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EQUIVARIANT CLASSIFICATION OF 2-TORUS MANIFOLDS

BY

ZHI LÜ (Shanghai) and MIKIYA MASUDA (Osaka)

Abstract. We consider locally standard 2-torus manifolds, which are a generalization of small covers of Davis and Januszkiewicz and study their equivariant classification. We formulate a necessary and sufficient condition for two locally standard 2-torus manifolds over the same orbit space to be equivariantly homeomorphic. This leads us to count the equivariant homeomorphism classes of locally standard 2-torus manifolds with the same orbit space.

1. Introduction. A 2-torus manifold is a closed smooth manifold of dimension n with a non-free effective action of a 2-torus group $(\mathbb{Z}_2)^n$ of rank n, and it is said to be locally standard if it is locally isomorphic to a faithful representation of $(\mathbb{Z}_2)^n$ on $\mathbb{R}^n$. The orbit space Q of a locally standard 2-torus M under this action is a nice manifold with corners. When Q is a simple convex polytope, M is called a small cover and studied in [4]. A typical example of a small cover is a real projective space $\mathbb{R}P^n$ with a standard action of $(\mathbb{Z}_2)^n$. Its orbit space is an n-simplex. On the other hand, a typical example of a compact non-singular toric variety is a complex projective space $\mathbb{C}P^n$ with a standard action of $(\mathbb{C}^*)^n$ where $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. $\mathbb{C}P^n$ has complex conjugation and its fixed point set is $\mathbb{R}P^n$. More generally, any compact non-singular toric variety admits complex conjugation and its fixed point set often provides an example of a small cover. Similarly to the theory of toric varieties, an interesting connection between topology, geometry and combinatorics is discussed for small covers in [4], [5] and [7]. Although locally standard 2-torus manifolds form a much wider class than small covers, one can still expect such a connection. See [9] for the study of 2-torus manifolds from the viewpoint of cobordism.

The orbit space Q of a locally standard 2-torus manifold M contains a lot of topological information on M. For instance, when Q is a simple convex polytope (in other words, when M is a small cover), the Betti numbers of M (with $\mathbb{Z}_2$ coefficients) are described in terms of face numbers of Q ([4]). This is not the case for a general Q, but the Euler characteristic of M can be described in terms of Q (Lemma 4.1). Although Q contains a lot of

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topological information on M, Q is not sufficient to reproduce M, i.e., there are many locally standard 2-torus manifolds with the same orbit space in general. We need two data to recover M from Q:

(1) One is a characteristic function on Q introduced in [4], which is a map from the set of codimension-one faces of Q to $(\mathbb{Z}_2)^n$ satisfying a certain linear independence condition. Roughly speaking, a characteristic function provides information on the set of non-free orbits in M.
(2) The other one is a principal $(\mathbb{Z}_2)^n$-bundle over Q which provides information on the set of free orbits in M.

It turns out that the orbit space Q together with these two data uniquely determines a locally standard 2-torus manifold up to equivariant homeomorphism (Lemma 3.1). When Q is a simple convex polytope, any principal $(\mathbb{Z}_2)^n$-bundle over it is trivial; so only a characteristic function matters in this case ([4]).

The set of isomorphism classes in all principal $(\mathbb{Z}_2)^n$-bundles over Q can be identified with $H^1(Q;(\mathbb{Z}_2)^n)$. Let $\Lambda(Q)$ be the set of all characteristic functions on Q. Then each element in $H^1(Q;(\mathbb{Z}_2)^n)\times\Lambda(Q)$ determines a locally standard 2-torus manifold with orbit space Q. However, different elements in the product may produce equivariantly homeomorphic locally standard 2-torus manifolds. Let $\mathrm{Aut}(Q)$ be the group of self-homeomorphisms of Q as a manifold with corners. It naturally acts on $H^1(Q;(\mathbb{Z}_2)^n)\times\Lambda(Q)$ and one can see that equivariant homeomorphism classes in locally standard 2-torus manifolds with orbit space Q can be identified with the coset space $(H^1(Q;(\mathbb{Z}_2)^n)\times\Lambda(Q))/\mathrm{Aut}(Q)$ (see Proposition 5.5). This space is a finite set and we are led to count its elements. However, this is not easy in general. We investigate the case where Q is a compact surface with only one boundary component. In this case, codimension-one faces sit in the boundary circle (i.e. an m-gon with $m\geq 2$), so a characteristic function on Q is a coloring on the m-gon with three colors. This shows that simple and nice combinatorics can still be related to topology.

It should be pointed out that a torus manifold is introduced in [8] as an extended notion of a toric or quasitoric manifold. A torus manifold is a closed smooth manifold of dimension $2n$ with an effective smooth action of a compact n-dimensional torus $(S^1)^n$ having a fixed point. (More precisely, an orientation data on M called an omniorientation in [2] is incorporated in the definition.) There is also the notion of local standardness in this setting ([4]). Although many interesting examples of torus manifolds are locally standard (e.g. this is the case for compact non-singular toric varieties with restricted action of the compact torus, more generally for torus manifolds with vanishing odd degree cohomology, [11]), the local standardness is not assumed in [8] because a combinatorial object called a multi-fan can be defined without assuming it (see also [10]). As for a 2-torus manifold, we do not require the existence of a fixed point but require that the action be non-free.

The argument developed in this paper for locally standard 2-torus manifolds also works with some modification for locally standard torus manifolds. But the number of locally standard torus manifolds (up to a certain equivalence relation such as equivariant homeomorphism) over a fixed manifold with corners is infinite in general while it is always finite in the 2-torus case so that the counting problem makes sense. This is why we restrict our concern to the 2-torus case.

The paper is organized as follows. In Section 2, we introduce the notion of locally standard 2-torus manifold and give several examples. Following Davis and Januszkiewicz [4], we define a characteristic function and construct a locally standard 2-torus manifold from a characteristic function and a principal bundle in Section 3. In Section 4 we describe the Euler characteristic of a locally standard 2-torus manifold in terms of its orbit space. Section 5 discusses three equivalence relations among locally standard 2-torus manifolds and identify them with some coset spaces. We count the number of colorings on a circle in Section 6. Applying this result, we find in Section 7 the number of equivariant homeomorphism classes in locally standard 2-torus manifolds when the orbit space is a compact surface with only one boundary component.

