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ON B-INJECTORS OF SYMMETRIC GROUPS S_n AND ALTERNATING GROUPS A_n: A NEW APPROACH

BY

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Abstract. The aim of this paper is to introduce the notion of BG-injectors of finite groups and invoke this notion to determine the B-injectors of S_n and A_n and to prove that they are conjugate. This paper provides a new, more straightforward and constructive proof of a result of Bialostocki which determines the B-injectors of the symmetric and alternating groups.

1. Introduction. N-injectors in a finite group G are maximal nilpotent subgroups which share many properties with Sylow subgroups. N-injectors were first defined by B. Fischer et al. [7] as follows: A subgroup A of G is an *N-injector* if for each $H \lhd\lhd G$, $A \cap H$ is a maximal nilpotent subgroup of H. A. Mann [10] proved that if $C_G(F(G)) \subseteq F(G)$, then G contains N-injectors, they form a conjugacy class, and they can be characterized as the maximal nilpotent subgroups which contain $F(G)$, the Fitting subgroup of G. If G is of odd order, a subgroup S of G is an N-injector if and only if S is a nilpotent subgroup of G of maximal order. (See A. Bialostocki [6, Cor. 5] and A. Mann [10, Thm. 1]). A. Bialostocki [4] defines a *B-injector* in a finite group G to be any maximal nilpotent subgroup B of G satisfying $d_2(B) = d_2(G)$, where $d_2(X) := \max\{|A| \mid A \leq X$ and A is nilpotent of class at most $2\}$. Bender [3] showed that if G is *N-constrained*, that is, $C_G(F(G)) \subseteq F(G)$, then A is an N-injector of G if and only if A is a maximal nilpotent subgroup of G containing an element of $a_2(G)$ where $a_2(G)$ is the set of all nilpotent subgroups of G, of class at most 2 and having order $d_2(G)$.

Sometimes B-injectors are called B-N-injectors or nilpotent injectors (see M. I. AlAli, Ch. Hering and A. Neumann [2], P. Flavell [8]). B-injectors and N-injectors of a finite group G are equivalent if G is N-constrained, and B-injectors are N-injectors for any finite group G (A. Neumann [11]).

B-injectors lead to theorems similar to Glaubermann's ZJ-Theorem and it is hoped that they will provide tools and arguments for a modified and

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shortened proof of the classification theorem for finite simple groups. This paper is a part of a greater programme of investigating the B-injectors in arbitrary groups, more precisely, investigating in which groups the B-injectors are conjugate. The symmetric groups S_n and the alternating groups A_n turn out to be critical in answering the question whether the B-injectors are conjugate or not.

2. General definitions and notations. Our notation is fairly standard. Throughout all groups are finite. If G is a group, $Z(G)$ denotes the center of G. If H and X are subsets of G, then $C_H(X)$ and $N_H(X)$ denote respectively the centralizer and normalizer of X in H.

The *generalized Fitting group* $F^*(G)$ is defined to be $F(G)E(G)$ where $E(G) = \langle L \mid L \lhd\lhd G \text{ and } L \text{ is quasisimple}\rangle$ is a subgroup of G. A group L is called *quasisimple* if $L' = L$ where L' is the derived group of L, and $L'/Z(L)$ is non-abelian simple. $O_p(G)$ denotes the unique maximal normal p-subgroup of G; it is the Sylow p-subgroup of $F(G)$, and $O_{p'}(G) = \prod O_q(G)$, where $q \neq p$ and q is prime. If Ω is a finite set, we denote by S_Ω, A_Ω the symmetric and alternating groups of Ω. If $|\Omega| = n$, we sometimes write S_n and A_n. Moreover, $\Phi(G)$ denotes the *Frattini subgroup* of G, the intersection of all maximal subgroups of G. The *Fitting subgroup* of G is the largest normal nilpotent subgroup of G and is denoted by $F(G)$. A permutation representation $\pi : H \to \mathrm{Sym}(Y)$ is *semiregular* if the identity element is the only element of H fixing points of Y. Equivalently $H_y = 1$ for all y in Y. The integer part of the real number x is denoted by $[x]$.

Definition 2.1. A nilpotent subgroup U of a group G is called a *BG-injector* of G if U contains every nilpotent subgroup of G that is normalized by U.

3. Preliminaries

Theorem 3.1 (A. Mann [10]). *Let U be a B-injector of G. Then U contains every nilpotent subgroup of G which is normalized by U.*

Corollary 3.1. *B-injectors are BG-injectors.*

Remark 3.1. It is clear that BG-injectors are maximal nilpotent and contain the Fitting group of G. Also if U is a BG-injector of G and if $U \leq H \leq G$, then U is a BG-injector of H.

We shall overview the BG-injectors in S_n and A_n, and single out the B-injectors among the BG-injectors. This works rather smoothly as the centralizers of elements of prime order in S_n have an easily accessible structure.

The following lemmas on BG-injectors are needed.

Lemma 3.1. *Let G be a finite group, and $U \le G$ be a BG-injector of G.*

(1) *If $Z \le Z(G)$ then $Z \le U$ and U/Z is a BG-injector of G/Z.*
(2) *If $F^*(G) = O_p(G)$ for some prime p, then U is a Sylow p-subgroup of G.*

Proof. (1) Let X/Z be a nilpotent subgroup of G/Z and $U/Z \le N_{G/Z}(X/Z)$. As $Z \le Z(G)$ and X/Z is nilpotent, X is nilpotent. Since U/Z normalizes X/Z, we see that U normalizes X. Thus $U \le N(X)$, and hence $X \le U$ and $X/Z \le U/Z$.

