COLLOQUIUM MATHEMATICUM

VOL. 115 2009 NO. 2

LIMITS OF TILTING MODULES

BY

CLEZIO A. BRAGA (Cascavel) and FLÁVIO U. COELHO (São Paulo)

Abstract. We study the problem of when a direct limit of tilting modules is still a tilting module.

Tilting theory first appeared in the context of finitely generated modules over artin algebras [12, 18] (see also [5]). Due to its success in this setting, several generalizations were considered. In this work we shall investigate when a direct limit of tilting modules is still a tilting module.

The motivation for the construction of such direct limits was inspired by the work of Buan and Solberg [13] who established conditions for an inverse limit of finitely generated cotilting modules to be still a cotilting module. Unfortunately, their proof cannot be dualized to tilting modules. In this paper, we shall use the notion of special preenvelope to prove a similar result for tilting modules (see 1.4 for definitions).

Let R be a ring with unity. We say that a (not necessarily finitely generated) R-module T is *tilting* provided: $\operatorname{pd} T < \infty$; $\operatorname{Ext}_R^i(T, T^{(I)}) = 0$ for each $i \geq 1$ and all sets I; and there exists an exact sequence

$$0 \to R \xrightarrow{f_0} T_0 \xrightarrow{f_1} \cdots \xrightarrow{f_r} T_r \to 0$$

with $T_i \in \operatorname{Add} T$ for each $0 \leq i \leq r$ (see [1, 2]). Our main result is as follows (see Section 1 for further definitions).

THEOREM. *Let R be a ring and $\{T^i\}_{i\in\mathbb{N}}$ be a sequence of tilting modules such that $\operatorname{Add} T^i \neq \operatorname{Add} T^j$ if $i \neq j$, $T^{i+1} \in (T^i)^{\perp}$ and $\operatorname{pd} T^i \leq n$. Then there exists another sequence $\{\overline{T}^i\}_{i\in\mathbb{N}}$ of tilting modules with $\operatorname{Add} \overline{T}^i = \operatorname{Add} T^i$, $\overline{T}^{i+1} \in (\overline{T}^i)^{\perp}$ and $\operatorname{pd} \overline{T}^i \leq n$ for some $n \geq 1$. This latter sequence is a direct system of monomorphisms such that $T = \varinjlim_{i\in\mathbb{N}} \overline{T}^i$ is a tilting module in $\operatorname{Mod} R$ and $\operatorname{pd} T \leq n+1$.*

In [11], we apply this result to the class of tilted algebras to construct infinitely generated tilting modules. The paper is organized as follows. After some preliminaries in Sections 1 and 2, we prove the above result in Section 3.

2000 *Mathematics Subject Classification*: 16D90, 16E30, 16G10.
Key words and phrases: tilting modules, direct limits.

DOI: 10.4064/cm115-2-6

1. Preliminaries

1.1. Throughout this work, R will denote a ring with unity. We shall denote by $\operatorname{Mod} R$ ($\operatorname{mod} R$) the category of all (finitely generated, respectively) left R-modules, while $\mathrm{FP}_\infty(R)$ will denote the (full) subcategory of $\operatorname{mod} R$ generated by the modules which admit countable projective resolutions in $\operatorname{mod} R$.

Given $M \in \operatorname{Mod} R$ and $i \in \mathbb{N}$, denote by $\Omega^i(M)$ the class of all ith syzygy modules occurring in a projective resolution of M. Set $\Omega^0 = \{M\}$ and $\Omega(M) = \bigcup_{i\in\mathbb{N}} \Omega^i(M)$. Analogously, denote by $\Omega^{-i}(M)$ the class of all ith cosyzygy modules occurring in an injective coresolution of M.

1.2. Let $\mathcal{C} \subseteq \operatorname{Mod} R$ be a class of modules. We say that $\mathcal{C}$ is *resolving* (or *coresolving*) provided: (i) $\mathcal{C}$ contains all projective (injective, respectively) modules; (ii) $\mathcal{C}$ is closed under direct summands and extensions; and (iii) $\mathcal{C}$ is closed under kernels of epimorphisms (cokernels of monomorphisms, respectively).