2. 2-torus manifolds. We denote the quotient additive group $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}_2$ throughout this paper. The natural action of a 2-torus $(\mathbb{Z}_2)^n$ of rank n on $\mathbb{R}^n$ defined by

$$(x_1, \dots, x_n) \mapsto ((-1)^{g_1}x_1, \dots, (-1)^{g_n}x_n), \qquad (g_1, \dots, g_n) \in (\mathbb{Z}_2)^n,$$

is called the *standard representation* of $(\mathbb{Z}_2)^n$. The orbit space is a positive cone $\mathbb{R}^n_{\geq 0}$. Any real n-dimensional faithful representation of $(\mathbb{Z}_2)^n$ is obtained from the standard representation by composing with a group automorphism of $(\mathbb{Z}_2)^n$, up to isomorphism. Therefore the orbit space of the faithful representation space can also be identified with $\mathbb{R}^n_{\geq 0}$.

A 2-torus manifold M is a closed smooth manifold of dimension n with a non-free effective smooth action of $(\mathbb{Z}_2)^n$. We say that M is *locally standard* if for each point x in M, there is a $(\mathbb{Z}_2)^n$-invariant neighborhood V_x of x such that V_x is equivariantly homeomorphic to an invariant open subset of a real n-dimensional faithful representation space of $(\mathbb{Z}_2)^n$.

For a locally standard 2-torus manifold M, the orbit space Q of M naturally becomes a manifold with corners (see [3] for the details on manifolds with corners), and it has a non-empty boundary since the action on M is assumed to be non-free. Therefore the notion of a face can be defined for Q. In this paper we assume that a face is connected. We call a face of dimension 0 a *vertex*, a face of dimension one an *edge* and a codimension-one face a *facet*. We also understand that Q itself is a face of Q of codimension zero.

An n-dimensional convex polytope P is said to be *simple* if exactly n facets meet at each of its vertices. Each point of a simple convex polytope P has a neighborhood which is affine isomorphic to an open subset of the positive cone $\mathbb{R}^n_{\geq 0}$, so P is an n-dimensional manifold with corners. A locally standard 2-torus manifold M is said to be a *small cover* when its orbit space is a simple convex polytope (see [4]).

We call a closed, connected, codimension-one submanifold of M *characteristic* if it is a connected component of the set fixed pointwise by some $\mathbb{Z}_2$ subgroup. Since M is compact, M has only finitely many characteristic submanifolds. The action of $(\mathbb{Z}_2)^n$ is free outside the union of all characteristic submanifolds, in other words, a point of M with non-trivial isotropy subgroup is contained in some characteristic submanifold of M.

Through the quotient map $M \to Q$, a characteristic submanifold of M corresponds to a facet of Q. A connected component of the intersection of k characteristic submanifolds of M corresponds to a codimension-k face of Q, so a codimension-k face of Q is a connected component of the intersection of k facets. In particular, any codimension-two face of Q (if it exists) is a connected component of the intersection of two facets of Q, which means that Q is *nice* (see [3]).

We now give examples of locally standard 2-torus manifolds.

Example 2.1. A real projective space $\mathbb{R}P^n$ with the standard $(\mathbb{Z}_2)^n$-action defined by

$$[x_0, x_1, \dots, x_n] \mapsto [x_0, (-1)^{g_1}x_1, \dots, (-1)^{g_n}x_n], \qquad (g_1, \dots, g_n) \in (\mathbb{Z}_2)^n,$$

is a locally standard 2-torus manifold. It has $n+1$ isolated points and $n+1$ characteristic submanifolds. The orbit space of $\mathbb{R}P^n$ for this action is an n-simplex, so this locally standard 2-torus manifold is actually a small cover.

Example 2.2. Let S^1 denote the unit circle in the complex plane $\mathbb{C}$ and consider two involutions on $S^1 \times S^1$ defined by

$$t_1 : (z, w) \mapsto (-z, w), \qquad t_2 : (z, w) \mapsto (z, \overline{w}).$$

Since t_1 and t_2 commute, they define a $(\mathbb{Z}_2)^2$-action on $S^1 \times S^1$, and it is easy to see that $S^1 \times S^1$ with this action is a locally standard 2-torus manifold. It has no fixed point and the orbit space is $\mathbb{R}P^1 \times I = S^1 \times I$ where I is a closed interval.

EXAMPLE 2.3. If M_1 and M_2 are both locally standard 2-torus manifolds of the same dimension, then their equivariant connected sum along their free orbits produces a new locally standard 2-torus manifold. For example, we take $\mathbb{R}P^2$ of Example 2.1 and $S^1 \times S^1$ of Example 2.2 and do the equivariant connected sum of them along their free orbits. The orbit space of the resulting locally standard 2-torus manifold M is the connected sum of a 2-simplex with $S^1 \times I$ at their interior points. M has five characteristic submanifolds and three of them have a fixed point but the other two have no fixed point.

If M is a locally standard 2-torus manifold of dimension n and a subgroup of $(\mathbb{Z}_2)^n$ has an isolated fixed point, then the isolated point must be fixed by the entire group $(\mathbb{Z}_2)^n$. This follows from the local standardness of M. The following is an example of a closed n-manifold with an effective $(\mathbb{Z}_2)^n$-action which is not a locally standard 2-torus manifold.

EXAMPLE 2.4. Consider two involutions on the unit sphere S^2 of $\mathbb{R} \times \mathbb{C}$ defined by

$$t_1 : (x, z) \mapsto (-x, -z), \qquad t_2 : (x, z) \mapsto (x, \overline{z}).$$

Since t_1 and t_2 commute, they define a $(\mathbb{Z}_2)^2$-action on S^2. But S^2 with this action is not a locally standard 2-torus manifold because the fixed point set of t_1t_2 consists of two isolated points $(0, \pm\sqrt{-1})$, which are not fixed by the entire group $(\mathbb{Z}_2)^2$.

3. Characteristic functions and principal bundles. Let Q be an n-dimensional nice manifold with corners having a non-empty boundary. We denote by $\mathcal{F}(Q)$ the set of facets of Q. We call a map

$$\lambda : \mathcal{F}(Q) \to (\mathbb{Z}_2)^n$$

a *characteristic function* on Q if it satisfies the following linear independence condition:

> whenever the intersection of k facets $F_1, \dots, F_k$ is non-empty, the elements $\lambda(F_1), \dots, \lambda(F_k)$ are linearly independent when viewed as vectors of the vector space $(\mathbb{Z}_2)^n$ over the field $\mathbb{Z}_2$.

We denote by G_F the subgroup of $(\mathbb{Z}_2)^n$ generated by $\lambda(F_1), \dots, \lambda(F_k)$, where F is a connected component of the intersection of $F_1, \dots, F_k$ and has codimension k.