(2) As $F^*(G) = O_p(G)$ and U is nilpotent, it follows that $O_p(G) \le F(G) \le U$ and $U = O_p(U) \times O_{p'}(G)$. So $O_{p'}(U) \le C_G(O_p(G)) = C_G(F^*(G)) \le F^*(G) = O_p(G)$. This implies that $O_{p'}(U) = 1$. Thus $U = O_p(U)$ and hence U is a p-group. As U is maximal nilpotent it follows that U is a Sylow p-subgroup.

Lemma 3.2. *Let G be a finite group, $U \le G$ be a BG-injector of G, and suppose that G is the central product of two subgroups G_1 and G_2, that is, $G = G_1G_2$, $[G_1, G_2] = 1$. Then $U = (U \cap G_1)(U \cap G_2)$ and $U \cap G_i$ is a BG-injector of G_i for $i = 1, 2$.*

Proof. As $G = G_1G_2$ and $[G_1, G_2] = 1$, it follows that $G_1 \le C_G(G_2)$, $G_2 \trianglelefteq G$ and $G_1 \cap G_2 \le Z(G)$. Define

$$U_1 = \{g_1 \in G_1 \mid \text{there exists } g_2 \in G_2 \text{ such that } g_1g_2 \in U\},$$
$$U_2 = \{g_2 \in G_2 \mid \text{there exists } g_1 \in G_1 \text{ such that } g_1g_2 \in U\}.$$

Then it can be easily seen that $U_i \le G_i$ for $i = 1, 2$. Also both U_i are nilpotent. We show that U_1 is nilpotent; the proof for U_2 is analogous.

As $G_1 \triangleleft G$ and $UG_2 = U_1G_2$, it follows that $G_2 \trianglelefteq UG_2$ and $UG_2/G_2 = U_1G_2/G_2$. So $U_1/U_1 \cap G_2 \cong U_1G_2/G_2 = UG_2/G_2 = U/U \cap G_2$. Since U is nilpotent, so is $U/U \cap G_2$, hence $U_1/U_1 \cap G_2$ is nilpotent. As $U_1 \cap G_2 \le G_1 \cap G_2 \le Z(G)$, it follows that $U_1 \cap G_2 \le Z(U_1)$. Hence U_1 is nilpotent.

So U_1, U_2 are nilpotent and hence U_1U_2 is nilpotent. Also it is clear that $U = U_1U_2$ and it follows that $U_i = U \cap G_i$, $i = 1, 2$. Thus $U = (U \cap G_1)(U \cap G_2)$. It remains to prove that $U \cap G_1$ is a BG-injector of G_1.

So let $X \le G_1$ be such that X is nilpotent with $U_1 \le N_{G_1}(X)$. Since $U = U_1U_2$ and $U_2 \le G_2$, it follows that U_2 centralizes G_1 and X. So $U_1 \le C(X) \le N(X)$, which implies that $U = U_1U_2 \le N(X)$. As U is a BG-injector, it follows that $X \le U$ and hence $X \le U \cap G_1 = U_1$. So $X \le U_1$. Thus U_1 is a BG-injector of G_1, and likewise U_2 is a BG-injector of G_2.

Remark 3.2. Let $\Omega = \{1, \dots, n\}$ and let $(A_1, \dots, A_m)$ be a partition of Ω, that is, Ω is a disjoint union of nonempty subsets $A_1, \dots, A_m$. If $H = \{g \in S_\Omega \mid A_i^g = A_i, i = 1, \dots, m\}$, then $H = H_1 \times \dots \times H_m$ where $H_i = \{g \in S_n \mid g \text{ leaves } A_i \text{ invariant and fixes any point outside}\}$. It is clear that $H_i \cong S_{A_i}$. So if $U \leq S_n$ with orbits $A_1, \dots, A_m$, it follows that $U \leq H_1 \times \cdots \times H_m \cong S_{A_1} \times \cdots \times S_{A_m}$.

If U is a BG-injector of S_n, then U is a BG-injector of H and by Lemma 3.2, we have $U = (U \cap H_1) \times \cdots \times (U \cap H_m)$ and $U \cap H_i$ is a BG-injector of $H_i \cong S_{A_i}$.

LEMMA 3.3. *Suppose that* $G = G_1 \times G_2$.

(1) *If* $A \in a_2(G)$, *then* $A = (A \cap G_1) \times (A \cap G_2)$ *and* $A \cap G_i \in a_2(G_i)$, $i = 1, 2$.
(2) *If* B *is a* B*-injector of* G, *then* $B = (B \cap G_1) \times (B \cap G_2)$ *and* $B \cap G_i$ *is a* B*-injector of* G_i, $i = 1, 2$.
(3) *If* $a_{2,p}(G) = \{X \leq G \mid X$ *is a* p*-group of class* ≤ 2 *and of maximal order*$\}$ *and if* $A \in a_{2,p}(G)$, *then* $A = (A \cap G_1) \times (A \cap G_2)$ *and* $A \cap G_i \in a_{2,p}(G_i)$, $i = 1, 2$.

Proof. Easy and hence omitted.

REMARK 3.3. Let H be a finite group such that $H \cong Z_p \wr S_k$, the wreath product of the cyclic group Z_p, p prime, with S_k. Then $F^*(H) = O_p(H)$.

Proof. See [9].

REMARK 3.4. For a partition $\Sigma = (A_1, \dots, A_m)$ of a finite set Ω, $Y_\Sigma = \{g \in S_\Omega \mid A_i^g = A_i \text{ for all } i\}$ is the *Young subgroup* of Ω.