Define, for each $i \geq 1$,

$$\mathcal{C}^{\perp_i} = \operatorname{Ker}\operatorname{Ext}^i(\mathcal{C}, \_), \qquad {}^{\perp_i}\mathcal{C} = \operatorname{Ker}\operatorname{Ext}^i(\_, \mathcal{C}),$$

and

$$\mathcal{C}^{\perp} = \bigcap_{i\geq 1} \operatorname{Ker}\operatorname{Ext}^i(\mathcal{C}, \_), \qquad {}^{\perp}\mathcal{C} = \bigcap_{i\geq 1} \operatorname{Ker}\operatorname{Ext}^i(\_, \mathcal{C}).$$

Clearly, ${}^{\perp}\mathcal{C}$ is a resolving subcategory. Observe also that ${}^{\perp}\mathcal{C} \cap \operatorname{mod} R \subseteq \mathrm{FP}_\infty(R)$ (see [3, Lemma 1.1]).

1.3. We shall now recall the notions of preenvelope and precover introduced by Enochs [15] and independently by Auslander and Smalø [8] under the names of left and right approximations.

Let $\mathcal{C} \subseteq \operatorname{Mod} R$ be a class of modules and $X \in \operatorname{Mod} R$. A *$\mathcal{C}$-preenvelope* of X is a morphism $f : X \to M$ with $M \in \mathcal{C}$ such that the induced morphism $\operatorname{Hom}(M, Y) \xrightarrow{f_*} \operatorname{Hom}(X, Y)$ is surjective for all Y in $\mathcal{C}$. If, moreover, such an f is a monomorphism and $\operatorname{Coker}(f) \in {}^{\perp_1}\mathcal{C}$, then we say that f is a *special $\mathcal{C}$-preenvelope*, and denote it also by (M, f). Finally, $\mathcal{C}$ is said to be a *preenveloping class* provided each $X \in \operatorname{Mod} R$ has a special preenvelope.

Dually, we can define (special) precovers and precovering class (see [4] for details).

1.4. Let $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ be a pair of classes of modules in $\operatorname{Mod} R$. We say that $\mathcal{C}$ is a *cotorsion pair* provided $\mathcal{A} = {}^{\perp_1}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp_1}$, and we say that $\mathcal{C}$ is *generated* by a class (or by a set) $\mathcal{S}$ if $\mathcal{B} = \mathcal{S}^{\perp_1}$.

If $\mathcal{S} \subseteq \operatorname{Mod} R$ is closed under syzygies, then $\mathcal{S}^{\perp_1} = \mathcal{S}^{\perp}$ and since $\mathcal{S}^{\perp}$ is coresolving, it is easy to see that ${}^{\perp_1}(\mathcal{S}^{\perp}) = {}^{\perp}(\mathcal{S}^{\perp})$. Therefore, $({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$ is a cotorsion pair generated by $\mathcal{S}$. Observe also that if $M \in \operatorname{Mod} R$, then $({}^{\perp}(M^{\perp}), M^{\perp})$ is a cotorsion pair generated by the set $\Omega(M)$.

We say that a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *complete* provided $\mathcal{B}$ is a preenveloping class and $\mathcal{A}$ is a precovering class. Observe that any cotorsion pair $(\mathcal{A}, \mathcal{B})$ which is generated by a set of modules is complete (see [14, Theorem 10]).

We say that a class $\mathcal{B}$ of $\operatorname{Mod} R$ is of *finite type* provided there exists a class of modules $\mathcal{S} \subseteq \mathrm{FP}_\infty(R)$ such that $\mathcal{B} = \mathcal{S}^{\perp_1}$.

1.5. The next result will be important in our considerations. We observe that this result was proved for $i = 1$ in [14, Lemma 17].

LEMMA 1.1. *Let $C \in \operatorname{Mod} R$ and $i \in \mathbb{N}$. Let $(A_\alpha \mid \alpha \leq \mu)$ be a sequence of modules and $(f_{\alpha\beta} \mid \alpha \leq \beta \leq \mu)$ be a sequence of monomorphisms such that $\{(A_\alpha, f_{\alpha\beta}) \mid \alpha \leq \beta \leq \mu\}$ is a continuous direct system. If*

$$\operatorname{Ext}^i(A_{\alpha+1}/f_{\alpha,\alpha+1}(A_\alpha), C) = 0 \quad \textit{for each } \alpha + 1 \leq \mu,$$

then $\operatorname{Ext}^i(A_\mu, C) = 0$.

