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A NOTE ON STEINHORN'S OMITTING TYPES THEOREM

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Abstract. Let $p(x)$ be a nonprincipal type. We give a sufficient condition for a model M to have a proper elementary extension omitting $p(x)$. As a corollary, we obtain a generalization of Steinhorn's omitting types theorem to the supersimple case.

1. Introduction. The well-known omitting types theorem states that if $p(x)$ is a nonprincipal type in a countable theory T then there is a model of T that omits $p(x)$. There are many variants of this theorem. Among such is that of Steinhorn [5]. He proved the following:

(∗) Let T be superstable and M a countable model of T containing an infinite indiscernible set. If $p(x)$ is a type over a finite set in M that is omitted in M then there is a proper elementary extension N of M that also omits $p(x)$.

In this paper we generalize this result to the case where T is not necessarily stable. As the original proof uses the notion of average types, it cannot be applied to unstable cases. We take a quite different approach. We directly use the ordinary omitting types theorem. With this approach, we can generalize Steinhorn's theorem to the supersimple case, and the proof becomes very short.

In this paper, we say that a model M is finitely generated if there is a finite tuple a in M such that any element in M is algebraic over a. In this terminology, one of our results is a slight generalization of the following:

THEOREM. *Let $p(x)$ be a type over $\emptyset$. Let M be a countable model omitting $p(x)$. Suppose that M is not finitely generated. Then there is a proper elementary extension N of M that also omits $p(x)$.*

2. Preliminaries. Our notations and definitions are standard. We briefly explain some of them. Throughout, T is a countable complete theory formulated in a countable language L. We work in a big model $\mathcal{M}$ of T. (In some situations, we work in $\mathcal{M}^{\mathrm{eq}}$.) $M, N, \dots$ are used to denote

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elementary submodels of $\mathcal{M}$. They are countable unless otherwise stated. $A, B, \ldots$ are used to denote small subsets of $\mathcal{M}$. Finite tuples of $\mathcal{M}$ are denoted by a, b and so forth. We write $a \in A$ if every member of a belongs to A.

Formulas are denoted by φ, ψ and so forth. If the parameters of φ are contained in A, then φ is called an *$L(A)$-formula.* A set $p(x)$ of formulas with the free variable x is called a *type* if $p(x)$ is finitely satisfiable in $\mathcal{M}$. In addition, if $p(x)$ consists of only $L(A)$-formulas, then we say that $p(x)$ is a *type over A*. We say that a type $p(x)$ is *isolated by a consistent formula* $\varphi(x)$ if for any $\psi(x) \in p(x)$ we have

$$\mathcal{M} \models \forall x[\varphi(x) \to \psi(x)].$$

A formula $\varphi(x)$ is said to be *algebraic* if there are only finitely many solutions to $\varphi(x)$. An element $a \in \mathcal{M}$ is said to be *algebraic over A* if a satisfies some algebraic $L(A)$-formula. $\mathrm{acl}(A)$ is the set of all algebraic elements over A. If A is countable, then so is $\mathrm{acl}(A)$. The set of all solutions to $\varphi(x)$ in M is denoted by $\varphi(x)^M$. If $\varphi(x)$ is an $L(M)$-formula, then a set D of the form $\varphi(x)^M$ is called a *definable set* in M. If $\varphi(x)$ is an $L(A)$-formula, then D is said to be A-*definable* or *definable over A*.

Definition 1. Let $D \subset M$ be an infinite definable set. We say that D is *finitely generated* if there is a finite tuple $m \in M$ such that $D \subset \mathrm{acl}(m)$.

Remark 2.

(i) If M is finitely generated, then it is countable. If M is not finitely generated, then no expansion of M by a finite set of constants is finitely generated.

(ii) If T is superstable and $D \subset M$ is finitely generated, then D is finitely generated by elements from D: Choose $m \in M$ with $D \subset \mathrm{acl}(m)$. By superstability, there is a finite tuple $a \in D$ such that $p = \mathrm{tp}(m/D)$ does not fork over a. For each $d \in D$, choose an algebraic formula $\varphi_d(x, m) \in \mathrm{tp}(d/m)$. By our choice of a, using definability of p, we can find a formula $\theta_d(x, e)$ with $e \in \mathrm{acl}^{\mathrm{eq}}(a)$ such that

$$M \models \varphi_d(b, m) \;\Leftrightarrow\; M \models \theta_d(b) \qquad (\forall b \in D).$$

Let $e_1, \ldots, e_k$ be all the conjugates of e over a. Then $\bigvee_{i=1,\ldots,k} \theta_d(x, e_i)$ is an algebraic formula in $L(a)$. This shows that D is finitely generated by $a \in D$. (Here D need not be a definable set.)

For understanding the main theorem, only a basic knowledge of model theory is necessary (see [1]). But for corollaries, the reader is assumed to have some knowledge of stability and simplicity (see [3], [4] or [7]).