It is obvious that $Y_\Sigma = Y_{A_1} \times \cdots \times Y_{A_m} \leq S_\Omega$, where

$$Y_{A_i} = \{g \in S_\Omega \mid g \text{ fixes all points not in } A_i\}$$

and $Y_{A_i} \cong S_{A_i}$. Further, we define $Y^*_{A_i} = Y_{A_i} \cap A_\Omega$ and

$$Y^*_\Sigma = \langle Y^*_{A_1}, \dots, Y^*_{A_m} \rangle = Y^*_{A_1} \times \cdots \times Y^*_{A_m} \leq A_\Omega.$$

Consider an element $g \in S_{A_i}$ of prime order $p \neq 2$.

Let $A = \{\alpha \in \Omega \mid \alpha^g \neq \alpha\}$ and $\Gamma = \{\alpha \in \Omega \mid \alpha^g = \alpha\}$. So $\Sigma = (A, \Gamma)$ is a partition of Ω. If $|A| = p^k$, then g is a product of k pairwise commuting p-cycles $t_1, \dots, t_k$ and $t_i \in Y_A$ corresponding to the orbits of g in A. Also $C_{S_\Omega}(g)$ permutes these t_i's and in particular normalizes $V = \langle t_1, \dots, t_k \rangle \cong Z_p^k$; hence $V \subseteq O_p(C_{S_\Omega}(g))$. So $C_{S_\Omega}(g) \leq Y_Z = Y_A \times \Gamma$, and thus $C_{S_\Omega}(g) = C_{Y_A}(g) \times Y_\Gamma$. As $C_{Y_A}(g) \cong Z_p \wr S_k$, Remark 3.3 implies $F^*(C_{Y_A}(g)) = O_p(C_Y(g))$ and $C(V) = V \times Y_\Gamma$. We then exploit the structure of $C(g)$ to investigate the BG-injectors of S_Ω and A_Ω. So we prove the following lemma.

Lemmas 3.6 and 3.7 were proved in [2]; to keep the paper self-contained we repeat the proof.

LEMMA 3.4. *Let* U *be a* BG*-injector in* S_Ω, $g \in Z(U)$ *of prime order* $p \neq 2$, *and let* Γ *and* A *be as defined in Remark* 3.4. *Then* $U = (U \cap Y_A) \times (U \cap Y_\Gamma)$,

$U \cap Y_A$ is a Sylow p-subgroup of Y_A, $U \cap Y_A$ is a BG-injector of Y_A, and $U \cap Y_\Gamma$ is a BG-injector of $Y_\Gamma \cong S_\Gamma$.

Proof. As $g \in Z(U)$ is of prime order $p \neq 2$, we have $p \mid |A|$, so

$$U \leq C_{S_\Omega}(g) = C_{Y_A}(g) \times Y_\Gamma.$$

As U is a BG-injector of S_Ω and $U \leq C_{Y_A}(g) \times Y_\Gamma \leq S_\Omega$, it follows that U is a BG-injector of $C_{Y_A}(g) \times Y_\Gamma \leq Y_A \times Y_\Gamma$. By Lemma 3.2, we have

$$U = (U \cap C_{Y_A}(g)) \times (U \cap Y_\Gamma) = (U \cap Y_A) \times (U \cap Y_\Gamma)$$

and $U \cap C_{Y_A}(g)$ is a BG-injector in $C_{Y_A}(g)$, $U \cap Y_\Gamma$ is a BG-injector in $Y_\Gamma \cong S_\Omega$ and $U \cap C_{Y_A}(g) = U \cap Y_A$. Furthermore, as $F^*(C_{Y_A}(g)) = O_p(C_{Y_A}(g))$ (use Remark 3.3), Lemma 3.2 implies that $U \cap Y_A$ is a Sylow p-subgroup of Y_A.

We can prove a similar result for A_Ω.

Lemma 3.5. *Let U be a BG-injector in A_Ω and let $g \in Z(U)$ with prime order $p \neq 2$. Then $U = (U \cap C_{Y_A^*}(g)) \times (U \cap Y_\Gamma^*)$.*

Proof. Since $g \in Z(U)$, we have

$$U \leq C_{A_\Omega}(g) \leq C_{S_\Omega}(g) = C_{Y_A}(g) \times Y_\Gamma \leq Y_A \times Y_\Gamma.$$

If V is as defined above, it follows that $V \subseteq O_p(C_{S_\Omega}(g)) = O_p(C_{A_\Omega}(g))$ as p is odd. As U is a BG-injector of $C_{A_\Omega}(g)$, this implies that $V \subseteq O_p(C_{A_\Omega}(g)) \subseteq U$; but U is nilpotent, so $U = O_p(U) \times O_{p'}(U)$.

Also $V \subseteq O_p(U)$ and $O_{p'}(U) \subseteq C(O_p(U))$, thus $O_{p'}(U) \subseteq C_{A_\Omega}(V)$. So $O_{p'}(U) \leq C_{S_\Omega}(V) = V \times Y_\Gamma$. As $U \leq A_\Omega$ and $V \subset A_\Omega$ ($p \neq 2$), we have

$$O_{p'}(U) = O_{p'}(U) \cap A_\Omega \leq (V \times Y_\Gamma) \cap A_\Omega = V \times (Y_\Gamma \cap A_\Omega) = V \times Y_\Gamma^*.$$

Thus $O_{p'} \leq Y_\Gamma^*$ as $p \mid |V|$, and therefore $U = O_p(U) \times O_{p'}(U) \leq C_{Y_A}(g) \times Y_\Gamma^*$; this implies that $U \leq C_{Y_A^*}(g) \times Y_\Gamma^*$, as $p \neq 2$. Hence Lemma 3.3 yields the conclusion.