Proof. Consider the exact sequence obtained from an injective coresolution of C,

$$0 \to C \to I_0 \to I_1 \to \cdots \to I_{i-1} \to Q_{i-1} \to 0.$$

By dimension shifting we get

$$0 = \operatorname{Ext}^{i+1}(A_{\alpha+1}/f_{\alpha,\alpha+1}(A_\alpha), C) \cong \operatorname{Ext}^1(A_{\alpha+1}/f_{\alpha,\alpha+1}(A_\alpha), Q_{i-1}).$$

Now, by [14, Lemma 17], $\operatorname{Ext}^1(A_\mu, Q_{i-1}) = 0$. Using again dimension shifting, we finally have $\operatorname{Ext}^i(A_\mu, C) = 0$. ∎

1.6. Let n be a positive integer and $T \in \operatorname{Mod} R$. Following [1] (see also [2], we say that T is *n-tilting* provided:

(T_1) $\operatorname{pd} T \leq n$;
(T_2) $\operatorname{Ext}^i_R(T, T^{(I)}) = 0$ for each $i \geq 1$ and all sets I;
(T_3) there exists an exact sequence

$$0 \to R \xrightarrow{f_0} T_0 \xrightarrow{f_1} \cdots \xrightarrow{f_r} T_r \to 0$$

with $T_i \in \operatorname{Add} T$ for each $0 \leq i \leq r$.

A class of modules $\mathcal{T}$ is called *n-tilting* if there exists an n-tilting module T such that $\mathcal{T} = T^{\perp}$. Observe that an n-tilting module T is of finite type (that is, $T^{\perp}$ is a finite type class; see [9, 10]). Moreover, since each cotorsion pair generated by a set is complete, the cotorsion pair $\mathcal{C} = ({}^{\perp}\mathcal{T}, \mathcal{T})$ is complete.

Dually, one defines n-cotilting modules and classes.

2. Tilting theory

2.1. We initially present some generalizations on tilting theory from finitely to infinitely generated tilting modules. For later reference, we mention the following result and its dual. The proofs can be found in [1, p. 247].

PROPOSITION 2.1. *Let T be an r-tilting module. Then there exists an exact sequence*

$$0 \to R \xrightarrow{f_0} T_0 \xrightarrow{f_1} \cdots \xrightarrow{f_k} T_k \to 0$$

with $T_i \in \operatorname{Add} T$ for each i and such that

(a) $k \le r$,
(b) *each f_i is the composition of* $\operatorname{coker}(f_{i-1})$ *with a special $T^{\perp}$-preenvelope of* $\operatorname{coker}(f_{i-1})$,
(c) $\operatorname{Add}(\coprod_{i=0}^{k} T_i) = \operatorname{Add} T$.

We will call a sequence as in the above proposition a *T-coresolution* of R. If the T_i's are finitely generated, we refer to it as a *finitely generated T-coresolution* for R.

COROLLARY 2.2 ([13, Lemma 1]). *Let R be an artin algebra and $T \in \operatorname{mod} R$ be a tilting module. Then there exists a T-coresolution of R*

$$0 \to R \xrightarrow{f_0} T_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{r-1}} T_{r-1} \to T_r \to 0$$

with $T_i \in \operatorname{add} T$ and $\operatorname{add} T = \operatorname{add}(\coprod_{i=0}^{r} T_i)$.

2.2. We shall now present some results which will help us to relate the orthogonal classes of two tilting (or cotilting) modules T and N and to associate a T-coresolution of R to an N-coresolution of R (or T- and N-resolutions of an injective cogenerator Q in the cotilting case).

LEMMA 2.3. *Let U and T be two tilting modules in $\operatorname{Mod} R$ such that $U \in T^{\perp}$. Then $U^{\perp} \subseteq T^{\perp}$ and $\operatorname{pd} T \le \operatorname{pd} U$.*

Proof. Let $X \in U^{\perp}$. Then, by [17, 5.1.9], there exists an exact sequence

$$\cdots \to U_1 \xrightarrow{f_1} U_0 \xrightarrow{f_0} X \to 0 \tag{1}$$

with $U_i \in \operatorname{Add} U$ for each i.

