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INCIDENCE COALGEBRAS OF INTERVALLY FINITE POSETS, THEIR INTEGRAL QUADRATIC FORMS AND COMODULE CATEGORIES

BY

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Abstract. The incidence coalgebras $C = K^{\square}I$ of intervally finite posets I and their comodules are studied by means of their Cartan matrices and the Euler integral bilinear form $b_C : \mathbb{Z}^{(I)} \times \mathbb{Z}^{(I)} \to \mathbb{Z}$. One of our main results asserts that, under a suitable assumption on I, C is an Euler coalgebra with the Euler defect $\partial_C : \mathbb{Z}^{(I)} \times \mathbb{Z}^{(I)} \to \mathbb{Z}$ zero and $b_C(\mathbf{lgth}\, M, \mathbf{lgth}\, N) = \chi_C(M,N)$ for any pair of indecomposable left C-comodules M and N of finite K-dimension, where $\chi_C(M,N)$ is the Euler characteristic of the pair M, N and $\mathbf{lgth}\, M \in \mathbb{Z}^{(I)}$ is the composition length vector. The structure of minimal injective resolutions of simple left C-comodules is described by means of the inverse $\mathfrak{C}_I^{-1} \in \mathbb{M}_I^{\preceq}(\mathbb{Z})$ of the incidence matrix $\mathfrak{C}_I \in \mathbb{M}_I(\mathbb{Z})$ of the poset I. Moreover, we describe the Bass numbers $\mu_m^I(S_I(a), S_I(b))$, with $m \geq 0$, for any simple $K^{\square}I$-comodules $S_I(a)$, $S_I(b)$ by means of the coefficients of the bth row of $\mathfrak{C}_I^{-1}$. We also show that, for any poset I of width two, the Grothendieck group $\mathbf{K}_0(K^{\square}I\text{-Comod}_{\mathrm{fc}})$ of the category of finitely copresented $K^{\square}I$-comodules is generated by the classes $[S_I(a)]$ of the simple comodules $S_I(a)$ and the classes $[E_I(a)]$ of the injective covers $E_I(a)$ of $S_I(a)$, with $a \in I$.

1. Introduction. Throughout this paper, we fix a field K. Given a nonempty set I, we denote by $\mathbb{M}_I(K)$ the set of all I by I matrices $\lambda = [\lambda_{pq}]_{p,q\in I}$ with $\lambda_{pq} \in K$. The set $\mathbb{M}_I(K)$ is equipped with the usual K-vector space structure and (partial) matrix multiplication (which is not associative and not everywhere defined if I is infinite); see [7, 2.1] and [36].

We denote by $\mathbb{M}_I^{\bullet}(K) \subseteq \mathbb{M}_I(K)$ the associative matrix K-algebra consisting of all matrices $\lambda = [\lambda_{pq}] \in \mathbb{M}_I(K)$ such that $\lambda_{pq} = 0$ for all but finitely many $p, q \in I$. Obviously, $\mathbb{M}_I^{\bullet}(K)$ has an identity element if and only if I is a finite set.

Let $I \equiv (I, \preceq)$ be a *poset*, that is, I is a partially ordered set with respect to the partial order relation $\preceq$ (see [25]). The relation $\preceq$ is uniquely determined by the *incidence matrix* (see [25])

$$(1.1) \qquad \mathfrak{C}_I = [c_{ij}]_{i,j\in I} \in \mathbb{M}_I^{\preceq}(\mathbb{Z}), \qquad c_{ij} = \begin{cases} 1 & \text{for } i \preceq j, \\ 0 & \text{for } i \not\preceq j, \end{cases}$$

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where the abelian group

$$\mathbb{M}_I^{\preceq}(\mathbb{Z}) = \{c = [c_{pq}]_{p,q\in I} \in \mathbb{M}_I(\mathbb{Z});\ c_{pq} = 0 \text{ if } p \not\preceq q\} \tag{1.2}$$

is viewed as a partial subalgebra of $\mathbb{M}_I(\mathbb{Z})$.

The poset I is defined to be *intervally finite* if for any $a \preceq b$, the *interval* $[a,b] = \{s \in I;\ a \preceq s \preceq b\}$ is a finite set.

If I is intervally finite then, for any $\lambda' = [\lambda'_{ij}], \lambda'' = [\lambda''_{ij}] \in \mathbb{M}_I^{\preceq}(\mathbb{Z})$, their product $\lambda' \cdot \lambda'' = [\lambda_{ab}]_{a,b\in I}$, where $\lambda_{ab} = \sum_{j\in I} \lambda'_{aj}\lambda''_{jb} = \sum_{a\preceq j\preceq b} \lambda'_{aj}\lambda''_{jb}$, is well defined and lies in $\mathbb{M}_I^{\preceq}(\mathbb{Z})$. Hence, $\mathbb{M}_I^{\preceq}(\mathbb{Z})$ is an associative K-algebra and the matrix E, with 1's on the main diagonal and zeros elsewhere, is the identity of $\mathbb{M}_I^{\preceq}(\mathbb{Z})$.

To any intervally finite poset I, we associate the incidence K-coalgebra $K^{\square}I = (KI, \varDelta_I, \varepsilon_I)$, where $KI \subseteq \mathbb{M}_I^{\bullet}(K)$ is the incidence K-algebra (2.1) of I, $\varDelta_I$ is the comultiplication and ε_I is the counity (see Definition 2.2).

We show that the coalgebra $C = K^{\square}I$ is basic, $c\ell$-hereditary, Hom-computable in the sense of [30], left locally artinian (hence left cocoherent), the Cartan matrix ${}_CF = {}_C\widehat{F} \in \mathbb{M}_I(\mathbb{Z})$ [30, (4.3)] is the transpose of $\mathfrak{C}_I$, ${}_CF$ has a (unique) left and right inverse ${}_CF^{-1} = [c^{-}_{ij}]_{i,j\in I}$ (2.10) in the partial algebra $\mathbb{M}_I^{\preceq}(\mathbb{Z})$, and the Euler integral bilinear form

$$b_C : \mathbb{Z}^{(I)} \times \mathbb{Z}^{(I)} \to \mathbb{Z} \tag{1.3}$$

[30, (4.6)] is defined by the formula $b_C(x,y) = x \cdot \mathfrak{C}_I^{-1} \cdot y^{\mathrm{tr}}$ for all $x, y \in \mathbb{Z}^{(I)}$, where $\mathbb{Z}^{(I)}$ is the direct sum of I-copies of the group $\mathbb{Z}$. Propositions 2.9 and 2.12 give simple formulae for c^{-}_{ab} in terms of the poset I (see (2.11) and (2.13)). We also show that, for any intervally finite poset I without infinitely many pairwise incomparable elements, C is an Euler coalgebra, the Euler defect $\partial_C : \mathbb{Z}^{(I)} \times \mathbb{Z}^{(I)} \to \mathbb{Z}$ is zero [30, (4.14)], the Euler characteristic

