x_{n_i}^*(x_{n_j})| < \epsilon_j$, if $i \neq j$.*

If (x_n) is a sequence and $(y_j) \subseteq \mathrm{co}\{(x_{n_j})\}$ for each j, then we say that (y_j) has *pairwise disjoint support* if $N_j \cap N_k = \emptyset$ whenever $j \neq k$ and $y_j = \sum_{i\in N_j} \alpha_i x_i$, with $\sum_{i \in N_j} \alpha_i = 1$, $\alpha_i \geq 0$.

LEMMA 2.11. *Let (x_n) be a bounded sequence in X such that $\{x_n : n \geq 1\}$ is not a V^*-set. Then there is a subsequence (x_{n_j}) of (x_n) such that if $(y_k) \subseteq \mathrm{co}\{(x_{n_j})\}$ is a sequence having pairwise disjoint support, then $\{y_k : k \geq 1\}$ is not a V^*-set.*

Proof. Let $\epsilon > 0$ and $\sum x_n^*$ be wuc in X^* such that $\langle x_n^*, x_n\rangle > \epsilon$. By Lemma 2.10, there is a subsequence $(x_{n_j}^*, x_{n_j})$ of (x_n^*, x_n) such that $|\langle x_{n_j}^*, x_{n_i}\rangle| < \epsilon/2^{i+3}$ for all $i \neq j$. Let $(y_k) \subseteq \mathrm{co}\{(x_{n_j})\}$ be a sequence having pairwise disjoint support. Suppose that $y_k = \sum_{i\in N_k} \alpha_i \, x_{n_i}$ with $\alpha_i \geq 0$, $i \in N_k$, and $\sum_{i\in N_k} \alpha_i = 1$. Let $y_k^* = \sum_{i \in N_k} x_{n_i}^*$ for each k. Then $\sum y_k^*$ is wuc in X^* and

$$\begin{aligned}\langle y_k^*, y_k\rangle &= \Big\langle \sum_{i\in N_k} x_{n_i}^*, \sum_{i\in N_k} \alpha_i\, x_{n_i}\Big\rangle \\ &\geq \sum_{i\in N_k} \alpha_i \langle x_{n_i}^*, x_{n_i}\rangle - \sum_{i\in N_k} \alpha_i \Big(\sum_{j\in N_k,\, j\neq i} |\langle x_{n_i}^*, x_{n_j}\rangle|\Big) > \epsilon - \epsilon/2 = \epsilon/2. \ \blacksquare\end{aligned}$$

We now have a version of Lemma 2.1 for V^*-sets.

LEMMA 2.12. *Let A be a bounded subset of X. Then A is a V^*-set if and only if for any sequence (x_n) in A, there is a sequence (z_n) so that $z_n \in \mathrm{co}\{x_i : i \geq n\}$ for each n and $\{z_n : n \geq 1\}$ is a V^*-set.*

Proof. Suppose A is a V^*-set and let (x_n) be a sequence in A. Set $z_n = x_n$. Then (z_n) satisfies the required conditions.

Conversely, suppose that A is not a V^*-set. Let (x_n) be a sequence in A such that $\{x_n : n \geq 1\}$ is not a V^*-set. Use Lemma 2.11 to choose a subsequence (x_{n_j}) of (x_n) such that if $(y_k) \subseteq \mathrm{co}\{(x_{n_j})\}$ is a sequence having pairwise disjoint support, then $\{y_k : k \geq 1\}$ is not a V^*-set. Let $z_j \in \mathrm{co}\{x_{n_i} : i \geq j\}$ for each $j \in \mathbb{N}$. Let (z_{j_k}) be a subsequence having pairwise disjoint support. Then $\{z_{j_k} : k \geq 1\}$ is not a V^*-set, and thus $\{z_i : i \geq 1\}$ is not a V^*-set. $\blacksquare$

COROLLARY 2.13. *Suppose that X has property (wV^*). Then a subset A of $L_1(\mu, X)$ is a V^*-set if and only if A is bounded, uniformly integrable, and for any sequence (f_n) in A, there exists a sequence (g_n) with $g_n \in \mathrm{co}\{f_i : i \geq n\}$ for each n such that for a.e. $\omega \in \Omega$, $(g_n(\omega))$ is a V^*-set.*

Proof. Suppose that A is a V^*-set. Then A is bounded and uniformly integrable ([Bom, Proposition 3.1]). Since X has property (wV^*), $L_1(\mu, X)$ has property (wV^*) ([Ra]). Then A is weakly precompact. Let (f_n) be a sequence in A. By Theorem 2.3, there exists a sequence (g_n) with $g_n \in \mathrm{co}\{f_i : i \geq n\}$ for each n such that for a.e. $\omega \in \Omega$, $(g_n(\omega))$ is weakly Cauchy. Then for a.e. $\omega \in \Omega$, $(g_n(\omega))$ is a V^*-set ([Pe]).