Let $K_i = \operatorname{Ker}(f_i)$. If $\operatorname{pd} T = r$, it follows from the above sequence that $\operatorname{Ext}^i(T, X) \cong \operatorname{Ext}^{r+i}(T, K_r) = 0$, and so $X \in T^{\perp}$. Hence $U^{\perp} \subseteq T^{\perp}$ and ${}^{\perp}(U^{\perp}) \supseteq {}^{\perp}(T^{\perp})$. Therefore $X \in {}^{\perp}(T^{\perp})$ implies that $X \in {}^{\perp}(U^{\perp})$. So $\operatorname{pd} X \le \operatorname{pd} U$. In particular, $\operatorname{pd} T \le \operatorname{pd} U$. ∎

The above result still holds for the category of finitely generated modules over an artin algebra Λ since $T^{\perp} \cap \operatorname{mod} \Lambda$ is a preenveloping class and ${}^{\perp}(T^{\perp}) \cap \operatorname{mod} \Lambda$ is a precover class in $\operatorname{mod} \Lambda$. For more details, see [6].

LEMMA 2.4. *Let U and T be two tilting modules in* $\operatorname{Mod} R$ *such that* $U \in T^{\perp}$. *Assume* $\operatorname{pd} U = r$ *for some* $r \in \mathbb{N}$. *Let*

$$0 \to R \xrightarrow{\gamma} T_0 \xrightarrow{f_0} T_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{s-1}} T_s \to 0$$

be a T-coresolution of the regular module R. Then $s \leq r$ and there is a U-coresolution

$$0 \to R \xrightarrow{\delta} U_0 \xrightarrow{g_0} U_1 \xrightarrow{g_1} \cdots \xrightarrow{g_{r-1}} U_r \to 0$$

of R such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\gamma"] \arrow[d, equal] & T_0 \arrow[r] \arrow[d] & \cdots \arrow[r] & T_s \arrow[r] \arrow[d] & 0 \arrow[d] & & & \\
0 \arrow[r] & R \arrow[r, "\delta"] & U_0 \arrow[r, "g_0"] & \cdots \arrow[r] & U_s \arrow[r, "g_s"] & U_{s+1} \arrow[r] & \cdots \arrow[r] & U_r \arrow[r] & 0
\end{tikzcd}$$

commutes and each vertical map is a special $U^{\perp}$-preenvelope or zero.

Proof. Let

$$0 \to R \xrightarrow{\lambda} \widetilde{U}_0 \xrightarrow{z_1} \widetilde{U}_1 \xrightarrow{z_2} \widetilde{U}_2 \to \cdots \xrightarrow{z_r} \widetilde{U}_r \to 0$$

be a U-coresolution of R, which exists by Lemma 2.1, and set $K_0 = \operatorname{Coker}(\lambda)$. Then there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\gamma"] \arrow[d, equal] & T_0 \arrow[r] \arrow[d, "s_0"] & L_0 \arrow[r] \arrow[d, "\overline{s}_0"] & 0 \\
0 \arrow[r] & R \arrow[r, "\lambda"] & \widetilde{U}_0 \arrow[r, "p_0"] & K_0 \arrow[r] & 0
\end{tikzcd}$$

Indeed, since $\widetilde{U}_0 \in T^{\perp}$, there is an $s_0 : T_0 \to \widetilde{U}_0$ such that $\lambda = s_0 \circ \gamma$. By cokernel properties, the morphism $\overline{s}_0 : L_0 \to K_0$ makes the above diagram commutative.

Let $L_i = \operatorname{Coker}(f_i)$ and $K_i = \operatorname{Coker}(z_i)$ for each $i > 0$. Using induction, we get, for each $i > 0$, the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & L_{i-1} \arrow[r, "\gamma_i"] \arrow[d, "\overline{s}_{i-1}"] & T_i \arrow[r] \arrow[d, "s_i"] & L_i \arrow[r] \arrow[d, "\overline{s}_i"] & 0 \\
0 \arrow[r] & K_{i-1} \arrow[r, "\lambda_i"] & \widetilde{U}_i \arrow[r] & K_i \arrow[r] & 0
\end{tikzcd}$$

As before, for each $i > 0$, the pair (T_i, γ_i) is a special $T^{\perp}$-preenvelope of L_{i-1}, and since $\widetilde{U}_i \in T^{\perp}$, there exists $s_i : T_i \to \widetilde{U}_i$ such that $\lambda_i \overline{s}_{i-1} = s_i \circ \gamma_i$.

By Lemma 2.3 and the definition of T-coresolution we obtain $s \leq \operatorname{pd} T \leq r$.