$$\chi_C(M,N) = \sum_{j=0}^{\infty} (-1)^j \dim_K \operatorname{Ext}^j_C(M,N) \tag{1.4}$$

is well defined, and $b_C(\mathbf{lgth}\, M, \mathbf{lgth}\, N) = \chi_C(M,N)$ for any indecomposable left C-comodules M and N of finite K-dimension. In this case, a minimal injective resolution of any simple left C-comodule $S_I(a)$, with $a \in I$, is socle-finite and describes the ath column of $\mathfrak{C}_I^{-1} \in \mathbb{M}_I^{\preceq}(\mathbb{Z})$. The structure of that resolution is described in Theorems 5.3 and 6.2. In particular, in Section 6 we describe the Bass numbers $\mu^I_m(S_I(a), S_I(b))$ (6.1), with $m \geq 0$, of any simple $K^{\square}I$-comodules $S_I(a)$, $S_I(b)$ by means of the entries of the bth row of $\mathfrak{C}_I^{-1}$.

In Section 5, we prove that, for any poset I of width at most two, the Grothendieck group $\mathbf{K}_0(K^{\square}I\text{-Comod}_{\mathrm{fc}})$ of the category of finitely copresented left $K^{\square}I$-comodules has the form

$$\mathbf{K}_0(K^{\square}I\text{-Comod}_{\mathrm{fc}}) = \mathbf{K}_0(K^{\square}I\text{-comod}) + \mathbf{K}_0(K^{\square}I\text{-inj}),$$

that is, it is generated by the classes $[S_I(a)]$ of the simple comodules $S_I(a)$ and the classes $[E_I(a)]$ of the injective covers $E_I(a)$ of $S_I(a)$, with $a \in I$ (cf. the group $K_0^+(C)$ defined in [30, (4.8)]).

Some of the results of this paper are applied in [32], where we present: (a) a characterisation of the incidence coalgebras $K^{\square}I$ that are representation-directed, (b) a description of posets I such that $K^{\square}I$ is representation-directed, and (c) a characterisation of the incidence coalgebras $K^{\square}I$ that are left pure semisimple. We show in [32] that every such coalgebra $K^{\square}I$ is tame of discrete comodule type (see [27] and [28]) and $\mathrm{gl.dim}\, K^{\square}I \leq 2$.

Throughout this paper we use the coalgebra representation theory notation and terminology introduced in [27], [28], and [37]. The reader is referred to [18] and [35] for the coalgebra and comodule terminology, and to [1], [2], [10] and [25] for the representation theory terminology and notation.

Given a K-coalgebra C, we denote by C-Comod and C-comod the categories of left C-comodules and left C-comodules of finite K-dimension, respectively. The corresponding categories of right C-comodules are denoted by Comod-C and comod-C. Further, we denote by C-inj the category of socle finite injective left C-comodules. Given a K-coalgebra C with comultiplication $\Delta : C \to C \otimes C$ and counity $\varepsilon : C \to K$, the coalgebra C^{op} opposite to C is the K-vector space C equipped with the same counity $\varepsilon : C \to K$ and comultiplication $\Delta^{\mathrm{op}} = \tau \circ \Delta : C \to C \otimes C$, where $\tau : C \otimes C \to C \otimes C$ is the twist map defined by $\tau(x \otimes y) = y \otimes x$ for $x, y \in C$. It is clear that the category Comod-C of right C-comodules is just the category C^{op}-Comod of left C^{op}-comodules.

Following [24, p. 404], the K-coalgebra C is defined to be *basic* if the left C-comodule $\mathrm{soc}\,_C C$ has a direct sum decomposition $\mathrm{soc}\,_C C = \bigoplus_{j \in I_C} S(j)$, where I_C is a set, $S(j)$ are simple comodules and $S(i) \not\cong S(j)$ for all $i \neq j$. It is shown in [27] that the definition is left-right symmetric and the notion of basic coalgebra introduced in [6] is equivalent to the above one.

Let $\mathbf{lgth}\, M = (\ell_j(M))_{j \in I_C} \in \mathbb{Z}^{(I_C)}$ be the *composition length vector* of a comodule M in C-comod, where $\ell_j(M) \in \mathbb{N}$ is the number of simple composition factors of M isomorphic to $S(j)$. We recall from [27] that the map $M \mapsto \mathbf{lgth}\, M$ extends to a group isomorphism

$$\mathbf{lgth} : \mathbf{K}_0(C) \xrightarrow{\simeq} \mathbb{Z}^{(I_C)},$$

where $\mathbf{K}_0(C) = \mathbf{K}_0(C\text{-comod})$ is the Grothendieck group of the category C-comod (see [27]). If $\dim_K S(j) = 1$, then $\ell_j(M) = \dim_K \mathrm{Hom}_C(M, E(j))$, where $E(j)$ is the injective envelope of $S(j)$ [30, Proposition 2.6]. The coalgebra C is defined to be *Hom-computable* if

$$\ell_{ij} := \ell_j(E(i)) = \dim_K \operatorname{Hom}_C(E(i), E(j))$$

is finite for all $i, j \in I_C$ [30, Proposition 2.9].

Following [29] and [30], a comodule N in C-Comod is said to be *finitely cogenerated* (or *socle-finite*) if N is a subcomodule of a finite direct sum of indecomposable injective comodules, or equivalently, $\dim_K \operatorname{soc} N$ is finite. We say that N is *finitely copresented* if there is an exact sequence $0 \to N \to E \to E'$ in C-Comod, where E and E' are each a finite direct sum of indecomposable injective comodules. We denote by C-$\mathrm{Comod}_{\mathrm{fc}}$ the full subcategory of C-Comod whose objects are the finitely copresented comodules.

We call a coalgebra C *left cocoherent* if any finitely cogenerated epimorphic image N of an indecomposable injective C-comodule E is finitely copresented (see [13]). Note that the class of left cocoherent coalgebras contains the right semiperfect coalgebras, hereditary coalgebras and left locally artinian coalgebras (i.e. coalgebras C with left indecomposable injectives artinian); see [13].