Conversely, let (f_n) be a sequence in A. Choose a sequence (g_n) with $g_n \in \mathrm{co}\{f_i : i \geq n\}$ for each n such that for a.e. $\omega \in \Omega$, $(g_n(\omega))$ is a V^*-set. By Corollary 2.4(ii), $\{g_n : n \geq 1\}$ is a V^*-set. By Lemma 2.12, A is a V^*-set. $\blacksquare$

COROLLARY 2.14. *Suppose that X^* has the Schur property. Then a subset A of $L_1(\mu, X)$ is a DP set if and only if A is bounded, uniformly integrable, and for any sequence (f_n) in A, there exists a sequence (g_n) with $g_n \in \mathrm{co}\{f_i : i \geq n\}$ for each n such that for a.e. $\omega \in \Omega$, $(g_n(\omega))$ is a DP set.*

Proof. We note that X^* has the Schur property if and only if X has the DPP and X contains no copy of ℓ_1 ([Di2]).

Suppose that A is a DP set. Then A is bounded, weakly precompact, and uniformly integrable [An]. Let (f_n) be a sequence in A. By Theorem 2.3, there exists a sequence (g_n) with $g_n \in \mathrm{co}\{f_i : i \geq n\}$ for each n such that for a.e. $\omega \in \Omega$, $(g_n(\omega))$ is weakly Cauchy in X, and hence a DP set ([Di2]).

Conversely, if A is bounded and uniformly integrable, then A is a DP set, by [An, Corollary 4]. ∎

We will need the following lemmas. The first of them is similar to a result of Grothendieck about relatively weakly compact sets ([Di1, p. 227]).

LEMMA 2.15. *Let A be a bounded subset of X. If for any $\epsilon > 0$ there exists a weakly precompact subset A_ϵ of X such that $A \subseteq A_\epsilon + \epsilon B_X$, then A is weakly precompact.*

Proof. Let (x_n) be a sequence in A. Choose a weakly precompact subset A_1 of X, a sequence (y_n^1) in A_1, and a sequence (z_n^1) in B_X so that $x_n = y_n^1 + z_n^1$ for $n \geq 1$. We observe that (y_n^1) has a weakly Cauchy subsequence. Let $\phi_1 : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function so that $(y^1_{\phi_1(n)})$ is weakly Cauchy.

Now consider the sequence $(x_{\phi_1(n)})$. Choose a weakly precompact subset A_2 of X, a sequence (y_n^2) in A_2, and a sequence (z_n^2) in $(1/2)B_X$ so that $x_{\phi_1(n)} = y_n^2 + z_n^2$ for $n \geq 1$. Then (y_n^2) has a weakly Cauchy subsequence. Let $\phi_2 : \phi_1(\mathbb{N}) \to \phi_1(\mathbb{N})$ be a strictly increasing function so that $(y^2_{\phi_2(n)})$ is weakly Cauchy.

Consider the sequence $(x_{\phi_2\phi_1(n)})$. Choose a weakly precompact subset A_3 of X, a sequence (y_n^3) in A_3, and a sequence (z_n^3) in $(1/3)B_X$ so that $x_{\phi_2\phi_1(n)} = y_n^3 + z_n^3$, $n \geq 1$. Let $\phi_3 : \phi_2\phi_1(\mathbb{N}) \to \phi_2\phi_1(\mathbb{N})$ be a strictly increasing function so that $(y^3_{\phi_3(n)})$ is weakly Cauchy and consider the sequence $(x_{\phi_3\phi_2\phi_1(n)})$. Choose a weakly precompact subset A_4 of X and use the hypotheses to continue this process.

Now consider the subsequence $w_1 = x_{\phi_1(1)}$, $w_2 = x_{\phi_2\phi_1(2)}$, $w_3 = x_{\phi_3\phi_2\phi_1(3)}$, $\ldots$ of (x_n). Let $\epsilon > 0$. Choose $i \in \mathbb{N}$ so that $2/i < \epsilon/2$, and let $x^* \in X^*$, $\|x^*\| \leq 1$. Choose $N \in \mathbb{N}$ so that if $p, q > N$, then

$$|x^*(y^i_{\phi_i(p)}) - x^*(y^i_{\phi_i(q)})| < \epsilon/2.$$

If $s, t > N + i$, then $w_s = y^i_{\phi_i(p)} + z^i_{\phi_i(p)}$ and $w_t = y^i_{\phi_i(q)} + z^i_{\phi_i(q)}$ for some $p, q > N$. Consequently,

$$\begin{aligned}|x^*(w_s) - x^*(w_t)| &\leq |x^*(y^i_{\phi_i(p)}) - x^*(y^i_{\phi_i(q)})| + |x^*(z^i_{\phi_i(p)}) - x^*(z^i_{\phi_i(q)})| \\ &< \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