REMARK. When $n \leq 2$, it is easy to see that any Q admits a characteristic function. When $n = 3$, Q admits a characteristic function if the boundary of Q is a union of 2-spheres, which follows from the Four Color Theorem, but Q may not admit a characteristic function otherwise. When $n \geq 4$, there is a simple convex polytope which admits no characteristic function (see [4, Nonexamples 1.22]).

A characteristic function arises naturally from a locally standard 2-torus manifold M of dimension n with orbit space Q. A facet of Q is the image of a characteristic submanifold of M under the quotient map $\pi\colon M \to Q$. To each element $F \in \mathcal{F}(Q)$ we assign the non-zero element of $(\mathbb{Z}_2)^n$ which fixes pointwise the characteristic submanifold $\pi^{-1}(F)$. The local standardness of M implies that this assignment satisfies the linear independence condition above required for a characteristic function.

Besides the characteristic function, a principal $(\mathbb{Z}_2)^n$-bundle over Q will be associated with M as follows. We take a small invariant open tubular neighborhood for each characteristic submanifold of M and remove their union from M. Then the $(\mathbb{Z}_2)^n$-action on the resulting space is free and its orbit space can naturally be identified with Q, so it gives a principal $(\mathbb{Z}_2)^n$-bundle over Q.

We have associated a characteristic function and a principal $(\mathbb{Z}_2)^n$-bundle with a locally standard 2-torus manifold. Conversely, one can reproduce the locally standard 2-torus manifold from these two data. This is done by Davis–Januszkiewicz [4] when Q is a simple convex polytope, but their construction still works in our setting. Let $\xi = (E, \kappa, Q)$, where $\kappa\colon E \to Q$, be a principal $(\mathbb{Z}_2)^n$-bundle over Q and let $\lambda : \mathcal{F}(Q) \to (\mathbb{Z}_2)^n$ be a characteristic function on Q. We define an equivalence relation $\sim$ on E as follows: for $u_1, u_2 \in E$,

$$u_1 \sim u_2 \Leftrightarrow \kappa(u_1) = \kappa(u_2) \text{ and } u_1 = u_2 g \text{ for some } g \in G_F$$

where F is the face of Q containing $\kappa(u_1) = \kappa(u_2)$ in its relative interior and G_F is the subgroup of $(\mathbb{Z}_2)^n$ defined at the beginning of this section. Then the quotient space $E/{\sim}$, denoted by $M(\xi, \lambda)$, naturally inherits the $(\mathbb{Z}_2)^n$-action from E.

The following is proved in [4] when Q is a simple convex polytope, but the same proof works in our setting.

Lemma 3.1. *If M is a locally standard 2-torus manifold over Q with ξ as the associated principal $(\mathbb{Z}_2)^n$-principal bundle and λ as the characteristic function, then there is an equivariant homeomorphism from $M(\xi, \lambda)$ to M which covers the identity on Q.*

4. Euler characteristic of a locally standard 2-torus manifold. The following formula describes the Euler characteristic $\chi(M)$ of a locally standard 2-torus manifold M in terms of its orbit space, and it is a special case of a much more general result. Here we are carrying out a standard exercise.

Lemma 4.1. *If M is a locally standard 2-torus manifold over Q, then*

$$\chi(M) = \sum_F 2^{\dim F}\chi(F,\partial F) = \sum_F 2^{\dim F}(\chi(F) - \chi(\partial F))$$

where F runs over all faces of Q.

Proof. As observed in Section 3, M can be decomposed into the disjoint union of $2^{\dim F}$ copies of $F\setminus\partial F$ over all faces F of Q. This implies the former identity in the theorem. The latter identity is well-known. In fact, it follows from the homology exact sequence for the pair $(F,\partial F)$. ∎

When $\dim M = 2$, Q is a surface with boundary and each boundary component is a circle with at least two vertices if it has a vertex.

Corollary 4.2. *If $\dim M = 2$ and Q has m vertices, then $\chi(M) = 4\chi(Q) - m$.*

Proof. Since ∂Q is a union of circles, $\chi(Q,\partial Q) = \chi(Q)$. If a boundary circle has no vertex, then it is an edge without boundary and its Euler characteristic is zero. So we may neglect it. If F is an edge with a vertex, then it has two endpoints and $\chi(F,\partial F) = \chi(F) - \chi(\partial F) = -1$, and if F is a vertex, then $\chi(F,\partial F) = \chi(F) = 1$. Since the number of edges with a vertex and the number of vertices are both m, it follows from Lemma 4.1 that $\chi(M) = 2^2\chi(Q) - 2m + m = 4\chi(Q) - m$. ∎

Remark. When $\dim M = 2$, it is not difficult to see that M is orientable if and only if Q is orientable and the characteristic function $\lambda\colon \mathcal{F}(Q) \to (\mathbb{Z}_2)^2$ associated with M assigns exactly two elements to each boundary component of Q with a vertex (cf. [12]). Therefore one can find the homeomorphism type of M from the corollary above and the characteristic function λ.

5. Classification of locally standard 2-torus manifolds. In this section we introduce three notions of equivalence in locally standard 2-torus manifolds over Q and identify each set of equivalence classes with the coset space of $H^1(Q;(\mathbb{Z}_2)^n) \times \Lambda(Q)$ under some action.

Following Davis and Januszkiewicz [4] we say that two locally standard 2-torus manifolds M and M' over Q are *equivalent* if there is a homeomorphism $f : M \to M'$ together with an element $\sigma \in \mathrm{GL}(n,\mathbb{Z}_2)$ such that

(1) $f(gx) = \sigma(g)f(x)$ for all $g \in (\mathbb{Z}_2)^n$ and $x \in M$,
(2) f induces the identity on the orbit space Q.

When we classify locally standard 2-torus manifolds up to the above equivalence, it suffices to consider locally standard 2-torus manifolds of the form $M(\xi,\lambda)$ by Lemma 3.1. We denote by ξ^σ the principal $(\mathbb{Z}_2)^n$-bundle ξ with $(\mathbb{Z}_2)^n$-action through $\sigma \in \mathrm{GL}(n,\mathbb{Z}_2)$. Then it is obvious that $M(\xi',\lambda')$ is equivalent to $M(\xi,\lambda)$ if and only if there exists $\sigma \in \mathrm{GL}(n,\mathbb{Z}_2)$ such that $\xi' = \xi^\sigma$ and $\lambda' = \sigma\circ\lambda$.