3. Main results

THEOREM 3. *Let $p(x)$ be a type over a finite set of M, where M is a countable model omitting $p(x)$. Suppose that $D = \delta^M$ is not finitely generated. Then there is a proper elementary extension N of M with the following properties*:

(i) $\delta^N \supsetneq D$;

(ii) *N also omits $p(x)$.*

Proof. We can assume that $p(x)$ is a type over $\emptyset$ and that $\delta(x)$ is an L-formula. Now we prove the contraposition. Assume that $p(x)$ is realized in any proper elementary extension N of M with $\delta^N \supsetneq D$. We shall show that D is finitely generated. Let T^* be the elementary diagram of M. By the omitting types theorem, we can find a formula $\varphi(x, y)$ with parameter m from M with the following properties:

- $T^* \cup \{\delta(y)\} \cup \{y \neq a : a \in D\} \cup \{\varphi(x, y)\}$ is consistent;
- $T^* \cup \{\delta(y)\} \cup \{y \neq a : a \in D\} \cup \{\varphi(x, y)\} \vdash p(x)$.

Choose L-formulas $\theta_n(x)$ $(n \in \omega)$ such that $\{\theta_n(x) : n \in \omega\}$ is equivalent to $p(x)$ and such that $T \vdash \forall x(\theta_{n+1}(x) \to \theta_n(x))$ $(n \in \omega)$. Then we can find an increasing sequence $\{A_n : n \in \omega\}$ of finite subsets of D such that

$$T^* \cup \{\delta(y)\} \cup \Big\{ \bigwedge_{a \in A_n} y \neq a \Big\} \cup \{\varphi(x, y)\} \vdash \theta_n(x).$$

We may assume that the A_n's were chosen as small as possible. Then, by rewriting the above, we have

$$M \models \exists x[\delta(y) \wedge \varphi(x, y) \wedge \neg\theta_n(x)] \leftrightarrow \bigvee_{a \in A_n} y = a.$$

If there were $d \in D \setminus \bigcup_{n\in\omega} A_n$, then $\varphi(x, d)$ would isolate $p(x)$. So we must have $D = \bigcup_{n\in\omega} A_n$. Then D is covered by algebraic formulas $\exists x[\delta(y) \wedge \varphi(x, y) \wedge \neg\theta_n(x)]$, all of which are L-formulas. So $D \subset \operatorname{acl}(\emptyset)$. ∎

In the above, we treated the case where only one type $p(x)$ is considered. The theorem can be easily generalized to the case of countably many types.

COROLLARY 4. *Let S be a countable set of types over $\emptyset$. Let M be a countable model omitting all types in S. Suppose that $D = \delta^M$ is not finitely generated. Then there is a proper elementary extension N of M with the following properties*:

(i) $\delta^N \supsetneq D$;

(ii) *N also omits all types in S.*

The following corollary generalizes the main result of Steinhorn [5] to the supersimple case. Basic properties of simplicity can be found in [2] and [7]. However, the only property of supersimplicity which we need is that every finite tuple of elements has a finite weight. Namely, if T is supersimple and I is an infinite independent set then for any tuple a there is $b \in I$ such that a and b are independent.

Corollary 5. *Let T be supersimple. Suppose that M contains an infinite independent* (*nonalgebraic*) *set I. Suppose also that M omits $p(x)$. Then there is a proper elementary extension N of M that also omits $p(x)$.*

Proof. By Theorem 3, it is sufficient to show that M is not finitely generated. Suppose otherwise. Then M is finitely generated by a finite tuple m. In particular, for any $a \in I$, the tuples a and m are dependent. This shows that the weight of m is infinite, contradicting the supersimplicity. ■

Remark 6. Suppose that a model M of supersimple theory contains an infinite indiscernible sequence $I = \{a_i\}_{i\in\omega}$. Then, by supersimplicity, there is $n \in \omega$ such that $I \setminus \{a_i\}_{i<n}$ is independent over $\{a_i\}_{i<n}$. So Corollary 5 generalizes Steinhorn's omitting types theorem.

Corollary 7. *Let T be supersimple. Suppose that M contains an infinite* (*nontrivial*) *indiscernible sequence. Suppose also that M omits $p(x)$. Then there is a proper elementary extension N of M that also omits $p(x)$.*

The following theorem is an analogue of Corollary 3.2 in [5].

Theorem 8. *Let T be a small supersimple theory. Suppose that $M \models T$ is not finitely generated. Let S be the set of all types in $S(\emptyset)$ that are not realized in M. Then there is an uncountable elementary extension M^* of M that also omits S.*

Proof. We will construct an elementary chain $\{M_i : i \in \omega_1\}$ of countable models with the following properties:

- $M_0 = M$, $M_i \precneqq M_{i+1}$ ($i \in \omega_1$).
- M_i does not realize a type in S.
- M_i is not finitely generated.