Hence (w_n) is weakly Cauchy, and A is weakly precompact. ∎

LEMMA 2.16 ([Bom, Corollary 1.7]). *Let A be a bounded subset of X. If for any $\epsilon > 0$ there exists a V^*-subset A_ϵ of X such that $A \subseteq A_\epsilon + \epsilon B_X$, then A is a V^*-set.*

LEMMA 2.17. *Let A be a bounded subset of X. If for any $\epsilon > 0$ there exists a DP subset A_ϵ of X such that $A \subseteq A_\epsilon + \epsilon B_X$, then A is a DP set.*

Proof. We recall the following characterization of DP sets obtained in [An]: a subset A of X is a DP set if and only if every weakly compact operator $T : X \to c_0$ maps A into a relatively compact set. Let $T : X \to c_0$ be a weakly compact operator with $\|T\| \leq 1$. For each $\epsilon > 0$, choose a DP subset A_ϵ of X such that $A \subseteq A_\epsilon + \epsilon B_X$. Then

$$T(A) \subseteq T(A_\epsilon) + \epsilon T(B_X) \subseteq T(A_\epsilon) + \epsilon B_{c_0},$$

and $T(A_\epsilon)$ is relatively compact ([An]). Therefore $T(A)$ is relatively compact ([Di1, p. 5]), and thus A is a DP set ([An]). ∎

Recall that $W = \{f \in L_1(\mu, X) : \text{for a.e. } \omega \in \Omega, \|f(\omega)\| \leq 1\}$. The following theorem is motivated by [Ul2, Theorem 8].

THEOREM 2.18. *Let A be a bounded subset of $L_1(\mu, X)$.*

(i) *If A is uniformly integrable, then for every $\epsilon > 0$, there exist a positive integer N and a subset H of NW such that $A \subseteq H + \epsilon B(0)$.*
(ii) *A is weakly precompact if and only if for every $\epsilon > 0$, there exist a positive integer N and a weakly precompact subset H of NW such that $A \subseteq H + \epsilon B(0)$.*

Proof. (i) Let $\epsilon > 0$. Since A is uniformly integrable, there is a $\delta > 0$ such that if $B \in \Sigma$, $\mu(B) < \delta$, then

$$\sup_{f \in A} \int_B \|f(\omega)\| \, d\mu < \epsilon.$$

Using the boundedness of A, we can find a positive integer N such that for each $f \in A$, $\mu(\{\omega \in \Omega : \|f(\omega)\| > N\}) < \delta$.

For $f \in A$, let $f_N = f \cdot \chi_{E_f}$, where $E_f = \{\omega \in \Omega : \|f(\omega)\| \leq N\}$. Note that $\|f - f_N\| < \epsilon$ for all $f \in A$. Let $H = \{f_N : f \in A\}$. Then $H \subseteq W(N) = NW$ and $A \subseteq H + \epsilon B(0)$. For all $\omega \in \Omega$, $H(\omega) \subseteq A(\omega) \cup \{0\}$.

(ii) Suppose A is weakly precompact. Then A is uniformly integrable ([DU, Theorem IV.2.4]). Let $\epsilon > 0$. By (i), there exist a positive integer N and a subset H of NW such that $A \subseteq H + \epsilon B(0)$. By [Bou, Proposition 10], the set $\{f \cdot \chi_E : f \in A, E \in \Sigma\}$ is weakly precompact, since A is a weakly precompact subset of $L_1(\mu, X)$ and $\{\chi_E : E \in \Sigma\}$ is a bounded subset of L_∞. Since $H \subseteq \{f \cdot \chi_E : f \in A, E \in \Sigma\}$, H is weakly precompact.

The converse follows from Lemma 2.15. ∎

Corollary 2.19. *If X has property (wV^*), then A is a V^*-set if and only if for every $\epsilon > 0$, there exist a positive integer N and a V^*-subset H of NW such that $A \subseteq H + \epsilon B(0)$.*

Proof. Suppose X has property (wV^*) and let A be a V^*-set in $L_1(\mu, X)$. Since X has property (wV^*), $L_1(\mu, X)$ has property (wV^*) ([Ra]). Then A is weakly precompact. Let $\epsilon > 0$. By Theorem 2.18(ii), there exist a positive integer N and a weakly precompact subset H of NW such that $A \subseteq H + \epsilon B(0)$. Since H is bounded and weakly precompact, H is a V^*-set ([Pe]).