2. Incidence coalgebras of intervally finite posets. Let $I \equiv (I, \preceq)$ be a poset. We write $i \prec j$ if $i \preceq j$ and $i \neq j$. The poset I is said to be *left locally bounded* if for any $b \in I$, the *left cone* ${}^{\trianglelefteq}b = \{p \in I;\ p \preceq b\}$ does not have infinitely many pairwise incomparable elements. The poset I is said to be *right locally bounded* if for any $a \in I$, the *right cone* $a^{\trianglelefteq} = \{q \in I;\ a \preceq q\}$ does not have infinitely many pairwise incomparable elements. The *width* $\mathbf{w}(I)$ of I is defined to be the maximal number of pairwise incomparable elements of I, if it is finite; otherwise $\mathbf{w}(I) = \infty$ (see [25]). We say that I is *connected* if it is not a disjoint union of two subposets I' and I'' with all $i' \in I'$ and $i'' \in I''$ incomparable in I. A subposet I' of a poset I is defined to be *convex*, or intervally closed, if for any $a \preceq b$ in I', the interval $[a, b] = \{s \in I;\ a \preceq s \preceq b\} = a^{\trianglerighteq} \cap {}^{\trianglelefteq}b$ is also contained in I'.

Following Rota [22], given an arbitrary poset I, we define the *incidence K-algebra $KI \subseteq \mathbb{M}_I^{\bullet}(K)$* of I to be the K-algebra (see [25])

$$\text{(2.1)} \quad KI = \mathbb{M}_I^{\preceq}(K) \cap \mathbb{M}_I^{\bullet}(K) = \{\lambda = [\lambda_{pq}] \in \mathbb{M}_I^{\bullet}(K);\ \lambda_{pq} = 0 \text{ if } p \npreceq q\}.$$

We call the unitary K-algebra $\mathbb{M}_I^{\preceq}(K)$ the *complete incidence algebra* of the poset $(I, \preceq)$ with coefficients in K (see Proposition 4.3).

It is easy to see that KI is an associative K-subalgebra of $\mathbb{M}_I^{\bullet}(K)$, and the matrix units e_{pq}, with $p \preceq q$, having the identity in the (p, q) entry and zeros elsewhere, form a K-basis of KI. Given $j \in I$, the matrix unit $e_j = e_{jj} \in KI$ is a primitive idempotent of KI, and $\{e_j\}_{j \in I}$ is a complete set of pairwise orthogonal primitive idempotents of KI. Obviously, the algebra KI has an identity element if and only if I is finite.

In most of this paper, we assume that I is a connected intervally finite poset. It follows that I is finite or countable (and therefore can be identified with a subset of $\mathbb{Z}$). Hence, if I is countable then $\mathfrak{C}_I$ is an integral $\mathbb{Z} \times \mathbb{Z}$ matrix and the K-dimension of KI is countable.

We recall from [10] and [25] that the *Hasse quiver* of I is the quiver $Q_I = (Q_0^I, Q_1^I)$, where $Q_0^I = I$ is the set of points of Q_I and there is a unique arrow $p \to q$ from $p \in I$ to $q \in I$ in Q_1^I if and only if $p \prec q$ and there is no $t \in I$ such that $p \prec t \prec q$.

For example, if $I = \mathbb{Z}$ with the linear order opposite to the natural one, then

$$Q_I : \cdots \leftarrow -2 \leftarrow -1 \leftarrow 0 \leftarrow 1 \leftarrow 2 \leftarrow \cdots \leftarrow r{+}1 \rightarrow r{+}2 \leftarrow r{+}3 \leftarrow \cdots$$

The algebra KI consists of the lower triangular matrices $\lambda \in \mathbb{M}_{\mathbb{Z}}^{\bullet}(K)$, and the matrix $\mathfrak{C}_I = [c_{pq}] \in \mathbb{M}_{\mathbb{Z}}^{\preceq}(\mathbb{Z})$ has ones on the main diagonal and below it, and zeros above the diagonal.

Following [12], we introduce the following definition (see also [18] and [28]).

Definition 2.2. Let I be an intervally finite poset. The *incidence K-coalgebra* of I is the triple

$$K^{\square}I = (KI, \Delta_I, \varepsilon_I), \tag{2.3}$$

where KI is the incidence K-algebra of I, and the counit $\varepsilon_I : KI \to K$ and comultiplication $\Delta_I : KI \to KI \otimes KI$ are defined by the formulae

$$\Delta_I(e_{pq}) = \sum_{p \preceq t \preceq q} e_{pt} \otimes e_{tq}, \qquad \varepsilon_I(e_{pq}) = \begin{cases} 0 & \text{for } p \neq q, \\ 1 & \text{for } p = q. \end{cases}$$

Since I is intervally finite, the K-linear map Δ_I is well-defined. We recall that $\dim_K K^{\square}I \le \aleph_0$ if the poset I is connected.

We start with the following useful observations.

Lemma 2.4. *Let I be an intervally finite poset. Let $I^* = (I, \preceq^*)$ be the poset opposite to $I \equiv (I, \preceq)$, that is, $p \preceq^* q$ if and only if $q \preceq p$.*

(a) *The K-linear map $\widehat{\mathrm{tr}} : K^{\square}I \xrightarrow{\simeq} K^{\square}I^*$ that associates to any matrix λ its transpose λ^{tr} defines an isomorphism of the K-coalgebra $K^{\square}I^*$ with the K-coalgebra $(K^{\square}I)^{\mathrm{op}}$.*

(b) *The coalgebra isomorphism $(K^{\square}I)^{\mathrm{op}} \cong K^{\square}I^*$ defined in* (a) *induces the category isomorphisms*

$$K^{\square}I^*\text{-Comod} \cong \text{Comod-}K^{\square}I \quad \text{and} \quad K^{\square}I^*\text{-comod} \cong \text{comod-}K^{\square}I.$$

(c) *If U is a convex subposet of I then $K^{\square}U$ is a subcoalgebra of $K^{\square}I$ and $K^{\square}U$-*comod *is an extension closed subcategory of $K^{\square}I$-*comod.

Proof. (a) The underlying K-vector spaces of $K^{\Box}I^{*}$ and $(K^{\Box}I)^{\mathrm{op}}$ are subspaces of the K-algebra $\mathbb{M}_I^{\bullet}(K)$ of all matrices $\lambda = [\lambda_{pq}] \in \mathbb{M}_I(K)$ such that $\lambda_{pq} = 0$ for all but finitely many $p, q \in I$. Transposition $\widehat{\mathrm{tr}} : \mathbb{M}_I^{\bullet}(K) \to \mathbb{M}_I^{\bullet}(K)$ carries the matrix unit $e_{pq} \in KI$ with $p \preceq q$ to the matrix unit $e_{qp} \in KI^{*}$, with $q \preceq^{*} p$. Moreover, $\widehat{\mathrm{tr}}$ induces the coalgebra isomorphism $\widehat{\mathrm{tr}} : K^{\Box}I \xrightarrow{\simeq} (K^{\Box}I^{*})^{\mathrm{op}}$. Indeed, it is easy to check that $\varepsilon_{I^{*}}(\widehat{\mathrm{tr}}(e_{pq})) = \varepsilon_I(e_{pq})$ and $(\widehat{\mathrm{tr}} \otimes \widehat{\mathrm{tr}})\Delta_I^{\mathrm{op}}(e_{pq}) = \Delta_{I^{*}}\widehat{\mathrm{tr}}(e_{pq}) = \Delta_{I^{*}}(e_{qp})$ for any $p, q \in I$ such that $p \preceq q$.