Iterating the above process, we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d, equal] & T_0 \arrow[r] \arrow[d] & T_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & T_s \arrow[r] \arrow[d] & 0 & & \\
0 \arrow[r] & R \arrow[r, "\lambda"] & \widetilde{U}_0 \arrow[r, "z_1"] & \widetilde{U}_1 \arrow[r, "z_2"] & \cdots \arrow[r] & \widetilde{U}_s \arrow[r] & \cdots \arrow[r] & \widetilde{U}_r \arrow[r] & 0
\end{tikzcd}$$

It remains to show that the vertical maps are special $U^{\perp}$-preenvelopes or zeros.

Let $j_0 : L_0 \to \overline{U}_0$ be a special $U^{\perp}$-preenvelope of L_0. First we observe that $L_0 \in {}^{\perp}(T^{\perp}) \subset {}^{\perp}(U^{\perp})$.

Since $\mathrm{Coker}(j_0) \in {}^{\perp}(U^{\perp})$, then $\overline{U}_0 \in U^{\perp} \cap {}^{\perp}(U^{\perp}) = \mathrm{Add}\, U$. So, we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\gamma"] \arrow[d, equal] & T_0 \arrow[r] \arrow[d, "(s_0, j_0 \circ \mathrm{coker}(\gamma))"] & L_0 \arrow[r] \arrow[d, "(\overline{s}_0, j_0)"] & 0 \\
0 \arrow[r] & R \arrow[r, "(\lambda, 0)"] & \widetilde{U}_0 \amalg \overline{U}_0 \arrow[r, "\begin{pmatrix} p_0 & 0 \\ 0 & \mathrm{id} \end{pmatrix}"'] & K_0 \amalg \overline{U}_0 \arrow[r] & 0.
\end{tikzcd}$$

Since j_0 is a monomorphism, so is $(\overline{s}, j_0)$. It now follows from the five lemma that $(s_0, j_0 \circ \mathrm{coker}(\gamma))$ is a monomorphism.

An easy calculation shows that $(\widetilde{U}_0 \amalg \overline{U}_0, (s_0, j_0 \circ \mathrm{coker}(\gamma)))$ is a special $U^{\perp}$-preenvelope of T_0.

Observe now that $\mathrm{Coker}(\lambda, 0) \cong \mathrm{Coker}(\lambda) \amalg U_0$ is in ${}^{\perp}(U^{\perp})$, and so $(\widetilde{U}_0 \amalg \overline{U}_0, (\lambda, 0))$ is also a special $U^{\perp}$-preenvelope of R.

By induction, we have

$$\begin{tikzcd}
0 \arrow[r] & L_{i-1} \arrow[r] \arrow[d, "(\overline{s}_{i-1}, j_{i-1})"] & T_i \arrow[r] \arrow[d, "(s_i, \gamma_i)"] & L_i \arrow[r] \arrow[d, "(\overline{s}_i, j_i)"] & 0 \\
0 \arrow[r] & K_{i-1} \amalg \overline{U}_{i-1} \arrow[r] & \widetilde{U}_{i-1} \amalg \overline{U}_{i-1} \amalg \overline{U}_i \arrow[r] & K_i \amalg \overline{U}_i \arrow[r] & 0
\end{tikzcd}$$

with $(\widetilde{U}_{i-1} \amalg \overline{U}_{i-1} \amalg \overline{U}_i, (s_i, \gamma_i))$ a special $U^{\perp}$-preenvelope of T_i, as required. ∎

3. Tilting limits. In [1], Angeleri-Hügel and Coelho presented an example of an infinitely generated n-tilting module constructed from a direct sum of copies of a finitely generated n-tilting module M (see [1, 2.1]). Based on this idea, Buan and Solberg built in [13] a dual example for n-cotilting modules. They used a sequence of finitely generated n-cotilting modules and built an inverse system of n-cotilting modules (not exactly the same as in the first sequence) which has as inverse limit an infinitely generated n-cotilting module.

Our aim here is to use this process for a general direct limit of n-tilting modules to get a new infinitely generated $(n+1)$-tilting module.

3.1. *A direct system of tilting modules.* This section is devoted to constructing a direct system of tilting modules. The procedure we describe below is dual to that used by Buan and Solberg in [13], but the modules we consider do not need to be finitely generated.

Let R be a ring and $\{T^0, T^1, \dots\}$ be a sequence of tilting modules such that $\operatorname{Add} T^i \neq \operatorname{Add} T^j$ if $i \neq j$. Suppose that $T^{i+1} \in (T^i)^\perp$ and there exists an n such that $\operatorname{pd} T^i \leq n$ for each $i \in \mathbb{N}$.