The converse follows from Lemma 2.16. ∎

Corollary 2.20. *If X^* has the Schur property, then A is a DP set if and only if for every $\epsilon > 0$, there exist a positive integer N and a DP subset H of NW such that $A \subseteq H + \epsilon B(0)$.*

Proof. Suppose X^* has the Schur property and let A be a DP set in $L_1(\mu, X)$. Let $\epsilon > 0$. Since A is weakly precompact ([Ro, p. 377]), there exist a positive integer N and weakly precompact subset H of NW such that $A \subseteq H + \epsilon B(0)$ (by Theorem 2.18(ii)). Since $L_1(\mu, X)$ has the DPP ([An]), H is a DP set ([Di2]).

The converse follows from Lemma 2.17. ∎

REFERENCES

[AK] F. Albiac and N. J. Kalton, *Topics in Banach Space Theory*, Grad. Texts in Math. 233, Springer, 2006.

[An] K. T. Andrews, *Dunford–Pettis sets in the space of Bochner integrable functions*, Math. Ann. 241 (1979), 35–41.

[BDS] R. G. Bartle, N. Dunford and J. Schwartz, *Weak compactness and vector measures*, Canad. J. Math. 7 (1955), 289–305.

[BL] E. M. Bator and P. W. Lewis, *Properties (V) and (wV) on $C(\Omega, X)$*, Math. Proc. Cambridge Philos. Soc. 117 (1995), 469–477.

[Bom] F. Bombal, *On (V^*) sets and Pełczyński's property (V^*)*, Glasgow Math. J. 32 (1990), 109–120.

[Bou] J. Bourgain, *An averaging result for l^1-sequences and applications to weakly conditionally compact sets in L^1_X*, Israel J. Math. 32 (1979), 289–298.

[Di1] J. Diestel, *Sequences and Series in Banach Spaces*, Grad. Texts in Math. 92, Springer, 1984.

[Di2] J. Diestel, *A survey of results related to the Dunford–Pettis property*, in: Contemp. Math. 2, Amer. Math. Soc., 1980, 15–60.

[DRS] J. Diestel, W. M. Ruess and W. Schachermayer, *Weak compactness in $L^1(\mu, X)$*, Proc. Amer. Math. Soc. 118 (1993), 447–453.

[DU] J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., 1977.

[Din] N. Dinculeanu, *Vector Measures*, Pergamon Press, 1967.

[DP] N. Dunford and B. J. Pettis, *Linear operations on summable functions*, Trans. Amer. Math. Soc. 47 (1940), 323–392.

[Em] G. Emmanuele, *On the Banach spaces with the property* (V^*) *of Pełczyński*, Ann. Mat. Pura Appl. 152 (1988), 171–181.

[GL] I. Ghenciu and P. Lewis, *Almost weakly compact operators*, Bull. Polish Acad. Sci. Math. 54 (2006), 237–256.

[Gr] A. Grothendieck, *Sur les applications linéaires faiblement compactes d'espaces du type* $C(K)$, Canad. J. Math. 5 (1953), 129–173.

[IT] A. Ionescu Tulcea and C. Ionescu Tulcea, *Topics in the Theory of Lifting*, Ergeb. Math. Grenzgeb. 48, Springer, 1969.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II*, Springer, 1979.

[Pe] A. Pełczyński, *Banach spaces on which every unconditionally converging operator is weakly compact*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 10 (1962), 641–648.

[Ra] N. Randrianantoanina, *Complemented copies of* ℓ_1 *and Pełczyński's Property* (V^*) *in Bochner function spaces*, Canad. J. Math. 48 (1996), 625–640.

[Ro] H. P. Rosenthal, *Point-wise compact subsets of the first Baire class*, Amer. J. Math. 99 (1977), 362–377.

[Sch] T. Schlumprecht, *Limited sets in Banach spaces*, dissertation, Münich, 1987.

[Ta] M. Talagrand, *Weak Cauchy sequences in* $L^1(E)$, Amer. J. Math. 106 (1984), 703–724.

[Tz] L. Tzafriri, *Reflexivity in Banach lattices and their subspaces*, J. Funct. Anal. 10 (1972), 1–18.

[Ul1] A. Ülger, *Continuous linear operators on* $C(K, X)$ *and pointwise weakly precompact subsets of* $C(K, X)$, Math. Proc. Cambridge Philos. Soc. 111 (1992), 143–150.

[Ul2] A. Ülger, *Weak compactness in* $L^1(\mu, X)$, Proc. Amer. Math. Soc. 113 (1991), 143–149.

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