Combining all these results, we obtain the following general lemma.

Lemma 3.6. *Let Ω be a finite set and let U be a BG-injector of S_Ω. Then there exists a partition $\Sigma = (A_1, \dots, A_m)$ of Ω such that*

(1) $U \leq Y_\Sigma = Y_{A_1} \times \cdots \times Y_{A_m}$.
(2) $U = (U \cap Y_{A_1}) \times \cdots \times (U \cap Y_{A_m})$.
(3) *For $i = 1, \dots, m$, there exists a prime p_i such that $U \cap Y_{A_i}$ is a Sylow p_i-subgroup of Y_{A_i} and also a BG-injector in Y_{A_i}.*
(4) (a) *If $p_i \neq 2$, then $p_i \mid |A|$.*
    (b) *If $p_i = 2$, then $|A_i| \not\equiv 3 \bmod 4$.*

Proof. We consider two cases:

Case 1: U is a 2-group. If Σ is a partition consisting of Ω alone, then $Y_\Sigma = S_\Omega$ and $U = U \cap Y_\Sigma$. As U is a BG-injector of S_Ω, it is maximal nilpotent and thus U is a Sylow 2-subgroup of S_Ω. So (1)–(3) follow, and 4(a) is also true. As U is a 2-group and a BG-injector, it cannot normalize a 3-cycle. Hence 4(b) follows.

CASE 2: U is not a 2-group. Then there exists a prime $p \neq 2$ such that $p \,|\, |U|$. As U is nilpotent, there exists $z \in Z(U)$ of order p. Let A_1 be the set of non-fixed points of $Z = Z(U)$ and Γ be the set of fixed points of Z. By Lemma 3.4, we have $U \le C_{S_\Omega}(z) \le Y_{A_1} \times Y_\Gamma$ and $p \,|\, |A_1|$, more precisely

$$U \le C_{S_\Omega}(z) = C_{Y_{A_1}}(z) \times Y_\Gamma \le Y_{A_1} \times Y_\Gamma.$$

Thus, by Lemma 3.2,

$$U = (U \cap C_{Y_{A_1}}(z)) \times (U \cap Y_\Gamma) = (U \cap Y_{A_1}) \times (U \cap Y_\Gamma)$$

and $U \cap C_{Y_{A_1}}(z)$ is a BG-injector of Y_{A_1}, and $U \cap Y_\Gamma$ is a BG-injector of Y_Γ. As $U \cap C_{Y_{A_1}}(z)$ is a BG-injector of $C_{Y_{A_1}}(z)$ and $\Gamma^*(C_{Y_{A_1}}(z)) = O_p(C_{Y_{A_1}}(z))$, we find that $U \cap C_{Y_{A_1}}(z)$ is a Sylow p-subgroup of $Y_{A_1} \cong S_{A_1}$ AND$U \cap Y_\Gamma$ is a BG-injector of $Y_\Gamma \cong S_\Gamma$. Repeating the argument for $U \cap Y_\Gamma$ and $Y_\Gamma \cong S_\Gamma$ yields the claim.

LEMMA 3.7. *Let Ω be a finite set and let U be a BG-injector of A_Ω. Then there exists a partition $\Sigma = (A_1, \ldots, A_m)$ of Ω such that*:

(1) $U \le Y^*_{A_1} \times \cdots \times Y^*_{A_m}$ *and* $U = (U \cap Y^*_{A_1}) \times \cdots \times (U \cap Y^*_{A_m})$.
(2) *For $i = 1, \ldots, m$, there exists a prime p_i such that $U \cap Y^*_{A_i}$ is a Sylow p_i-subgroup of $Y^*_{A_i}$.*
(3) *If $p_i \neq 2$, then $p_i \,|\, |A_i|$, and if $p_i = 2$, then $|A_i| \not\equiv 3 \bmod 4$.*

Proof. We argue as in the proof of Lemma 3.6.

COROLLARY 3.2. *Let B be a B-injector of S_Ω. Then there exists a partition $\Sigma = (A_1, \ldots, A_m)$ of Ω, such that $B \le Y_{A_i \cup A_j} \times Y_{\Omega \setminus (A_i \cup A_j)}$ for any $i \neq j$ and by Lemma 3.3, $B \cap Y_{A_i \cup A_j}$ is a BG-injector of $Y_{A_i \cup A_j}$. In particular*,

$$d_2(S_{A_i}) = d_2(Y_{A_i}) = d_2(B \cap Y_{A_i}) = d_{2,p_i}(S_{A_i}).$$

NOTE. If $n = n_1 + n_2$, where $n_i > 0$, then $d_2(S_n) \ge d_2(S_{n_1}) d_2(S_{n_2})$ because $S_{n_1} \times S_{n_2} \le S_n$ and so $d_2(S_{n_1}) d_2(S_{n_2}) = d_2(S_{n_1} \times S_{n_2}) \le d_2(S_n)$.