It then follows, by Lemma 2.3, that $(T^{i+1})^\perp \subseteq (T^i)^\perp$ for each $i \in \mathbb{N}$.

Now, using induction and Lemma 2.4, we get a commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d, equal] & T_1^0 \arrow[r] \arrow[d, "f_1^0"] & T_2^0 \arrow[r] \arrow[d, "f_2^0"] & T_3^0 \arrow[r] \arrow[d, "f_3^0"] & \cdots \arrow[r] & T_n^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R \arrow[r] \arrow[d, equal] & T_1^1 \arrow[r] \arrow[d, "f_1^1"] & T_2^1 \arrow[r] \arrow[d, "f_2^1"] & T_3^1 \arrow[r] \arrow[d, "f_3^1"] & \cdots \arrow[r] & T_n^1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R \arrow[r] \arrow[d, equal] & T_1^2 \arrow[r] \arrow[d, "f_1^2"] & T_2^2 \arrow[r] \arrow[d, "f_2^2"] & T_3^2 \arrow[r] \arrow[d, "f_3^2"] & \cdots \arrow[r] & T_n^2 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}
\tag{2}
$$

where some T_j^i may be zeros, and each nonzero morphism $f_j^i : T_j^i \to T_j^{i+1}$ is a special $(T^{i+1})^\perp$-preenvelope of T_j^i.

These morphisms form a direct system in each column of diagram (2).

Consider now, for $j = 1, \dots, n$, the direct limit $\varinjlim_{i \in \mathbb{N}} T_j^i$ and denote it by T_j. Clearly, $\mathbb{N}$ is a directed set and so the direct limit is an exact functor. So we have the exact sequence

$$0 \to R \to T_1 \to T_2 \to \cdots \to T_n \to 0.$$

Adding the morphisms f_j^i in each line, we get a direct system $\{\overline{T}^i, f^i\}_{i \in \mathbb{N}}$, where $f^i = \coprod_{j=1}^n f_j^i$ and $\overline{T}^i = \coprod_{j=1}^n T_j^i$. Set $T = \coprod_{j=1}^n T_j$. Then

$$T = \coprod_{j=1}^n T_j = \coprod_{j=1}^n \varinjlim_{i \in \mathbb{N}} T_j^i \cong \varinjlim_{i \in \mathbb{N}} \coprod_{j=1}^n T_j^i = \varinjlim_{i \in \mathbb{N}} \overline{T}^i.$$

Since $(T^i)^\perp \subseteq (T^l)^\perp$ if $i \geq l$, we have $\operatorname{Ext}^m(\overline{T}^l, \overline{T}^i) = 0$ for all $m > 0$ and $i \geq l$.

This is the dual diagram to that obtained in [13] for the inverse system of finitely generated cotilting modules.

3.2. *Direct limits of tilting modules.* We now prove our main result, that is, that the module $\varinjlim_{i\in\mathbb{N}} \overline{T}_i$, constructed as above, is an infinitely generated $(n+1)$-tilting module.

THEOREM 3.1. *Let R be a ring and $\{T^i\}_{i\in\mathbb{N}}$ be a sequence of tilting modules such that* $\operatorname{Add} T^i \neq \operatorname{Add} T^j$ *if $i \neq j$, $T^{i+1} \in (T^i)^\perp$ and* $\operatorname{pd} T^i \leq n$*. Then there exists another sequence of tilting modules $\{\overline{T}^i\}_{i\in\mathbb{N}}$ with* $\operatorname{Add} \overline{T}^i = \operatorname{Add} T^i$, $\overline{T}^{i+1} \in (\overline{T}^i)^\perp$ *and* $\operatorname{pd} \overline{T}^i \leq n$*. This latter sequence consists of a direct system of monomorphisms such that $T = \varinjlim_{i\in\mathbb{N}} \overline{T}^i$ is a tilting module in* $\operatorname{Mod} R$ *and* $\operatorname{pd} T \leq n+1$.

Proof. First, consider the sequence $\{\overline{T}^i\}_{i\in\mathbb{N}}$ obtained from the original sequence $\{T^i\}_{i\in\mathbb{N}}$ from diagram (2).