We denote by $\mathcal{P}(Q)$ the set of all principal $(\mathbb{Z}_2)^n$-bundles over Q. Since the classifying space of $(\mathbb{Z}_2)^n$ is an Eilenberg–MacLane space $K((\mathbb{Z}_2)^n,1)$, $\mathcal{P}(Q)$ can be naturally identified with $H^1(Q;(\mathbb{Z}_2)^n)$, and the action of σ sending ξ to ξ^σ is just the action on $H^1(Q;(\mathbb{Z}_2)^n)$ induced from the automorphism σ on the coefficient group $(\mathbb{Z}_2)^n$. With this understood, the above fact implies the following.

PROPOSITION 5.1. *The set of equivalence classes in locally standard* 2-*torus manifolds over Q bijectively corresponds to the coset space*

$$\mathrm{GL}(n,\mathbb{Z}_2)\backslash(H^1(Q;(\mathbb{Z}_2)^n)\times\Lambda(Q))$$

under the diagonal action.

The action of $\mathrm{GL}(n,\mathbb{Z}_2)$ on $H^1(Q;(\mathbb{Z}_2)^n)\times\Lambda(Q)$ is free when Q has a vertex by the following lemma.

LEMMA 5.2. *If Q has a vertex, then the action of* $\mathrm{GL}(n,\mathbb{Z}_2)$ *on $\Lambda(Q)$ is free and* $|\Lambda(Q)| = |\mathrm{GL}(n,\mathbb{Z}_2)\backslash\Lambda(Q)|\prod_{k=1}^n(2^n-2^{k-1})$.

Proof. Suppose that $\lambda=\sigma\circ\lambda$ for some $\lambda\in\Lambda(Q)$ and $\sigma\in\mathrm{GL}(n,\mathbb{Z}_2)$. Take a vertex of Q and let $F_1,\dots,F_n$ be the facets of Q meeting at the vertex. Then

$$(\lambda(F_1),\dots,\lambda(F_n)) = \sigma(\lambda(F_1),\dots,\lambda(F_n)).$$

Since the matrix $(\lambda(F_1),\dots,\lambda(F_n))$ is non-singular, σ is the identity matrix. This proves the former statement in the lemma. Then the latter statement follows from the well-known fact that $|\mathrm{GL}(n,\mathbb{Z}_2)| = \prod_{k=1}^n(2^n-2^{k-1})$ (see [1]). ∎

Lemma 5.2 is also helpful when counting the number of elements in $\Lambda(Q)$. Here is an example.

EXAMPLE 5.3 (The number of characteristic functions on a prism). There exist seven combinatorially inequivalent 3-polytopes with six vertices (see [6, Theorem 6.7]) and only one of them is simple, which is a prism P^3.

Let us count the number of characteristic functions on P^3. The prism P^3 has five facets, consisting of three square facets and two triangular facets. We denote the square facets by F_1,F_2,F_4, and the triangular facets by F_3,F_5. The facets F_1,F_2,F_3 intersect at a vertex and we may assume that a characteristic function λ on P^3 takes the standard basis $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ of $(\mathbb{Z}_2)^3$ to F_1,F_2,F_3 respectively through the action of $\mathrm{GL}(3,\mathbb{Z}_2)$ on $(\mathbb{Z}_2)^3$. The characteristic function λ must satisfy the linear independence condition at each vertex of P^3. This requires that the values of λ on the remaining facets

F_4, F_5 must be as follows:

$$(\lambda(F_4), \lambda(F_5)) = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3) \quad \text{or} \quad (\mathbf{e}_1 + \mathbf{e}_2, a\mathbf{e}_1 + b\mathbf{e}_2 + \mathbf{e}_3),$$

where $a, b \in \mathbb{Z}_2$. Therefore, $|\mathrm{GL}(3, \mathbb{Z}_2)\backslash\Lambda(P^3)| = 5$ and $|\Lambda(P^3)| = 5|\mathrm{GL}(3, \mathbb{Z}_2)| = 840$ by Lemma 5.2.

Another natural equivalence relation between locally standard 2-torus manifolds is equivariant homeomorphism. An *automorphism* of Q is a self-homeomorphism of Q as a manifold with corners, and we denote the group of automorphisms of Q by $\mathrm{Aut}(Q)$. Similarly, an *automorphism* of $\mathcal{F}(Q)$ is a bijection from $\mathcal{F}(Q)$ to itself which preserves the poset structure of $\mathcal{F}(Q)$ defined by inclusions of faces, and we denote the group of automorphisms of $\mathcal{F}(Q)$ by $\mathrm{Aut}(\mathcal{F}(Q))$. An automorphism of Q induces an automorphism of $\mathcal{F}(Q)$, so we have a natural homomorphism

$$\Phi\colon \mathrm{Aut}(Q) \to \mathrm{Aut}(\mathcal{F}(Q)). \tag{5.1}$$

We note that $\mathrm{Aut}(\mathcal{F}(Q))$ acts on $\Lambda(Q)$ by sending $\lambda \in \Lambda(Q)$ to $\lambda \circ h$ for $h \in \mathrm{Aut}(\mathcal{F}(Q))$.

LEMMA 5.4. *$M(\xi, \lambda)$ is equivariantly homeomorphic to $M(\xi', \lambda')$ if and only if there is an $h \in \mathrm{Aut}(Q)$ such that $\lambda' = \lambda \circ \Phi(h)$ and $h^*(\xi') = \xi$ in $\mathcal{P}(Q)$, where $h^*(\xi')$ denotes the bundle induced from ξ' by h.*

Proof. If $M(\xi, \lambda)$ is equivariantly homeomorphic to $M(\xi', \lambda')$, then there is an equivariant homeomorphism $H\colon M(\xi', \lambda') \to M(\xi, \lambda)$ and it is easy to see that the automorphism of Q induced from H is the desired h in the statement.

Conversely, suppose that there is an $h \in \Lambda(Q)$ such that $\lambda' = \lambda \circ \Phi(h)$ and $\xi' = h^*(\xi)$ in $\mathcal{P}(Q)$. Then there is a bundle map $\hat{h}\colon \xi' \to \xi$ which covers h, and $\hat{h}$ descends to a map H from $M(\xi', \lambda')$ to $M(\xi, \lambda)$ because $\lambda' = \lambda \circ \Phi(h)$. It is not difficult to see that H is an equivariant homeomorphism. ∎

$\mathrm{Aut}(Q)$ naturally acts on $H^1(Q; (\mathbb{Z}_2)^n)$, and the canonical bijection between $\mathcal{P}(Q)$ and $H^1(Q; (\mathbb{Z}_2)^n)$ is equivariant with respect to the actions of $\mathrm{Aut}(Q)$.