LEMMA 3.8. *Let Ω be a finite set of size n, and let $P \le S_\Omega$ be a p-subgroup of S_Ω of class ≤ 2. Then there exist integers $a, b \ge 0$ such that $n \ge p^{a+b}$ and $|P| \le p^{a+b+ab}$.*

Proof. Without loss of generality one can assume that P is transitive on Ω, $Z = Z(P)$ acts semiregularly on Ω, and since the class of P is ≤ 2, it follows that $P' \le Z(P)$, and if Z_α is the set of elements in Z which fix $\alpha \in \Omega$ then $(P_\alpha)' \le (P')_\alpha \le Z_\alpha = 1$. So P_α is abelian and hence $M = \langle Z, P_\alpha \rangle = Z \times P_\alpha$ is an abelian normal subgroup of P, as $P' \le Z \le M$ and $Z \cap Z_\alpha = Z_\alpha = 1$. Set $|P/M| = p^a$ and $|Z| = p^b$. Then there exist $t_1, \dots, t_a \in P$ such that $P/M = \langle Mt_1, \dots, Mt_a \rangle$. Define $\sigma : P_\alpha \to (P')^a$ by $\sigma(x) = ([x, t_1], \dots, [x, t_a])$. As class$(P) \leq 2$, it follows that σ is a homomorphism and is injective. Therefore $|P_\alpha| \leq |P'|^a \leq |Z(P)|^a = p^{ba}$ and

$$n = [P : P_\alpha] = [P : M][M : P_\alpha]$$

as $P_\alpha \leq M \leq P$. So

$$[P : P_\alpha] = p^a \frac{|M|}{|P_\alpha|} = p^a \frac{|Z|\,|P_\alpha|}{|P_\alpha|} = p^a p^b = p^{a+b}$$

and $|P| = n|P_\alpha| \leq np^{ab} = p^{a+b+ab}$. This completes the proof.

Corollary 3.3. *Let Ω be a finite set of size n, and let $P \leq S_\Omega$ be a transitive p-subgroup of class ≤ 2 on Ω.*

(1) *If $p \neq 2$, then $|P| \leq p^{n/p}$, where equality can hold for $n = p$ or $n = 9$ and $p = 3$.*
(2) *If $p = 2$, then $|P| = n = 2$ or $|P| \leq 8^{n/4}$. If $n > 2$ then $|P| < 8^{n/4}$.*

Proof. Consider two cases:

Case 1: $p \neq 2$. By Lemma 3.8, there exist integers $a, b \geq 0$ such that $n = p^{a+b}$ and $|P| \leq p^{a+b-1}$. As $p \neq 2$, it follows that $p^{a+b+ab} \leq p^{n/p}$ if and only if $a + b + ab \leq n/p = p^{a+b-1}$, where equality can only hold for $n = p$ or $n = 9$ and $p = 3$.

Case 2: $p = 2$. Then $|P| \leq 2^{a+b+ab}$. If $n > 2$, then $2^{a+b+ab} \leq 2^{3 \cdot n/4}$ if and only if $a + b + ab \leq 3 \cdot 2^{a+b-2}$.

Now we prove the following lemmas.

Lemma 3.9. *Let $P \leq S_\Omega$ be a p-subgroup with orbits $A_1, \dots, A_m$. Then $P \leq Y_\Sigma = Y_{A_1} \times \cdots \times Y_{A_m}$, where $\Sigma = (A_1, \dots, A_m)$ is a partition of Ω. Let $\zeta_i : Y_\Sigma \to Y_{A_i}$ be the projection. Then*:

(1) *$P \leq P^{\zeta_1} \times \cdots \times P^{\zeta_m}$ and $P^{\zeta_i} \leq Y_{A_i}$.*
(2) *Each P^{ζ_i} is transitive on A_i.*
(3) *$P \cap Y_{A_i} \leq P^{\zeta_i}$.*
(4) *If P is of class ≤ 2 and of maximal order $d_{2,p}(S_\Omega)$, then*
   (a) *$P = P^{\zeta_1} \times \cdots \times P^{\zeta_m}$.*
   (b) *$P \cap Y_{A_i} = P^{\zeta_i}$.*
   (c) *$P = (P \cap Y_{A_1}) \times \cdots \times (P \cap Y_{A_m})$.*

Proof. (1) As $Y_\Sigma = Y_{A_1} \times \cdots \times Y_{A_m}$, any $x \in Y_\Sigma$ can be uniquely written as $x = x_1 \cdots x_m$ with $x_i \in Y_{A_i}$ and $x^{\zeta_i} = x_i$. So $x = x^{\zeta_1} \cdots x^{\zeta_m}$. Hence $x \in P^{\zeta_1} \times \cdots \times P^{\zeta_m}$, and this proves (1).

(2) Let $\alpha, \beta \in A_i$. As P is transitive on A_i, there exists $x \in P$ such that $\alpha^x = \beta$. Let $x = x_1 \cdots x_m$ with $x_j \in Y_{A_j}$. By the definition of Y_{A_k}, if

$x_j \in Y_{A_i}$ for $j \neq i$, then x_j fixes all points not on A_j, hence all points in A_i as $A_i \subseteq \Omega \setminus A_j$. Thus $\alpha^{x_j} = \alpha$ and $\beta^{x_j} = \beta$ for all $j \neq i$. So

$$\beta = \alpha^x = \alpha^{x_1 x_2 \cdots x_{i-1} x_i x_{i+1} \cdots x_m} = \alpha^{x_i x_{i+1} \cdots x_m}$$

and $\alpha^{x_i} = \beta^{x_m^{-1} x_{m-1}^{-1} \cdots x_{i-1}^{-1}} = \beta$, which proves (2).

(3) Let $x \in P \cap Y_{A_i}$. Then the decomposition of x in $Y_{A_1} \times \cdots \times Y_{A_m}$ is

$$x = (1, \ldots, 1, \underset{\substack{\downarrow \\ i}}{x}, 1, \ldots, 1).$$

So $x = x^{\zeta_i} \in P^{\zeta_i}$. Hence $P \cap Y_{A_i} \leq P^{\zeta_i}$.