If we have such an elementary chain, then $M^* = \bigcup_{i\in\omega_1} M_i$ is an uncountable model omitting S. So suppose that we have constructed M_j's for $j < i$. If i is a limit ordinal, then we can simply put $M_i = \bigcup_{j<i} M_j$. So we can concentrate on the case $i = j + 1$. Since S is a countable set (by the smallness of T), by Corollary 4, there is a proper extension $M_i \supset M_j$ that omits all the types in S. It only remains to show the following:

Claim A. *M_i is not finitely generated.*

Suppose otherwise. Then there is $m \in M_i$ such that $M_i = \mathrm{acl}(m)$. Since M_j is not finitely generated, we have a sequence $\{a_k : k \in \omega\}$ of finite tuples of M_j such that $a_{k+1} \notin \mathrm{acl}(a_0, \ldots, a_k)$ for each k. If a_{k+1} and m were independent over $\{a_0, \ldots, a_k\}$, then we would have $a_{k+1} \in \mathrm{acl}(a_0, \ldots, a_k)$. So the types $p_k = \mathrm{tp}(m/a_0, \ldots, a_k)$ form a forking sequence of infinite length. This contradicts the supersimplicity. ■

4. Examples. The following example is from Example 2.9 in [5].

EXAMPLE 9. Let $M = (M; D^M, R^M, B^M, F^M, <^M)$, where the universe M is the disjoint union of three countable sets D^M, R^M and F^M; D^M and R^M are disjoint copies of ω; $<^M$ is the natural ordering of D^M, i.e. $(D^M, <^M) = (\omega, <)$; B^M is the set of all bijections $f : D^M \to R^M$ that are identical on a cofinite subset of ω; $F^M \subset B^M \times D^M \times R^M$ is defined by

$$M \models F(f, m, n) \;\Leftrightarrow\; f(m) = n.$$

It is easy to see that $T = \mathrm{Th}(M)$ is unstable. (In fact, T has both the strict order property and the independence property.) For any two finite sets $A, B \subset R^M$ of the same cardinality, there is an automorphism sending A to B. So R^M is an indiscernible set.

Let $p(x)$ be the type $\{D(x)\} \cup \{n < x : n \in D^M\}$. Each $n \in D^M$ is definable over $\emptyset$, so $p(x)$ is a type over $\emptyset$. In [5] it was shown that there is no proper elementary extension $N \succ M$ omitting $p(x)$. The argument there also shows that $M = \mathrm{dcl}(f)$. So M is finitely generated.

The following example shows that there is a finitely generated model which contains an infinite indiscernible sequence.

EXAMPLE 10. Let $L = \{U, V, R_0, R_1, \ldots\}$, where U and V are unary relation symbols, and each R_i $(i \in \omega)$ is a binary relation symbol. We define the following L-structure M.

- The universe is the disjoint union of U^M and V^M;
- $U^M = \omega$; V^M is the set of all bijections $f : \omega \to \omega$ that are identical on a cofinite set;
- $R_i^M \subset V^M \times U^M$, $M \models R_i(f, n) \Leftrightarrow f(n) = i$.

It is not hard to see that $T = \mathrm{Th}(M)$ is stable and not superstable. Moreover, M has the following properties.

1. *U^M is an indiscernible set.*

Let A and B be two finite subsets of U^M with $|A| = |B|$. We show that there is an automorphism σ of M with $\sigma(A) = B$. First choose a bijection $\sigma \in V$ such that $\sigma(A) = B$. We extend σ by defining $\sigma(f) = f \circ \sigma^{-1}$ for $f \in V^M$. Then we have $R_i(f, n) \Leftrightarrow f(n) = i \Leftrightarrow \sigma(f)(\sigma(n)) = i \Leftrightarrow R_i(\sigma(f), \sigma(n))$. This shows that σ is an automorphism of M.

2. *Let $f \in V^M$. Then $U^M \subset \mathrm{dcl}(f)$. In particular, U^M is finitely generated by f.*

For any $n \in U^M$, $R_{f(n)}(f, n)$ holds in M. Since f is one-to-one, n is the unique element satisfying the formula $R_{f(n)}(f, x)$. So we have $n \in \mathrm{dcl}(f)$.

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REFERENCES

[1] C. C. Chang and H. J. Keisler, *Model Theory*, Stud. Logic Found. Math. 73, North-Holland, 1990.
[2] B. Kim, *Forking in simple unstable theories*, J. London Math. Soc. 57 (1998), 257–267.
[3] A. Pillay, *An Introduction to Stability Theory*, Oxford Logic Guides, Clarendon Press, 1983.
[4] S. Shelah, *Classification Theory and the Number of Nonisomorphic Models*, rev. ed., Stud. Logic Found. Math. 92, North-Holland, 1990.
[5] C. Steinhorn, *A new omitting types theorem*, Proc. Amer. Math. Soc. 89 (1983), 480–486.
[6] A. Tsuboi, *Models omitting given complete types*, Notre Dame J. Formal Logic 49 (2008), 393–399.
[7] F. O. Wagner, *Simple Theories*, Math. Appl. 503, Kluwer, 2000.

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