We know that $\overline{T}^i$ is a tilting R-module and that $\operatorname{Add} \overline{T}^i = \operatorname{Add} T^i$ by Proposition 2.1, so $(T^i)^\perp = (\overline{T}^i)^\perp$. Hence $\overline{T}^{i+1} \in (T^i)^\perp = (\overline{T}^i)^\perp$.

It remains to show that T is an $(n+1)$-tilting module. Condition (T_3) is clear by the above construction.

In order to prove (T_2), we first observe that since $\operatorname{Ext}^m(\overline{T}^l, \overline{T}^i) = 0$ for $m > 0$ and $i \geq l$, and since $(\overline{T}^l)^\perp$ is closed under direct limits, it follows that $T^{(I)}$ belongs to $(\overline{T}^l)^\perp$ for any index set I.

Let C_k^{l+1} be the cokernel of f_k^l for each $l \geq 0$, and set $C_k^0 = T_k^0$. Hence, there exists for each $k \in \{1, \dots, n\}$ an exact sequence

$$0 \to T_k^l \xrightarrow{f_k^l} T_k^{l+1} \to C_k^{l+1} \to 0.$$

Taking the coproducts of these sequences we get another exact sequence

$$0 \to \overline{T}^l \xrightarrow{f^l} \overline{T}^{l+1} \to \coprod_{k=1}^{n} C_k^{l+1} \to 0. \tag{3}$$

Observe that f^l is a special $(\overline{T}^{l+1})^\perp$-preenvelope. Since $T^{(I)} \in (T^l)^\perp$ for all $l \leq 0$, applying the functor $\operatorname{Hom}(\_, T^{(I)})$ to this sequence, we obtain

$$\operatorname{Hom}(\overline{T}^{l+1}, T^{(I)}) \xrightarrow{(f^l)^*} \operatorname{Hom}(\overline{T}^l, T^{(I)}) \to \operatorname{Ext}^1\Big(\coprod_{k=1}^{n} C_k^{l+1}, T^{(I)}\Big) \to 0.$$

It then follows, using the fact that f^l is a special $(\overline{T}^{l+1})^\perp$-preenvelope, that $\operatorname{Ext}^1(\coprod_{k=1}^n C_k^{l+1}, T^{(I)}) = 0$. Using a similar argument we can deduce from the exact sequence in (3) that

$$\operatorname{Ext}^m\Big(\coprod_{k=1}^{n} C_k^0, T^{(I)}\Big) = 0 \quad \text{and} \quad \operatorname{Ext}^m\Big(\coprod_{k=1}^{n} C_k^{l+1}, T^{(I)}\Big) = 0,$$

for all $m > 1$. Condition (T_2) now follows from Lemma 1.1.

Now, since $\operatorname{pd} \overline{T}^i \leq n$ by construction, we deduce from sequence (3) that $\operatorname{pd}(\coprod_{k=1}^n C_k^{l+1}) \leq n+1$. Hence $\operatorname{Ext}^m(\coprod_{k=1}^n C_k^{l+1}, A) = 0$ for all $m > n+1$

and all modules A. By 1.1, we deduce that $\operatorname{Ext}^m(T,A)=0$ for all $m>n+1$ and all modules A. Hence $\operatorname{pd} T\le n+1$ and condition (T_1) is also proved.

PROPOSITION 3.2. *Let* $T=\varinjlim_{i\in\mathbb{N}}\overline{T}^i$ *be as in Theorem* 3.1. *Then*

$$T^{\perp}=\bigcap_{i\in\mathbb{N}}(T^i)^{\perp}.$$

Proof. By the proof of Theorem 3.1, $T=\varinjlim_{i\in\mathbb{N}}\overline{T}^i\in(\overline{T}^i)^{\perp}$ for all $i\in\mathbb{N}$. Then $T\in\bigcap_{i\in\mathbb{N}}(\overline{T}^i)^{\perp}$. Since each $\overline{T}^i$ is a tilting module, Lemma 2.3 yields $T^{\perp}\subset\bigcap_{i\in\mathbb{N}}(\overline{T}^i)^{\perp}$. Conversely, let $X\in\bigcap_{i\in\mathbb{N}}(\overline{T}^i)^{\perp}$. Then

$$\operatorname{Ext}^m\Big(\coprod_{k=1}^{n}C_k^l,X\Big)=0$$

for all $l\ge 0$, and so $\operatorname{Ext}^m(\varinjlim_{i\in\mathbb{N}}\overline{T}^i,X)=0$, by Lemma 1.1. Therefore $X\in T^{\perp}$. Hence $T^{\perp}=\bigcap_{i\in\mathbb{N}}(\overline{T}^i)^{\perp}$. ∎

For completeness, we state the similar result proved by Buan and Solberg [13] on inverse limits of cotilting modules.