(4) As ζ_i, $i = 1, \ldots, m$, are homomorphisms, we have class$(P^{\zeta_i}) \leq$ class$(P) \leq 2$, which implies that class$(P^{\zeta_1} \times \cdots \times P^{\zeta_m}) \leq 2$. So $P^{\zeta_1} \times \cdots \times P^{\zeta_m}$ is a p-subgroup of S_Ω of class ≤ 2. Thus $|P^{\zeta_1} \times \cdots \times P^{\zeta_m}| \leq d_{2,p}(S_\Omega) = |P|$. As $P \leq P^{\zeta_1} \times \cdots \times P^{\zeta_m}$, from (1) it follows that $|P| \leq |P^{\zeta_1} \times \cdots \times P^{\zeta_m}| \leq |P|$. Hence $P = P^{\zeta_1} \times \cdots \times P^{\zeta_m}$. So $P^{\zeta_i} \leq P$ and $P \cap Y_i \leq P^{\zeta_i} \leq P \cap Y_{A_i}$. Thus $P \cap Y_{A_i} = P^{\zeta_i}$, proving (4).

Lemma 3.10. *Let Ω be a finite set of size n.*

(1) *If $p \neq 2$, then $d_{2,p}(S_n) = d_{2,p}(A_n) = p^{[n/p]}$.*
(2) *If $p \neq 2$, then $d_{2,2}(S_n) = \varepsilon_n 8^{[n/4]}$, where*

$$\varepsilon_n = \begin{cases} 1, & n \equiv 0, 1 \bmod 4, \\ 2, & n \equiv 2, 3 \bmod 4, \end{cases}$$

and if $n > 1$, then $d_{2,2}(A_n) = \frac{1}{2} d_{2,2}(S_n) = \frac{1}{2}\varepsilon_n 8^{[n/4]}$. Furthermore, if $p \neq 3$, then:

(a) *All p-subgroups of S_n of class ≤ 2 and order $d_{2,p}(S_n)$ are conjugate.*
(b) *If $p > 3$, then these groups are elementary abelian.*
(c) *If $p = 2$, then these groups are isomorphic to $Z_{\varepsilon_n} \times D_8^{[n/4]}$, where D_8 denotes a Sylow 2-subgroup of S_4, which is a dihedral group of order 8.*

Proof. It can be easily seen that S_n contains subgroups of order $p^{[n/p]}$ for any prime p and generated by $[n/p]$ cycles with distinct supports and $p^{[n/p]} \leq d_{2,p}(S_n)$.

Also S_n contains 2-subgroups of order $\varepsilon_n 8^{[n/4]} \leq d_{2,2}(S_n)$. This can be explained as follows. Let $\pi = (A_1, \ldots, A_m, A)$ be a partition of Ω. Let $|A_i| = 4$, $i = 1, \ldots, m$, and $|A| = r$, where $n = 4m + r$, $0 \leq r \leq 4$. It follows that

$$H = Y_{A_1} \times \cdots \times Y_{A_m} \times Y_r \leq S_n$$

where $Y_{A_i} \cong S_4$ and $Y_r \cong Z_{\varepsilon_n}$. Hence $H \cong S_4^m \times S_r$ contains $D_8^m \times Z_{\varepsilon_n}$ of class ≤ 2. It remains to show that for $p \neq 3$, these groups are exactly all possible p-subgroups of class ≤ 2 and order $d_{2,p}(S_n)$.

We consider two cases:

CASE 1: $p \neq 2$. Let $|A_i| = n_i$. Then $p^{[n_i/p]} = p^{n_i/p} \le d_{2,p}(S_{A_i}) = |P \cap Y_{A_i}|$. By Corollary 3.3 we have $|P \cap Y_{A_i}| \le p^{n_i/p}$. Hence $p^{n_i/p} = d_{2,p}(S_{A_i}) = |P \cap Y_{A_i}|$. Again by Corollary 3.3, we have either $n_i = p$, or $n_i = 9$ and $p = 3$. So if $p \neq 3$, then all orbits of P have length 1 or p. Thus P is conjugate to the subgroup constructed above and hence $d_{2,p}(S_n) = p^{[n/p]}$. As $p \neq 2$, it follows that $d_{2,p}(S_n) = d_{2,p}(A_n)$.

CASE 2: $p = 2$. Let $P \in a_{2,2}(S_n)$ and let $P \le Y_\Sigma = Y_{A_1} \times \cdots \times Y_{A_m}$ where Y_{A_i}, $i = 1, \ldots, m$, are the Young subgroups corresponding to the partition $\Sigma = (A_1, \ldots, A_m)$. By Lemma 3.3, $P = (P \cap Y_{A_1}) \times \cdots \times (P \cap Y_{A_m})$ where $P \cap Y_{A_i} \in a_{2,2}(Y_{A_i})$, and by Lemma 3.9, $P \cap Y_{A_i}$ is a transitive subgroup of Y_{A_i}. By Corollary 3.3, $|A_i| = 1$ or 2 and $8^{n/4} \le d_2(S_{A_i}) = |P \cap Y_{A_i}| \le 8^{n/4}$. This implies that $|P \cap Y_{A_i}| = 8^{n/4}$, which occurs if and only if $n_i = 4$. Hence again P is a group conjugate to the group constructed above. As $P \not\le A_n$, this implies that $d_{2,2}(A_n) = \frac{1}{2} d_{2,2}(S_n)$.

Now we are in a position to prove the first main result.