(b) Recall that there is an isomorphism $(K^{\Box}I)^{\mathrm{op}}$-Comod $\cong$ Comod-$K^{\Box}I$ of categories that restricts to $(K^{\Box}I)^{\mathrm{op}}$-comod $\cong$ comod-$K^{\Box}I$. Hence (b) follows from (a).

(c) The first part follows immediately from the definition, but the second one is not immediate. However, the proof is left to the reader (consult [14, Section 2]). ∎

Now we collect the basic properties of the incidence coalgebra $K^{\Box}I$.

PROPOSITION 2.5. *Let I be an intervally finite poset I.*

(a) *The coalgebra $K^{\Box}I$ is basic, and it is connected* (*indecomposable*) *if and only if the poset I is connected. Moreover,* $\dim_K K^{\Box}I \leq \aleph_0$ *if I is connected.*

(b) *For $C = K^{\Box}I$ and each $j \in I_C$,*

$$S_I(j) = e_j \cdot (KI) \cdot e_j \cong Ke_j$$

is a one-dimensional simple left coideal (*and subcoalgebra*) *of C, the left ideal*

$$E_I(j) = KI \cdot e_j$$

of the K-algebra KI is a left coideal of the coalgebra C such that $\operatorname{soc} E_I(j) = S_I(j)$, $\mathrm{End}_C S_I(j) \cong K$, *and* $\mathrm{End}_C E_I(j) \cong K$. *Moreover, there are vector space isomorphisms*

$$\mathrm{Hom}_C(E_I(q), E_I(p)) \underset{\simeq}{\xrightarrow{\xi_{qp}}} \begin{cases} Ke_{pq} & \text{if } p \preceq q, \\ 0 & \text{if } p \not\preceq q. \end{cases} \tag{2.6}$$

(c) *There are left $K^{\Box}I$-comodule decompositions*

$$\operatorname{soc} K^{\Box}I = \bigoplus_{j \in I} S_I(j) \quad \text{and} \quad K^{\Box}I = \bigoplus_{j \in I} E_I(j), \tag{2.7}$$

(d) *The coalgebra C is* Hom-*computable, the composition length matrix* ${}_CF = [\ell_{pq}] \in \mathbb{M}_I^{\preceq}(\mathbb{Z})$ *coincides with the Cartan matrix* ${}_C\widehat{F} = [\widehat{\ell}_{pq}] \in \mathbb{M}_I^{\preceq}(\mathbb{Z})$ *with* $\ell_{pq} = \widehat{\ell}_{pq} = \dim_K \mathrm{Hom}_C(E_I(p), E_I(q))$, *and* ${}_CF^{\mathrm{tr}} = \mathfrak{C}_I$. *Given $p \in I$, the transpose of the vector* $\mathbf{lgth}\, E_I(p) = (\ell_{pq})_{q \in I} = (c_{qp})_{q \in I} \in \mathbb{Z}^{(I)}$ *of $E_I(p)$ is the pth column of the matrix $\mathfrak{C}_I$.*

Proof. The statement (a) is a consequence of (b).

To prove (b), (c), and (d), we note that the left ideal $E_I(j) = KI \cdot e_j$ of the K-algebra KI is the jth column left ideal of KI consisting of all matrices $\lambda = [\lambda_{pq}]$ such that $\lambda_{pq} = 0$ for $q \neq j$ and all $p \in I$. It follows that $K^{\Box}I = \bigoplus_{j\in I} E_I(j)$. By the definition of the comultiplication Δ_I, $E_I(j)$ is a left coideal of $C = K^{\Box}I$. Further, it is easy to see that there are K-vector space isomorphisms

$$\begin{aligned}\mathrm{Hom}_C(E_I(q), E_I(p)) &\cong \mathrm{Hom}_C(KI\cdot e_q, KI\cdot e_p) \cong e_p\cdot(KI)\cdot e_q\\ &\cong \begin{cases} Ke_{pq} & \text{if } p \preceq q,\\ 0 & \text{if } p \not\preceq q.\end{cases}\end{aligned}$$

Moreover, one easily shows that if $p \prec q \prec s$ and $\xi_{qp} : \mathrm{Hom}_C(E_I(q), E_I(p)) \xrightarrow{\sim} Ke_{pq}$ and $\xi_{sq} : \mathrm{Hom}_C(E_I(s), E_I(q)) \xrightarrow{\sim} Ke_{qs}$ are the composite isomorphisms then, for any $f \in \mathrm{Hom}_C(E_I(q), E_I(p))$ and $g \in \mathrm{Hom}_C(E_I(s), E_I(q))$,

$$\xi_{sp}(f\circ g) = \xi_{qp}(f)\cdot \xi_{sq}(g). \tag{2.8}$$

It follows that $\ell_{pq} = \widehat{\ell}_{pq} = \dim_K \mathrm{Hom}_C(E_I(p), E_I(q)) = c_{qp} \leq 1$ for all $p, q \in I$, that is, C is Hom-computable and ${}_CF^{\mathrm{tr}} = \mathfrak{C}_I$ [30, Proposition 2.9]. Moreover, there is an algebra isomorphism $\mathrm{End}_C E_I(j) \cong K$. Hence each $E_I(j)$ is an indecomposable injective left C-comodule containing the simple comodule $S_I(j)$, that is, $S_I(j) = \mathrm{soc}\, E_I(j)$, and $E_I(j)$ is the injective envelope of $S_I(j)$ in $K^{\Box}I$-Comod. ∎

Throughout, we make the identifications ${}_CF = {}_C\widehat{F} = \mathfrak{C}_I^{\mathrm{tr}}$.

Following [16], [23], and [31] we define a K-coalgebra C to be *left $c\ell$-hereditary* if every colocal epimorphic image of an injective left C-comodule is injective. Here, a comodule M is called *colocal* if M contains a unique simple subcomodule, or equivalently, M is isomorphic to a subcomodule of an indecomposable injective comodule (see also [31] and [32]). It is easy to check that C is left $c\ell$-hereditary if and only if every non-zero homomorphism $f : E \to E'$ between indecomposable injective left C-comodules E and E' is surjective. It is clear that hereditary coalgebras are left and right $c\ell$-hereditary.