PROPOSITION 5.5. *The set of equivariant homeomorphism classes in all locally standard 2-torus manifolds over Q bijectively corresponds to the coset space*

$$(H^1(Q, (\mathbb{Z}_2)^n) \times \Lambda(Q))/\mathrm{Aut}(Q)$$

under the diagonal action of $\mathrm{Aut}(Q)$. If Q is a simple convex polytope, then the set of equivariant homeomorphism classes in all small covers over Q bijectively corresponds to the coset space $\Lambda(Q)/\mathrm{Aut}(\mathcal{F}(Q))$.

Proof. The former statement in the proposition follows from Lemma 5.4. If Q is a simple polytope, then $H^1(Q; (\mathbb{Z}_2)^n) = 0$. Therefore, the latter statement follows if we prove that the map Φ in (5.1) is surjective when Q is a simple convex polytope.

A simple polytope Q has a simplicial polytope Q^* as its dual and the face poset $\mathcal{F}(Q)$ is the same as $\mathcal{F}(Q^*)$ with reverse inclusion. Therefore $\mathrm{Aut}(\mathcal{F}(Q))=\mathrm{Aut}(\mathcal{F}(Q^*))$. Since Q^* is simplicial, an element φ of $\mathrm{Aut}(\mathcal{F}(Q^*))$ is realized by a simplicial automorphism on the boundary of Q^*, so it extends to an automorphism of Q^*. Since Q is dual to Q^*, the automorphism of Q^* determines a bijection on the vertex set of Q and hence an automorphism of Q which induces the chosen φ. ∎

Our last equivalence relation is a combination of the previous two. We say that two locally standard 2-torus manifolds M and M' over Q are *weakly equivariantly homeomorphic* if there is a homeomorphism $f\colon M \to M'$ together with $\sigma \in \mathrm{GL}(n,\mathbb{Z}_2)$ such that $f(gx) = \sigma(g)f(x)$ for any $g \in (\mathbb{Z}_2)^n$ and $x \in M$. We note that f induces an automorphism of Q but it may not be the identity on Q. The observation above shows that $M(\xi,\lambda)$ and $M(\xi',\lambda')$ are weakly equivariantly homeomorphic if and only if there are $h \in \mathrm{Aut}(Q)$ and $\sigma \in \mathrm{GL}(n,\mathbb{Z}_2)$ such that $\xi' = h^*(\xi^\sigma)$ and $\lambda' = \sigma \circ \lambda \circ h$. This yields

PROPOSITION 5.6. *The set of weakly equivariant homeomorphism classes in locally standard* 2*-torus manifolds over* Q *bijectively corresponds to the double coset space*

$$\mathrm{GL}(n,\mathbb{Z}_2)\backslash(H^1(Q;(\mathbb{Z}_2)^n)\times \Lambda(Q))/\mathrm{Aut}(Q)$$

under the diagonal actions of $\mathrm{Aut}(Q)$ *and* $\mathrm{GL}(n,\mathbb{Z}_2)$. *If* Q *is a simple convex polytope*, *then the set of weakly equivariant homeomorphism classes in small covers over* Q *bijectively corresponds to the double coset space*

$$\mathrm{GL}(n,\mathbb{Z}_2)\backslash\Lambda(Q)/\mathrm{Aut}(\mathcal{F}(Q)). \tag{5.2}$$

REMARK. When Q is a right-angled regular hyperbolic polytope (such a Q is the dodecahedron, the 120-cell or an m-gon with $m \geq 5$), it is shown in [7, Theorem 3.3] that the double coset space (5.2) identifies with the set of hyperbolic structures in small covers over Q. This together with Mostow rigidity implies that when $\dim Q \geq 3$, that is, when Q is the dodecahedron or the 120-cell, (5.2) identifies with the set of homeomorphism classes in small covers over Q ([7, Corollary 3.4]), i.e., the natural surjective map from the double coset space to the set of homeomorphism classes in small covers over Q is bijective for such Q. However, this last statement does not hold for an m-gon Q with $m \geq 6$ although it holds for $m = 3, 4, 5$ (see the remark following Example 6.5 in the next section).

6. Counting the colorings on a circle. When $\dim Q = 2$, each boundary component is a circle with at least two vertices if it has a vertex, and any two non-zero elements in $(\mathbb{Z}_2)^2$ form a basis of $(\mathbb{Z}_2)^2$; so a characteristic function on Q is equivalent to coloring arcs on the boundary circles with three colors in such a way that any two adjacent arcs have different colors.

Let $S(m)$ be a circle with m (≥ 2) vertices. A coloring on $S(m)$ (with three colors) means to color arcs of $S(m)$ in such a way that any adjacent arcs have different colors. We denote by $\Lambda(m)$ the set of all colorings on $S(m)$ and set

$$A(m) := |\Lambda(m)|.$$

LEMMA 6.1. $A(m) = 2^m + (-1)^m 2$.

Proof. Let $L(m)$ be a segment with $m+1$ vertices, so $L(m)$ consists of m segments. The number of colorings of segments of $L(m)$ with three colors in such a way that any adjacent segments have different colors is $3 \cdot 2^{m-1}$. If the two end segments have different colors, then gluing the end points of $L(m)$ produces a coloring on $S(m)$; otherwise, a coloring on $S(m-1)$. Thus, $A(m) + A(m-1) = 3 \cdot 2^{m-1}$. On the other hand, a simple observation shows that $A(3) = A(2) = 6$. These imply the lemma. ∎

We think of $S(m)$ as the unit circle of $\mathbb{C}$ with m vertices $e^{2\pi k/m}$ $(k = 0, 1, \ldots, m-1)$. Let $\mathfrak{D}_m$ be the dihedral group of order $2m$ consisting of m rotations of $\mathbb{C}$ by angles $2\pi k/m$ $(k = 0, 1, \ldots, m-1)$ and m reflections in lines in $\mathbb{C}$ obtained by rotating the real axis by $\pi k/m$ $(k = 0, 1, \ldots, m-1)$. Then the action of $\mathfrak{D}_m$ on $S(m)$ preserves the vertices so that $\mathfrak{D}_m$ acts on the set $\Lambda(m)$. With this understood we have

THEOREM 6.2. *Let φ denote Euler's totient function, that is, $\varphi(1) = 1$ and $\varphi(N)$ for a positive integer N (≥ 2) is the number of positive integers less than N and coprime to N. Then*

$$|\Lambda(m)/\mathfrak{D}_m| = \frac{1}{2m}\Big(\sum_{2\leq d|m} \varphi(m/d)A(d) + \frac{1+(-1)^m}{2} \cdot 3 \cdot 2^{m/2} \cdot \frac{m}{2}\Big).$$