THEOREM 3.2. *Let Ω be a finite set of size n and let B be a B-injector of S_Ω.*

(1) *If* $n \equiv 3 \bmod 4$, *then* $B = \langle d \rangle \times T$ *where* d *is a* 3*-cycle, and* T *is a Sylow* 2*-subgroup of* $C_{S_\Omega}(d)$.
(2) *If* $n \not\equiv 3 \bmod 4$, *then* B *is a Sylow* 2*-subgroup. In particular, all the* B*-injectors of* S_Ω *are conjugate.*

Proof. As B is a B-injector of S_Ω, it is a BG-injector of S_Ω. By Lemma 3.6, there exists a partition $\Sigma = (A_1, \ldots, A_m)$ of Ω such that $B \le Y_\Sigma$ and $B = (B \cap Y_{A_1}) \times \cdots \times (B \cap Y_{A_m})$ and for $i = 1, \ldots, m$, there exist primes p_i such that $B \cap Y_{A_i}$ is a Sylow p_i-subgroup of Y_{A_i}, and hence, by Lemma 3.3, a B-injector of Y_{A_i}.

Let $p_i \neq 2$. Then $p_i \mid |A_i| = n_i$ and

$$\begin{aligned} \varepsilon_{n_i} 8^{[n_i/4]} &\le d_2(S_{A_i}) = d_2(Y_{A_i}) = d_2(B \cap Y_{A_i}) = d_{2,p_i}(B \cap Y_{A_i}) \\ &= p_i^{[n_i/p_i]} = p_i^{n_i/p_i}. \end{aligned}$$

This implies that $p_i = 3 = n_i$. Hence either $B \cap Y_{A_i}$ is a 2-group, or $|A_i| = 3$ and $B \cap Y_{A_i}$ is a 3-cycle. We have at most one i such that $|A_i| = 3$, because we assume that $|A_i| = |A_j| = 3$ for $i \neq j$. It follows that $(B \cap Y_{A_i}) \times (B \cap Y_{A_j}) \le Y_{A_i \cup A_j} \cong S_6$ and $(B \cap Y_{A_i}) \times (B \cap Y_{A_j})$ is again a B-injector of $Y_{A_i \cup A_j}$. Hence $d_2(S_6) = d_2((B \cap Y_{A_i}) \times (B \cap Y_{A_j})) = 3^2 = 9$, which is a contradiction, as $16 = \varepsilon_6 8^{[6/4]} = d_{2,2}(S_6) \le d_2(S_6) = 9$, so $d_2(S_2) > 9$. Hence either B is a Sylow 2-group (if no $|A_i|$ is 3), or $b = \langle d \rangle \times T$ for some 3-cycle. If B is a Sylow 2-group, then $n \not\equiv 3 \bmod 4$ as observed above. If $n \equiv 3 \bmod 4$, then a Sylow 2-group T of S_n has a fixed point and an orbit of length 2. So $T = Z_2 \times T_1$ where T_1 is a Sylow 2-group of S_{n-3}, and we deduce that

$$\begin{aligned} d_{2,2}(S_n) &= d_{2,2}(T) = d_{2,2}(Z_2)d_{2,2}(T_1) = 2d_{2,2}(S_{n-3}) \\ &< 3d_{2,2}(S_{n-3}) = d_2(S_3)d_2(S_{n-3}) \leq d_2(S_n). \end{aligned}$$

As $d_{2,2}(S_n) < d_2(S_n)$, it follows that B-injectors cannot be 2-groups. So $B = \langle d \rangle \times T$, and this completes the description of the BG-injectors of S_n.

Now we discuss the B-injectors of A_n. First we give a lemma.

Lemma 3.11.

(1) *If p is prime, $p \geq 7$, then $p^k < 3^{[pk/3]}$ for all $k \geq 1$.*
(2) *$5^k < 3^{[5k/3]}$ for all $k \geq 3$.*
(3) *$3^k < \frac{1}{2}8^{[3k/4]}$ for all $k \geq 3$.*

Proof. Easy.

Now we prove the second main result.

Theorem 3.3. *Let B be a B-injector in $A_\Omega = A_n$.*

(1) *If $|\Omega| = 5$, then B is a Sylow 5-subgroup.*
(2) *If $|\Omega| = 6$, then B is a Sylow 3-subgroup.*
(3) *If $|\Omega| \neq 5, 6$, then there exists a B-injector B^* of S_Ω such that $B = B^* \cap A_\Omega$ (B^* is known by Theorem 3.2).*

Let B be a B-injector of $X = A_5$ or A_6, and let p be a prime divisor of $|X|$. If $z_p \in Z(B)$, then $d_2(X) = d_2(B) = d_2(C_X(z_p)) \leq |C_X(z_p)|$ as $B \leq C_X(z_p)$.

Let $X = A_5$. Then $2 \nmid |B|$, as otherwise $5 \leq d_2(A_5) \leq |C_X(z_2)| = 4$, a contradiction. Also $3 \nmid |B|$, as otherwise $5 \leq d_2(A_5) \leq |C_X(z_3)| = 3$, a contradiction. So B is a Sylow 5-subgroup.

Likewise if $X = A_6$, then B is a Sylow 3-subgroup.

Now we discuss the third case. Let B be a B-injector of A_Ω and $|\Omega| \neq 5, 6$.

Case 1: B is a 2-group. Then B is a Sylow 2-subgroup. So $B = B^* \cap A_\Omega$ for some Sylow 2-subgroup of S_Ω. As B is a BG-injector of A_Ω and is a 2-group, it cannot normalize a 3-cycle, and hence $|\Omega| \not\equiv 3 \bmod 4$, because in this case, Sylow 2-subgroups of S_Ω and A_Ω do normalize a 3-cycle. So B^* is a B-injector of S_Ω (B^* is known by Theorem 3.2), and the assertion follows.