COROLLARY 2.9. *Let I be an intervally finite poset, and $C = K^{\Box}I$.*

(a) *The coalgebra C is left and right $c\ell$-hereditary.*

(b) *The incidence matrix $\mathfrak{C}_I$ and the Cartan matrix ${}_CF = [\ell_{pq}] \in \mathbb{M}_I^{\preceq}(\mathbb{Z})$ with $\ell_{pq} = \dim_K \mathrm{Hom}_C(E_I(p), E_I(q))$ are two-sided invertible in the ring $\mathbb{M}_I^{\preceq}(\mathbb{Z})$, and ${}_CF^{\mathrm{tr}} = \mathfrak{C}_I$. More precisely, the matrix*

$$\mathfrak{C}_I^{-1} = [c^-_{pq}]_{p,q\in I} \in \mathbb{M}_I^{\preceq}(\mathbb{Z}), \tag{2.10}$$

defined by the formula (2.11) *below, is a unique right and unique left inverse of $\mathfrak{C}_I$ and ${}_CF^{-1} = (\mathfrak{C}_I^{-1})^{\mathrm{tr}}$.*

(c) *Given $a \prec b$ in I, the restriction $\mathbf{C}_I^{-1}|_{[a,b]}$ is the inverse of the restriction $\mathbf{C}_{[a,b]}$ of $\mathbf{C}_I$ to $[a,b]$.*

Proof. (a) Given $p \prec q$, let $\kappa_{qp} : E_I(q) \to E_I(p)$ be the C-comodule homomorphism such that $\xi_{qp}(\kappa_{qp}) = e_{pq}$, where $\xi_{qp} : \mathrm{Hom}_C(E_I(q), E_I(p)) \xrightarrow{\simeq} Ke_{pq}$ is the isomorphism (2.6).

We prove that C is left $c\ell$-hereditary by showing that, given $s, q \in I$, any non-zero homomorphism $g : E_I(s) \to E_I(q)$ is surjective. Since $f \neq 0$, the preceding proof yields $q \preceq s$. In view of Proposition 2.5(b), we may assume that $q \prec s$. If, to the contrary, g is not surjective then $E_I(q)/\mathrm{Im}\, g$ is non-zero and there exist $p \in I$ and a non-zero $h \in \mathrm{Hom}_C(E_I(q)/\mathrm{Im}\, g, E_I(p))$. It follows that $fg = 0$, where f is the composite homomorphism $E_I(q) \to E_I(q)/\mathrm{Img} \xrightarrow{h} E_I(p)$. Note that $p \prec q \prec s$ and $f = \mu' \cdot \kappa_{qp}$, $g = \mu \cdot \kappa_{sq}$ for some non-zero scalars $\mu, \mu' \in K$. By applying (2.8), we get

$$0 = \xi_{sp}(0) = \xi_{sp}(f \circ g) = \xi_{qp}(f) \cdot \xi_{sq}(g) = \mu\mu' e_{pq} \cdot e_{qs} = \mu\mu' e_{qs} \neq 0.$$

This contradiction finishes the proof of (a).

(b) Since, according to Proposition 2.5(d), ${}_C F^{\mathrm{tr}} = \mathbf{C}_I$, it is sufficient to prove that $\mathbf{C}_I$ has a left inverse that is also a right inverse. We define $\mathbf{C}_I^{-1} = [c^-_{pq}]_{p,q \in I}$ as follows. Given $a, b \in I$ such that $a \preceq b$, we view the interval $[a,b]$ as a subposet of I, and let $\mathbf{C}_{[a,b]} = [c_{pq}]_{p,q \in [a,b]} \in \mathbb{M}_{[a,b]}(\mathbb{Z})$ be the restriction of the matrix $\mathbf{C}_I$ to $[a,b]$. Since I is intervally finite, $[a,b]$ is finite, say $[a,b] = \{a_1 = a, a_2, \ldots, a_m = b\}$ with $a_i \prec a_j$ for $i < j$. It follows that by a simultaneous permutation of rows and columns of $\mathbf{C}_{[a,b]}$ we can reduce it to an upper triangular matrix in $\mathbb{M}_m(\mathbb{Z})$ with ones on the diagonal. It follows that $\det \mathbf{C}_{[a,b]} = 1$, and the matrix $\mathbf{C}_{[a,b]}$ has an inverse $\mathbf{C}_{[a,b]}^{-1} = [\widehat{c}^{ab}_{pq}]_{p,q \in [a,b]} \in \mathbb{M}_{[a,b]}(\mathbb{Z})$ such that $\widehat{c}^{ab}_{pp} = 1$ for any $p \in [a,b]$. It is easy to see that if $a \preceq c \preceq d \preceq b$, then the restriction of $\mathbf{C}_{[a,b]}^{-1}$ to $[c,d] \subseteq [a,b]$ is $\mathbf{C}_{[c,d]}^{-1}$, and in particular $\widehat{c}^{ab}_{cd} = \widehat{c}^{cd}_{cd}$.

For any $a, b \in I$, we define

$$c^-_{ab} = \begin{cases} 0 & \text{if } a \not\preceq b, \\ \widehat{c}^{ab}_{ab} & \text{if } a \preceq b. \end{cases} \tag{2.11}$$

Obviously, $\mathbf{C}_I^{-1} = [c^-_{ab}]_{a,b \in I} \in \mathbb{M}_I^{\preceq}(\mathbb{Z})$. To see that $\mathbf{C}_I^{-1} \cdot \mathbf{C}_I = [c'_{pq}]$ is the identity matrix $E \in \mathbb{M}_I^{\preceq}(\mathbb{Z})$, fix $a, b \in I$. Since $c_{pq} = c^-_{pq} = 0$ whenever $p \not\preceq q$, we get

$$\begin{aligned} c'_{ab} &= \sum_{s \in I} c^-_{as} c_{sb} = \sum_{a \preceq s \preceq b} c^-_{as} c_{sb} = \sum_{s \in [a,b]} \widehat{c}^{as}_{as} c_{sb} \\ &= \sum_{s \in [a,b]} \widehat{c}^{ab}_{as} c_{sb} = \begin{cases} 1 & \text{for } a = b, \\ 0 & \text{for } a \neq b, \end{cases} \end{aligned}$$

because $\mathfrak{C}_{[a,b]}^{-1} \cdot \mathfrak{C}_{[a,b]}$ is the identity matrix in $\mathbb{M}_{[a,b]}(\mathbb{Z})$. Similarly we show that $\mathfrak{C}_I \cdot \mathfrak{C}_I^{-1} = E$. The equality $({}_CF^{\mathrm{tr}})^{-1} = \mathfrak{C}_I^{-1}$ follows easily from ${}_CF^{\mathrm{tr}} = \mathfrak{C}_I$. Since (c) follows from the construction of $\mathfrak{C}_I^{-1}$, the proof is complete. ∎

Now we give an explicit formula for the entries c_{ij}^{-} of $\mathfrak{C}_I^{-1}$ in terms of I, and hence we get an explicit formula for the Euler $\mathbb{Z}$-bilinear form b_C.