Proof. The famous Burnside lemma or Cauchy–Frobenius lemma (see [1]) says that if G is a finite group and X is a finite G-set, then

$$|X/G| = \frac{1}{|G|}\sum_{g\in G} |X^g|,$$

where X^g denotes the set of g-fixed points in X. We apply this formula to our $\mathfrak{D}_m$-set $\Lambda(m)$. Let $a \in \mathfrak{D}_m$ be the rotation by $2\pi/m$ and $b \in \mathfrak{D}_m$ be the reflection in the real axis. Then

$$|\Lambda(m)/\mathfrak{D}_m| = \frac{1}{2m}\sum_{k=0}^{m-1}(|\Lambda(m)^{a^k}| + |\Lambda(m)^{a^k b}|). \tag{6.1}$$

Here, if d is the greatest common divisor of k and m, then $\Lambda(m)^{a^k} = \Lambda(m)^{a^d}$ because the subgroups generated by a^k and by a^d are the same. Since $\Lambda(m)^{a^d} = \Lambda(d)$ and $\Lambda(1)$ is empty, we have

$$\sum_{k=0}^{m-1} |\Lambda(m)^{a^k}| = \sum_{2\le d|m} \varphi(m/d)A(d). \tag{6.2}$$

On the other hand, since $a^k b$ is the reflection in the line in $\mathbb{C}$ obtained by rotating the real axis by $\pi k/m$, we have

$$|\Lambda(m)^{a^k b}| = \begin{cases} 3\cdot 2^{m/2} & \text{when } m \text{ is even and } k \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases} \tag{6.3}$$

Putting (6.2) and (6.3) into (6.1), we obtain the formula in the theorem. ∎

EXAMPLE 6.3. As is well known, $\varphi(p^n) = p^{n-1}(p-1)$ for any prime number p and positive integer n, and $\varphi(ab) = \varphi(a)\varphi(b)$ for a, b relatively prime. We set

$$B(m) := |\Lambda(m)/\mathfrak{D}_m|.$$

Using the formula in Theorem 6.2 together with Lemma 6.1, one finds that

$$B(2) = 3,\quad B(3) = 1,\quad B(4) = 6,\quad B(5) = 3,\quad B(6) = 13,$$
$$B(7) = 9,\quad B(8) = 30,\quad B(9) = 29,\quad B(10) = 78,$$

$$B(2^k) = 2^{2^k-k-1} + 3\cdot 2^{2^{k-1}-2} + \sum_{i=1}^{k} 2^{2^{i-1}-i-1},$$

$$B(p^k) = \sum_{i=1}^{k} \frac{1}{2p^i}\,(2^{p^i} - 2^{p^{i-1}}),$$

$$B(2p) = \frac{1}{4p}\,(4^p + (3p+1)2^p + 6p - 6),$$

$$B(pq) = \frac{1}{2pq}\,(2^{pq} - 2^p - 2^q + 2) + \frac{1}{2p}\,(2^p - 2) + \frac{1}{2q}\,(2^q - 2),$$

where p is an odd prime and q is another odd prime.

REMARK. The same argument works for coloring $S(m)$ with s colors. In this case the identity in Lemma 6.1 turns into

$$A_s(m) = (s-1)^m + (-1)^m(s-1)$$

and if we denote by $\Lambda_s(m)$ the set of all colorings on $S(m)$ with s colors, then the formula in Theorem 6.2 turns into

$$|\Lambda_s(m)/\mathfrak{D}_m| = \frac{1}{2m}\biggl(\sum_{2\le d|m} \varphi(m/d)A_s(d) + \frac{1+(-1)^m}{2}\cdot s\cdot (s-1)^{m/2}\cdot \frac{m}{2}\biggr).$$

The computation of $|\mathrm{GL}(2,\mathbb{Z}_2)\backslash\Lambda(m)/\mathfrak{D}_m|$ can be done in a similar fashion but is rather complicated. We note that the action of $\mathrm{GL}(2,\mathbb{Z}_2)$ on $\Lambda(m)$ permutes the three colors used to color $S(m)$. $\mathrm{GL}(2,\mathbb{Z}_2)$ consists of six elements; three of them are of order 2 and two are of order 3.

Theorem 6.4. *Let α and β be the functions defined as follows:*

$$\alpha(1)=1,\quad \alpha(2)=3,\quad \alpha(3)=2,\quad \alpha(6)=4,$$
$$\beta(1)=0,\quad \beta(2)=2,\quad \beta(3)=2,\quad \beta(6)=4.$$

Then $|\mathrm{GL}(2,\mathbb{Z}_2)\backslash\Lambda(m)/\mathfrak{D}_m|$ is given by

$$\frac{1}{2m}\bigg[\sum_{d|m}\Big\{\varphi(m/d)\cdot\frac{1}{6}\big(\alpha((m/d,6))A(d)+\beta((m/d,6))A(d-1)\big)\Big\}+E(m)\bigg],$$

where $(m/d,6)$ denotes the greatest common divisor of m/d and 6, $A(q)=2^q+(-1)^q2$ as before, and

$$E(m)=\begin{cases}(m/6)A((m+1)/2) & \text{if } m \text{ is odd},\\ m\cdot 2^{m/2-1} & \text{if } m \text{ is even.}\end{cases}$$

Proof. Applying the Burnside lemma to our $\mathfrak{D}_m$-set

$$\Gamma(m):=\mathrm{GL}(2,\mathbb{Z}_2)\backslash\Lambda(m),$$

we have

$$|\mathrm{GL}(2,\mathbb{Z}_2)\backslash\Lambda(m)/\mathfrak{D}_m|=\frac{1}{2m}\sum_{g\in\mathfrak{D}_m}|\Gamma(m)^g| \tag{6.4}$$

$$=\frac{1}{2m}\sum_{k=0}^{m-1}(|\Gamma(m)^{a^k}|+|\Gamma(m)^{a^kb}|)=\frac{1}{2m}\bigg[\sum_{d|m}\varphi(m/d)|\Gamma(m)^{a^d}|+\sum_{k=0}^{m-1}|\Gamma(m)^{a^kb}|\bigg].$$

We need to analyze $|\Gamma(m)^{a^d}|$ with $d\,|\,m$, and $|\Gamma(m)^{a^kb}|$.

First we deal with $|\Gamma(m)^{a^d}|$ with $d\,|\,m$. Note that $\lambda\in\Lambda(m)$ is a representative of $\Gamma(m)^{a^d}$ if and only if there is $\sigma\in\mathrm{GL}(2,\mathbb{Z}_2)$ such that

$$\sigma\circ\lambda=\lambda\circ a^d. \tag{6.5}$$

Since a^d is of order m/d, the repeated use of (6.5) shows that

$$\sigma^{m/d}=1. \tag{6.6}$$

The identity (6.5) implies that the λ satisfying (6.5) can be determined by the coloring restricted to the union of d consecutive arcs, say T, and it also tells us how to recover λ from the coloring on T.