Case 2: B is not a 2-group. By Lemma 3.7, there exists a partition $\pi = (A_1, \ldots, A_m)$ of Ω such that $B \leq Y^*_\pi = Y^*_{A_1} \times \cdots \times Y^*_{A_n}$, $B = (B \cap Y^*_{A_1}) \times \cdots \times (B \cap Y^*_{A_m})$, $B \cap Y^*_{A_i}$ is a B-injector of $Y^*_{A_i} \cong A_{A_i}$ and either $B \cap Y^*_{A_i}$ is a Sylow 2-subgroup if $|A_i| \not\equiv 3 \bmod 4$, or $B \cap Y^*_{A_i}$ is a Sylow p_i-subgroup for some prime $p_i \neq 2$ and $p_i \mid |A_i|$.

Let $p_i \neq 2$. Then as $B \cap Y^*_{A_i}$ is a B-injector of $Y^*_{A_i}$, one has: If $|A_i| = p_i k = n_i$ then

$$d_2(A_{A_i}) = d_2(Y^*_{A_i}) = d_2(B \cap Y^*_{A_i}) = d_{2,p_i}(A_{A_i}) = p_i^k,$$

and

$$3^{[p_i k/3]} = 3^{[n_i/3]} = d_{2,3}(A_{A_i}) \leq d_2(A_{A_i}) = p_i^k.$$

Also we have $\frac{1}{2} d_{2,2}(S_A) \leq d_{2,2}(A_{A_i}) \leq d_2(A_{A_i})$, thus $\frac{1}{2}\varepsilon_{n_i} 8^{[n_i/4]} \leq d_2(A_{A_i}) = p_i^k$. By Lemma 3.10, we have the following restrictions on p_i and $|A_i|$. As $3^{[p_i k/3]} \leq p_i^k$, it follows that $p_i = 3$ or 5 by Lemma 3.11(1). If $p_i = 5$, then $k = 1$ or 2 and hence $|A_i| = 3$ or 6 by Lemma 3.11(3). So we can renumber the components of π so that $\pi = (A_1, \dots, A_a, \Gamma_1, \dots, \Gamma_b, \Sigma)$ where $|A_i| = 3$ for $i = 1, \dots, a$, $|\Gamma_i| = 5$ for $i = 1, \dots, b$, and $|\Sigma| = m$ with $n = 3a + 5b + m$. Then

$$B = (B \cap Y^*_{A_1}) \times \cdots \times (B \cap Y^*_{A_n}) \times (B \cap Y^*_{\Gamma_1}) \times \cdots \times (B \cap Y^*_{\Gamma_b}) \times (B \cap Y^*_{\Sigma})$$

and hence

$$d_2(A_\Omega) = 3^a 5^b d_{2,2}(A_\Sigma) = 3^a 5^b d_{2,2}(A_\Sigma) = 3^a 5^b d_{2,2}(S_m)$$

and

$$\frac{1}{2} d_2(S_{3a+5b}) d_2(S_m) \leq \frac{1}{2} d_2(S_n) \leq d_2(A_n) = d_2(B) = 3^a 5^b d_{2,2}(A_\Sigma).$$

Hence if $m = 0$, then $\frac{1}{2} d_2(S_{3a+5b}) \leq 3^a 5^b$. If $m \neq 0$, then

$$\frac{1}{2} d_2(S_{3a+5b}) d_2(S_m) \leq q 3^a 5^b d_{2,2}(A_m) = 3^a 5^b \cdot \frac{1}{2} d_{2,2}(A_n) = 3^a 5^b \cdot \frac{1}{2} d_{2,2}(S_m)$$
$$\leq d_2(S_{3a+5b}) \frac{1}{2} d_2(S_m),$$

so $d_2(S_{3a+5b}) = 3^a 5^b$ and this implies $a \leq 1,\ b = 0$ and $d_2(S_m) = d_{2,2}(S_m)$. Hence, if $m \neq 0$, then B is a 2-group or $\langle d \rangle \times T$.

This completes the proof of the theorem.

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REFERENCES

[1] M. I. AlAli, Ch. Hering and A. Neumann, *On the B-injectors of sporadic groups*, Comm. Algebra 27 (1999), 2853–2863.

[2] —, —, —, *On B-injectors of the covering groups of A_n*, Illinois J. Math., to appear.

[3] Z. Arad and D. Chillag, *Injectors of finite solvable groups*, Comm. Algebra 7 (1979), 115–138.

[4] A. Bialostocki, *Nilpotent injectors in alternating groups*, Israel J. Math. 44 (1983), 335–344.

[5] A. Bialostocki, *Nilpotent injectors in symmetric groups*, ibid. 41 (1982), 261–273.
[6] —, *On products of two nilpotent subgroups of a finite group*, ibid. 20 (1975), 178–188.
[7] B. Fischer, W. Gaschütz and B. Hartley, *Injektoren endlicher auflösbarer Gruppen*, Math. Z. 102 (1967), 337–339.
[8] P. Flavell, *Nilpotent injectors in finite groups all whose local subgroups are N-constrained*, J. Algebra 149 (1992), 405–418.
[9] G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Encyclopedia Math. Appl. 16, Addison-Wesley, 1981.
[10] A. Mann, *Injectors and normal subgroups of finite groups*, Israel J. Math. 9 (1971), 554–558.
[11] A. Neumann, *Nilpotent injectors in finite groups*, Arch. Math. (Basel) 71 (1998), 337–340.

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