For simplicity of notation, we denote by $\ell(a,b)$ the length of the maximal path in the Hasse quiver $Q_{[a,b]}$ of the poset $[a,b]$, and we call it the *length of the interval* $[a,b]$.

PROPOSITION 2.12. *Let I be an intervally finite connected poset and let $\mathfrak{C}_I^{-1} = [c_{ij}^{-}]_{i,j\in I} \in \mathbb{M}_I^{\preceq}(\mathbb{Z})$ be the* (*left and right*) *inverse of $\mathfrak{C}_I = [c_{ij}]$. Then*

$$c_{ab}^{-} = \begin{cases} \sum\limits_{s=1}^{\ell(a,b)} (-1)^s \widehat{c}_{ab}^{(s)} & \textit{if } a \preceq b, \\ 0 & \textit{if } a \not\preceq b, \end{cases} \tag{2.13}$$

where

$$\widehat{c}_{ab}^{(s)} = \sum_{a=j_0 \prec j_1 \prec \cdots \prec j_s = b} c_{aj_0} c_{j_0 j_1} c_{j_1 j_2} \ldots c_{j_{s-1} b}.$$

In particular,

$$c_{ab}^{-} = \begin{cases} 1 & \textit{if } a = b, \\ -1 & \textit{if there is an arrow } a \to b \textit{ in the Hasse quiver } Q_I, \\ 0 & \textit{if } Q_{[a,b]} = \{a \to j_1 \to \cdots \to j_{s-1} \to j_s = b\} \textit{ and } s \geq 2. \end{cases}$$

Proof. It follows from (2.11) that $c_{ab}^{-} = 0$ if $a \not\preceq b$ in I, and $c_{ab}^{-} = \widehat{c}_{ab}^{ab}$ if $a \preceq b$, where $\widehat{c}_{ab}^{ab}$ is the (a,b)-entry of

$$\mathfrak{C}_{[a,b]}^{-1} = [\widehat{c}_{pq}^{ab}]_{p,q\in[a,b]} \in \mathbb{M}_J(\mathbb{Z}), \qquad J = [a,b].$$

Thus, we may assume that $I = [a,b]$ is a finite poset, where $a \preceq b$. Moreover, we may assume that $I = \{a = 1, 2, \ldots, m-1, m = b\}$ with $p \preceq q$ implying $p \leq q$ in the natural order of $\mathbb{Z}$. This means that the matrices $\mathfrak{C}_{[a,b]} = \mathfrak{C}_I = [c_{ij}] \in \mathbb{M}_m(\mathbb{Z})$ and $\mathfrak{C}_{[a,b]}^{-1} = \mathfrak{C}_I^{-1} = [c_{ij}^{-}] \in \mathbb{M}_m(\mathbb{Z})$ have the upper triangular forms

$$\mathfrak{C}_I = \begin{bmatrix} 1 & c_{12} & \ldots & c_{1m} \\ 0 & 1 & \ldots & c_{2m} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}, \qquad \mathfrak{C}_I^{-1} = \begin{bmatrix} 1 & c_{12}^{-} & \ldots & c_{1m}^{-} \\ 0 & 1 & \ldots & c_{2m}^{-} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}$$

with $c_{11} = \cdots = c_{mm} = 1$, $c_{11}^{-} = \cdots = c_{mm}^{-} = 1$, $c_{ij} = 0$ and $c_{ij}^{-} = 0$ if $i \not\preceq j$.

It follows that the group homomorphism $\widehat{\sigma}_I : \mathbb{Z}^m \to \mathbb{Z}^m$ defined by the formula $\widehat{\sigma}_I(x) = x \cdot \mathfrak{C}_I$ for $x = (x_1, \ldots, x_m) \in \mathbb{Z}^m$ is an isomorphism, and $\widehat{\sigma}_I^{-1}$ is defined by $\widehat{\sigma}_I^{-1}(x') = x' \cdot \mathfrak{C}_I^{-1}$ for $x' = (x'_1, \ldots, x'_m) \in \mathbb{Z}^m$.

It is easy to see that if $(x'_1, \ldots, x'_m) = \widehat{\sigma}_I(x)$ then $x'_j = \sum_{i \preceq j} c_{ij} x_i = \sum_{i \preceq j} x_i$ for all $j \in \{1, \ldots, m\}$. It follows that

$$\begin{aligned} x'_1 &= x_1, \\ x'_2 &= c_{12}x_1 + x_2, \\ x'_3 &= c_{13}x_1 + c_{23}x_2 + x_3, \\ &\ \vdots \qquad \vdots \\ x'_m &= c_{1m}x_1 + c_{2m}x_2 + \cdots + c_{m-1,m}x_{m-1} + x_m, \end{aligned}$$

and, since $x = \widehat{\sigma}_I^{-1}(x')$, we have

$$\begin{aligned} x_1 &= c^-_{11}x'_1, \\ x_2 &= c^-_{12}x'_1 + x'_2, \\ x_3 &= c^-_{13}x'_1 + c^-_{23}x'_2 + x'_3, \\ &\ \vdots \qquad \vdots \\ x_m &= c^-_{1m}x'_1 + c^-_{2m}x'_2 + \cdots + c^-_{m-1,m}x'_{m-1} + x'_m. \end{aligned}$$

On the other hand, the elimination procedure applied to the first system of equations yields

$$\begin{aligned} &x_1 = x'_1, && \text{hence } c^-_{11} = 1; \\ &x_2 = -c_{12}x'_1 + x'_2, && \text{hence } c^-_{12} = -c_{12},\ c^-_{22} = 1; \\ &x_3 = [-c_{13} + c_{12}c_{23}]x'_1 - c_{23}x'_2 + x'_3, && \text{hence } c^-_{13} = -c_{12} + c_{12}c_{23}, \\ & && \qquad\quad c^-_{23} = -c_{23},\ c^-_{33} = 1; \\ &x_4 = [-c_{14} + c_{12}c_{24} + c_{13}c_{34} - c_{12}c_{23}c_{34}]x'_1 \\ &\qquad + [-c_{24} + c_{23}c_{34}]x'_2 - c_{34}x'_3 + x'_4, && \text{hence } c^-_{14} = -c_{14} + c_{12}c_{24} \\ & && \qquad\qquad + c_{13}c_{34} - c_{12}c_{23}c_{34}, \\ & && \qquad\quad c^-_{24} = -c_{24} + c_{23}c_{34}, \\ & && \qquad\quad c^-_{34} = -c_{34},\ c^-_{44} = 1. \end{aligned}$$