Let μ be a coloring on T. We shall count the colorings λ on $S(m)$ which are extensions of μ and satisfy (6.5) for some $\sigma\in\mathrm{GL}(2,\mathbb{Z}_2)$. For each σ satisfying (6.6), there is a unique extension to $S(m)$ which satisfies (6.5). However, the extension may not be a coloring, i.e., two arcs meeting at a junction of T and its translations by rotations $(a^d)^r$ $(r = 1, \dots, m/d - 1)$ may have the same color. Let t and t' be the end arcs of T such that the rotation of t by a^{d-1} is t'. (Note: When $d = 1$, we understand $t = t'$ and then the subsequent argument works.) The extension is a coloring if and only if

$$\sigma(\mu(t)) \neq \mu(t'). \tag{6.7}$$

As is easily checked, the number of σ satisfying conditions (6.6) and (6.7) is $\alpha((m/d, 6))$ if $\mu(t) \neq \mu(t')$, and $\beta((m/d, 6))$ if $\mu(t) = \mu(t')$. On the other hand, the number of μ with $\mu(t) \neq \mu(t')$ is $A(d)$, and of those with $\mu(t) = \mu(t')$ is $A(d-1)$. It follows that the number of λ satisfying (6.5) for some σ is $\alpha((m/d, 6))A(d) + \beta((m/6, d))A(d-1)$. This proves that

$$|\Gamma(m)^{a^d}| = \tfrac{1}{6}\big(\alpha((m/d, 6))A(d) + \beta((m/6, d))A(d-1)\big) \tag{6.8}$$

since the action of $\mathrm{GL}(2, \mathbb{Z}_2)$ on $\Lambda(m)$ is free by Lemma 5.2, and the order of $\mathrm{GL}(2, \mathbb{Z}_2)$ is 6.

Next we deal with $|\Gamma(m)^{a^k b}|$. A similar argument shows that $\lambda \in \Lambda(m)$ is a representative of $\Gamma(m)^{a^k b}$ if and only if there is $\sigma \in \mathrm{GL}(2, \mathbb{Z}_2)$ such that

$$\sigma \circ \lambda = \lambda \circ a^k b. \tag{6.9}$$

Since $a^k b$ is of order two, the repeated use of (6.9) shows that

$$\sigma^2 = 1. \tag{6.10}$$

Suppose that m is odd. Then the line fixed by $a^k b$ goes through a vertex, say v, of $S(m)$ and the midpoint of the arc, say e', of $S(m)$ opposite to the vertex v. Let H be the union of $(m+1)/2$ consecutive arcs starting from v and ending at e'. Let e be the other end arc of H different from e'. The arc e has v as a vertex. Let ν be a coloring on H and let $\sigma \in \mathrm{GL}(2, \mathbb{Z}_2)$ satisfy (6.10). Then ν has an extension to a coloring of $S(m)$ satisfying (6.9) if and only if

$$\sigma(\nu(e)) \neq \nu(e) \quad \text{and} \quad \sigma(\nu(e')) = \nu(e').$$

It follows that $\nu(e)$ must be different from $\nu(e')$ and there is only one σ satisfying the two identities above for each such ν. Since the number of ν with $\nu(e) \neq \nu(e')$ is $A((m+1)/2)$, so is the number of $\lambda \in \Lambda(m)$ satisfying (6.9) for some σ. It follows that $|\Gamma(m)^{a^k b}| = \frac{1}{6}A((m+1)/2)$ and hence

$$\sum_{k=0}^{m-1} |\Gamma(m)^{a^k b}| = \frac{m}{6}\, A((m+1)/2). \tag{6.11}$$

Suppose that m is even and k is odd. Then the line fixed by $a^k b$ goes through the midpoints of two opposite arcs, say e and e', of $S(m)$. Let H be the union of $m/2 + 1$ consecutive arcs starting from e and ending at e'. Let ν be a coloring on H and let $\sigma \in \mathrm{GL}(2, \mathbb{Z}_2)$ satisfy (6.10). Then ν has an extension to a coloring of $S(m)$ satisfying (6.9) if and only if

$$\sigma(\nu(e)) = \nu(e) \quad \text{and} \quad \sigma(\nu(e')) = \nu(e').$$

If $\nu(e) \neq \nu(e')$ then such a σ must be the identity, and if $\nu(e) = \nu(e')$ then there are two such σ, one of which is the identity. Since the number of ν with $\nu(e) \neq \nu(e')$ is $A(m/2+1)$, and of those with $\nu(e) = \nu(e')$ is $A(m/2)$, the number of $\lambda \in \Lambda(m)$ satisfying (6.9) for some σ is $A(m/2+1) + 2A(m/2)$. It follows that

$$\sum_{k=0,\, k\,\mathrm{odd}}^{m-1} |\Gamma(m)^{a^k b}| = \frac{m}{12}\,(A(m/2+1) + 2A(m/2)). \tag{6.12}$$

Suppose that m is even and k is even. Then the line fixed by $a^k b$ goes through two opposite vertices, say v and v', of $S(m)$. Let H be the union of $m/2$ consecutive arcs starting from v and ending at v'. Let e and e' be the end arcs of H which respectively have v and v' as a vertex. Let ν be a coloring on H and let $\sigma \in \mathrm{GL}(2, \mathbb{Z}_2)$ satisfy (6.10). Then ν has an extension to a coloring of $S(m)$ satisfying (6.9) if and only if

$$\sigma(\nu(e)) \neq \nu(e) \quad \text{and} \quad \sigma(\nu(e')) \neq \nu(e').$$

If $\nu(e) \neq \nu(e')$ then there is only one such σ, and if $\nu(e) = \nu(e')$ then there are two. Since the number of ν with $\nu(e) \neq \nu(e')$ is $A(m/2)$, and of those with $\nu(e) = \nu(e')$ is $A(m/2-1)$, the number of $\lambda \in \Lambda(m)$ satisfying (6.9) for some σ is $A(m/2) + 2A(m/2-1)$. It follows that

$$\sum_{k=0,\, k\,\mathrm{even}}^{m-1} |\Gamma(m)^{a^k b}| = \frac{m}{12}\,(A(m/2) + 2A(m/2-1)). \tag{6.13}$$

Thus, when m is even, it follows from (6.12) and (6.13) that

$$\begin{aligned} \sum_{k=0}^{m-1} |\Gamma(m)^{a^k b}| &= \frac{m}{12}\,(A(m/2+1) + 3A(m/2) + 2A(m/2-1)) \\ &= m \cdot 2^{m/2-1} \end{aligned} \tag{6.14}$$

where we have used $A(q) = 2^q + (-1)^q 2$ at the latter identity.