THEOREM 3.3. *Let* $\{C_i\}_{i\in\mathbb{N}}$ *be a sequence of cotilting modules with* $\operatorname{id} C_i\le n$ *for some* $n>0$, *such that* $C_i\in{}^{\perp}C_{i-1}$ *for each* $i>0$. *Then the inverse limit* $X=\varprojlim_{i\in\mathbb{N}}C_i$ *is an infinitely generated cotilting module with* $\operatorname{id} X\le n$.

We finish this work by exhibiting an example to illustrate the above construction. Recall that an artin algebra Λ is *hereditary* if each submodule of a projective Λ-module is also projective, or equivalently, if $\operatorname{gldim}\Lambda\le 1$.

EXAMPLE. Let Λ be a representation-infinite, basic and indecomposable hereditary artin algebra. Consider the decomposition $T_0=\Lambda=\coprod_{i=1}^{n}P_i$ of Λ into indecomposable projective Λ-modules. For each $j\in\mathbb{N}$, we define $T_j=\coprod_{i=1}^{n}\tau^{-j}P_i$, where τ denotes the Auslander–Reiten translation (see [7]). Then the sequence $\{T_i\}_{i\in\mathbb{N}}$ satisfies the conditions of Theorem 3.1.

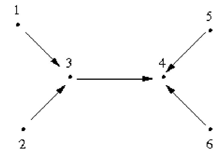
In fact, since Λ is hereditary, we have $\operatorname{pd} T_i\le 1$ for each $i\in\mathbb{N}$.

Let now j,l satisfy $1\le j\le l\le n$. Then

$$\operatorname{Ext}^1(T_j,T_l)\cong\coprod_{i,k}\operatorname{Ext}^1(\tau^{-j}P_i,\tau^{-l}P_k)\cong\coprod_{i,k}\operatorname{DHom}(\tau^{-l}P_k,\tau^{-j+1}P_i)=0.$$

by the Auslander–Reiten formula. Therefore T_j is selforthogonal and $T_l\in T_j^{\perp}$ for $j\le l$. Since all T_j decompose into sums of the same number of nonisomorphic simple Λ-modules, each T_i is a finitely generated 1-tilting module. Moreover, $\operatorname{add} T_{j+1}\ne\operatorname{add} T_j$ if $i\ne j$. By Azumaya decomposition ([16, Theorem 21.6]) we have $\operatorname{Add} T_i\ne\operatorname{Add} T_j$ if $i\ne j$. Hence, there exists an infinitely generated tilting module as that in 3.1 over hereditary algebras.

Let H be the finite-dimensional hereditary algebra given by the quiver



The picture below shows a sequence of tilting modules constructed as above in the postprojective component of the Auslander–Reiten quiver of H.

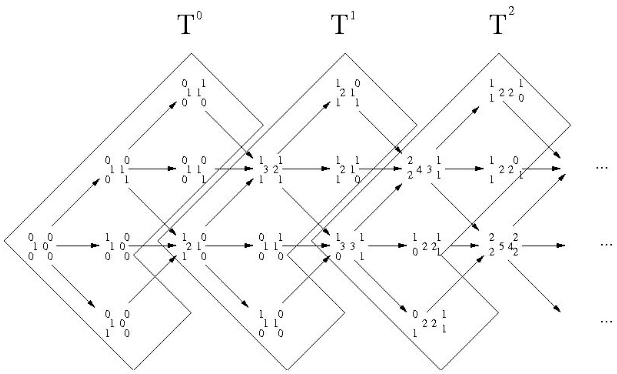


Fig. 1. Tilting sequence for the hereditary algebra H

Acknowledgments. This work is part of the PhD thesis written by the first author. Both authors would like to acknowledge partial support from the agencies CNPq and FAPESP.

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Centro de Ciências Exatas
e Tecnológicas – CCET
Universidade Estadual do Oeste do Paraná
Cascavel, PR, Brazil
E-mail: clezio@unioeste.br

Departamento de Matemática – IME
Universidade de São Paulo
São Paulo, Brazil
E-mail: fucoelho@ime.usp.br

Received 12 May 2008;
revised 9 October 2008 (5046)