We can show by induction that

$$x_m = c^-_{1m}x'_1 + c^-_{2m}x'_2 + \cdots + c^-_{m-1,m}x'_{m-1} + x'_m,$$

where $c^-_{mm} = 1$ and

$$c^-_{pm} = \sum_{s=1}^{m} (-1)^s \Big[\sum_{p = j_0 \prec j_1 \prec \cdots \prec j_s = m} c_{pj_0} c_{j_0 j_1} c_{j_1 j_2} \ldots c_{j_{s-1} m} \Big]$$

for $p = 1, \ldots, m-1$. Since we assume $a = 1$ and $b = m$, the coefficient $c_{ab}^- = c_{1m}^-$ has the desired form. ∎

3. Integral bilinear forms associated to intervally finite posets. Assume that I is a connected intervally finite poset and let $C = K^{\square}I$, with decompositions (2.7). Let $\mathbf{C}_I = [c_{ij}]_{i,j\in I} \in \mathbb{M}_I(\mathbb{Z})$ be the incidence matrix of I and let $\mathbf{C}_I^{-1} = [c_{ij}^-]_{i,j\in I} \in \mathbb{M}_I^{\preceq}(\mathbb{Z})$ be the right (and left) inverse of $\mathbf{C}_I$.

Following [8] and [26], we associate to I five $\mathbb{Z}$-bilinear forms

$$\widehat{b}_I, b_I, b_I^{\mathrm{tr}}, \overline{b}_I, \overline{b}_I^{\mathrm{tr}} : \mathbb{Z}^{(I)} \times \mathbb{Z}^{(I)} \to \mathbb{Z} \tag{3.1}$$

defined by the formulae

$$\widehat{b}_I(x,y) = \sum_{i\in I} x_i y_i + \sum_{j\prec i\in I^-} x_i y_j - \sum_{p\in\max I}\Big(\sum_{i\prec p} x_i\Big) y_p,$$

$$b_I(x,y) = \sum_{i\in I} x_i y_i + \sum_{i\prec j} x_i y_j = x\cdot \mathbf{C}_I \cdot y^{\mathrm{tr}},$$

$$b_I^{\mathrm{tr}}(x,y) = \sum_{i\in I} x_i y_i + \sum_{j\prec i} x_i y_j = x\cdot \mathbf{C}_I^{\mathrm{tr}} \cdot y^{\mathrm{tr}},$$

$$\overline{b}_I(x,y) = \sum_{i\in I} x_i y_i + \sum_{i\prec j} c_{ij}^- x_i y_j = x\cdot \mathbf{C}_I^{-1} \cdot y^{\mathrm{tr}},$$

$$\overline{b}_I^{\mathrm{tr}}(x,y) = \sum_{i\in I} x_i y_i + \sum_{j\prec i} c_{ji}^- x_i y_j = x\cdot \mathbf{C}_I^{-tr} \cdot y^{\mathrm{tr}},$$

where $\max I$ is the set of all maximal elements of I and $I^- = I \setminus \max I$ is viewed as a subposet of I. We call $\widehat{b}_I$ the *Tits* (*geometric*) *bilinear form* of I, b_I and b_I^{tr} the *ordinary* $\mathbb{Z}$*-bilinear forms* of I, and $\overline{b}_I$, $\overline{b}_I^{\mathrm{tr}}$ the *Euler* $\mathbb{Z}$*-bilinear forms* of I. The corresponding integral quadratic forms

$$\widehat{q}_I, q_I, \overline{q}_I : \mathbb{Z}^{(I)} \to \mathbb{Z}, \tag{3.2}$$

defined by $\widehat{q}_I(x) = \widehat{b}_I(x,x)$, $q_I(x) = b_I(x,x) = b_I^{\mathrm{tr}}(x,x)$, and $\overline{q}_I(x) = \overline{b}_I(x,x) = \overline{b}_I^{\mathrm{tr}}(x,x)$, are called the *Tits* (*geometric*) *integral quadratic form*, the *ordinary integral quadratic form*, and the *Euler integral quadratic form* of I, respectively.

We note that if I is infinite and has no maximal elements, then $\max I = \emptyset$, $I^- = I$ and we get $b_I^{\mathrm{tr}} = \widehat{b}_I$ and $q_I = \widehat{q}_I$, that is, the ordinary and the Tits quadratic forms of I coincide.

We recall from [8], [25], and [26] that the Tits form of a finite poset plays a crucial role in the study of matrix representations of posets and in describing the finite posets that are of finite or of tame prinjective type. Similarly, the Euler quadratic form of a coalgebra or a finite-dimensional algebra is one of the basic tools in determining the representation type (see [1], [4], [15], [21], [25]–[30], [32], [33], [34]).

We recall from [30, (4.6)] that the Euler $\mathbb{Z}$-bilinear form

$$b_C : \mathbb{Z}^{(I)} \times \mathbb{Z}^{(I)} \to \mathbb{Z} \tag{3.3}$$

of $C = K^{\square}I$ is defined by the formula $b_C(x,y) = x \cdot ({}_CF^{-1})^{\mathrm{tr}} \cdot y^{\mathrm{tr}}$ for $x, y \in \mathbb{Z}^{(I)}$, where ${}_CF = [\ell_{pq}] \in \mathbb{M}_I^{\preceq}(\mathbb{Z})$ is the Cartan matrix of C, with $\ell_{pq} = \dim_K \mathrm{Hom}_C(E_I(p), E_I(q))$. The Euler quadratic form $q_C : \mathbb{Z}^{(I)} \to \mathbb{Z}$ is defined by

$$q_C(x) = b_C(x,x) = x \cdot ({}_CF^{-1})^{\mathrm{tr}} \cdot x^{\mathrm{tr}} \quad \text{for } x \in \mathbb{Z}^{(I)}.$$

To get a matrix description of the Tits form $\widehat{b}_I$ and relate it to the Euler forms $\overline{b}_I$, $\overline{b}_I^{\mathrm{tr}}$, b_C, we introduce some concepts.

Definition 3.4. Let I be a connected intervally finite poset.