The theorem now follows from (6.4), (6.8), (6.11) and (6.14). ∎

Remark. When m is even, $\Lambda(m)$ contains exactly three colorings with two colors and it defines the unique element in the double coset space $\mathrm{GL}(2, \mathbb{Z}_2)\backslash\Lambda(m)/\mathfrak{D}_m$.

Example 6.5. We set

$$C(m) := |\mathrm{GL}(2, \mathbb{Z}_2)\backslash\Lambda(m)/\mathfrak{D}_m|.$$

Using the formula in Theorem 6.4, one finds that

$$C(2) = 1, \quad C(3) = 1, \quad C(4) = 2, \quad C(5) = 1, \quad C(6) = 4, \quad C(7) = 3,$$
$$C(8) = 8, \quad C(9) = 8, \quad C(10) = 18, \quad C(11) = 21, \quad C(12) = 48.$$

We conclude this section with a remark. When Q is an m-gon ($m \geq 3$), a small cover over Q is a closed surface with Euler characteristic $4 - m$ and the cardinality of the set of homeomorphism classes in small covers over Q is one (resp. two) when m is odd (resp. even). On the other hand, the double coset space (5.2) identifies with $\mathrm{GL}(2, \mathbb{Z}_2)\backslash\Lambda(m)/\mathfrak{D}_m$ and we see from Theorem 6.4 that its cardinality is strictly larger than 2 when $m \geq 6$. So, the natural surjective map from the double coset space (5.2) to the set of homeomorphism classes in small covers over Q is not injective when Q is an m-gon with $m \geq 6$. However, it is bijective when $m = 3, 4, 5$ (see Example 6.5).

7. Locally standard 2-torus manifolds of dimension two. We shall count the equivariant homeomorphism classes in locally standard 2-torus manifolds with orbit space Q when Q is a compact surface with only one boundary component.

THEOREM 7.1. *Suppose that Q is a compact surface with only one boundary component with m (≥ 2) vertices and set*

$$h(Q) := |H^1(Q; (\mathbb{Z}_2)^2)/\mathrm{Aut}(Q)|.$$

Then the number of equivariant homeomorphism classes in locally standard 2-torus manifolds over Q is $h(Q)B(m)$, where $B(m) = |\Lambda(m)/\mathfrak{D}_m|$ is the number discussed in the previous section.

Proof. By Corollary 5.5 it suffices to count the number of orbits in $H^1(Q; (\mathbb{Z}_2)^2) \times \Lambda(Q)$ under the diagonal action of $\mathrm{Aut}(Q)$. Since Q has only one boundary component and m vertices, $\Lambda(Q)$ can be identified with $\Lambda(m)$ of Section 6, and $\mathrm{Aut}(\mathcal{F}(Q))$ is isomorphic to the dihedral group $\mathfrak{D}_m$.

Let H be the normal subgroup of $\mathrm{Aut}(Q)$ which acts on $H^1(Q; (\mathbb{Z}_2)^2)$ trivially. We claim that the restriction of the natural homomorphism

$$\mathrm{Aut}(Q) \to \mathrm{Aut}(\mathcal{F}(Q)) \cong \mathfrak{D}_m \tag{7.1}$$

to H is still surjective. An automorphism of Q (as a manifold with corners) which rotates the boundary circle and fixes the exterior of its neighborhood is an element of H. Therefore H contains all rotations in $\mathfrak{D}_m$. It is not difficult to see that any closed surface admits an involution which has one-dimensional fixed point component and acts trivially on the cohomology with $\mathbb{Z}_2$ coefficients. Since Q is obtained from a closed surface by removing an invariant open disk centered at a point in the one-dimensional fixed point set, Q admits an involution which reflects the boundary circle and lies in H. This implies the claim.

Let K be the kernel of the homomorphism $\mathrm{Aut}(Q) \to \mathrm{Aut}(\mathcal{F}(Q))$. Then

$$\begin{aligned}(7.2)\quad &|(H^1(Q;(\mathbb{Z}_2)^2) \times \Lambda(Q))/\mathrm{Aut}(Q)| \\ &\qquad = |(H^1(Q;(\mathbb{Z}_2)^2)/K \times \Lambda(Q))/\mathrm{Aut}(Q)|.\end{aligned}$$

For any element g in $\mathrm{Aut}(Q)$, there is an element h in H such that gh lies in K because the map (7.1) restricted to H is surjective. Since H acts trivially on $H^1(Q;(\mathbb{Z}_2)^2)$, this shows that an $\mathrm{Aut}(Q)$-orbit in $H^1(Q;(\mathbb{Z}_2)^2)$ is the same as a K-orbit. This means that the induced action of $\mathrm{Aut}(Q)$ on $H^1(Q;(\mathbb{Z}_2)^2)/K$ is trivial. Therefore the right hand side at (7.2) reduces to

$$|H^1(Q;(\mathbb{Z}_2)^2)/\mathrm{Aut}(Q))|\,|\Lambda(Q)/\mathrm{Aut}(Q)|.$$

Here the first factor is $h(Q)$ by definition, and the second is $|\Lambda(m)/\mathfrak{D}_m| = B(m)$ by the surjectivity of (7.1), proving the theorem. ∎

EXAMPLE 7.2. $H^1(Q;(\mathbb{Z}_2)^2)$ is isomorphic to $H^1(Q;\mathbb{Z}_2) \oplus H^1(Q;\mathbb{Z}_2)$ and the action of $\mathrm{Aut}(Q)$ on the direct sum is diagonal. When Q is a disk, $h(Q) = 1$. When Q is a real projective plane with an open disk removed, $H^1(Q;\mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2$ and the action of $\mathrm{Aut}(Q)$ on it is trivial. Therefore, $h(Q) = 4$ in this case. When Q is a torus with an open disk removed, $H^1(Q;\mathbb{Z}_2)$ is isomorphic to $(\mathbb{Z}_2)^2$. The action of $\mathrm{Aut}(Q)$ on it is non-trivial and it is not difficult to see that $h(Q) = 5$ in this case.

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Institute of Mathematics
School of Mathematical Science
Fudan University
Shanghai 200433, P.R. China
E-mail: zlu@fudan.edu.cn

Department of Mathematics
Osaka City University
Sumiyoshi-ku, Osaka 558-8585, Japan
E-mail: masuda@sci.osaka-cu.ac.jp

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