(a) The *reduced incidence matrix* of I is the bipartite matrix

$$\mathfrak{C}_I^{\bullet} = \left[\begin{array}{c|c} \mathfrak{C}_{I^-} & 0 \\ \hline 0 & E \end{array}\right] \in \mathbb{M}_I^{\preceq}(\mathbb{Z}), \tag{3.5}$$

where $E \in \mathbb{M}_{\max I}(\mathbb{Z})$ is the identity matrix and $\mathfrak{C}_{I^-} \in \mathbb{M}^{\preceq}_{I^- \times I^-}(\mathbb{Z})$ is the incidence matrix of $I^- = I \setminus \max I$.

(b) The *Tits matrix* of I is the bipartite matrix

$$\widehat{\mathfrak{C}}_I = \mathfrak{C}_I^{\bullet} + (\mathfrak{C}_I^{\bullet})^{\mathrm{tr}} - \mathfrak{C}_I = \left[\begin{array}{c|c} \mathfrak{C}_{I^-}^{\mathrm{tr}} & -U \\ \hline 0 & E \end{array}\right] \in \mathbb{M}_I^{\preceq}(\mathbb{Z}), \tag{3.6}$$

where $U = [c_{jp}]_{j \in I^-, p \in \max I} \in \mathbb{M}^{\preceq}_{I^- \times \max I}(\mathbb{Z})$ has $c_{jp} = 1$ if $j \in I^-$, $p \in \max I$ and $j \prec p$, whereas $c_{jp} = 0$ otherwise.

The following proposition shows that the forms $\widehat{b}_I$, b_C and b_I^{tr}, b_C are $\mathbb{Z}$-congruent if, for each $a \in I$, the right cone $a^{\trianglerighteq}$ and the left cone ${}^{\trianglelefteq}a$ are finite.

Proposition 3.7. *Let I be a connected intervally finite poset, and $C = K^{\square}I$.*

(a) *For each $j \in I$, the jth column of $\mathfrak{C}_I$ is the length vector*

$$\mathbf{e}(j) = \mathbf{lgth}\, E_I(j) \in \mathbb{Z}^I$$

of the indecomposable injective comodule $E_I(j)$, that is,

$$\mathfrak{C}_I = [\ldots, \mathbf{e}(j)^{\mathrm{tr}}, \ldots]_{j \in I} \in \mathbb{M}_I^{\preceq}(\mathbb{Z}).$$

(b) *$b_C = \overline{b}_I$, that is, for all $x, y \in \mathbb{Z}^{(I)}$,*

$$b_C(x,y) = \overline{b}_I(x,y) = x \cdot \mathfrak{C}_I^{-1} \cdot y^{\mathrm{tr}} = \sum_{i \in I} x_i y_i + \sum_{i \prec p} c_{ip}^- x_i y_p.$$

(c) *For all* $x, y \in \mathbb{Z}^{(I)}$,

$$\widehat{b}_I(x,y) = x \cdot \widehat{\mathfrak{C}}_I \cdot y^{\mathrm{tr}} \quad \textit{and} \quad b_I^{\mathrm{tr}}(x,y) = x \cdot \mathfrak{C}_I^{\mathrm{tr}} \cdot y^{\mathrm{tr}}.$$

(d) *The bilinear forms* $\widehat{b}_I$, b_C, *and* b_I^{tr}, b_C *are* $\mathbb{Z}$*-congruent and the following diagrams are commutative*:

$$\begin{tikzcd} \mathbb{Z}^{(I)} \times \mathbb{Z}^{(I)} \arrow[r, "\widehat{b}_I"] \arrow[d, "\sigma_I^{\bullet} \times \sigma^{\bullet}"', "\cong"] & \mathbb{Z} \\ \mathbb{Z}^{(I)} \times \mathbb{Z}^{(I)} \arrow[ur, "b_{K\square I}"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z}^{(I)} \times \mathbb{Z}^{(I)} \arrow[r, "b_I^{\mathrm{tr}}"] \arrow[d, "\sigma_I^{\mathrm{tr}} \times \sigma^{\mathrm{tr}}"', "\cong"] & \mathbb{Z} \\ \mathbb{Z}^{(I)} \times \mathbb{Z}^{(I)} \arrow[ur, "b_{K\square I}"'] & \end{tikzcd} \tag{3.8}$$

if, *for each* $a \in I$, *the right cone* $a^{\trianglerighteq}$ *and the left cone* ${}^{\trianglelefteq}a$ *are finite*, *where the group homomorphisms* $\sigma_I^{\bullet}, \sigma_I^{\mathrm{tr}} : \mathbb{Z}^{(I)} \to \mathbb{Z}^{(I)}$ *defined by* $\sigma_I^{\bullet}(x) = x \cdot \mathfrak{C}_I^{\bullet}$ *and* $\sigma_I^{\mathrm{tr}}(x) = x \cdot \mathfrak{C}_I^{\mathrm{tr}}$, *for* $x \in \mathbb{Z}^{(I)}$, *are isomorphisms* (see [26, (2.2), (3.3)]).

Proof. (a) By Proposition 2.5(d), ${}_C F^{\mathrm{tr}} = \mathfrak{C}_I$ and (a) follows.

(b) By Corollary 2.9, $({}_C F^{\mathrm{tr}})^{-1} = \mathfrak{C}_I^{-1}$. Then $b_C(x,y) = \overline{b}_I(x,y) = x \cdot \mathfrak{C}_I^{-1} \cdot y^{\mathrm{tr}}$. The equality $x \cdot \mathfrak{C}_I^{-1} \cdot y^{\mathrm{tr}} = \sum_{i \prec p} c_{ip}^{-} x_i y_p$ for all $x, y \in \mathbb{Z}^{(I)}$ is verified by a direct calculation.

(c) The equalities are easily verified by a direct calculation.

(d) By Corollary 2.9, $\mathfrak{C}_I \in \mathbb{M}_I^{\preceq}(\mathbb{Z})$ and $\mathfrak{C}_{I^-} \in \mathbb{M}_{I^-}^{\preceq}(\mathbb{Z})$ are invertible. It follows that $\mathfrak{C}_I^{\bullet}$ and $\mathfrak{C}_I^{\mathrm{tr}}$ are invertible in $\mathbb{M}_I^{\preceq}(\mathbb{Z})$.

Assume that ${}^{\trianglelefteq}a$ is finite for each $a \in I$. Hence, in view of (a), each row of $\mathfrak{C}_I^{\mathrm{tr}}$ has a finite number of non-zero entries and therefore, for every $x \in \mathbb{Z}^{(I)}$, the vector $\sigma_I^{\mathrm{tr}}(x) = x \cdot \mathfrak{C}_I^{\mathrm{tr}} \in \mathbb{Z}^I$ lies in $\mathbb{Z}^{(I)}$. Thus, $\sigma_I^{\mathrm{tr}} : \mathbb{Z}^{(I)} \to \mathbb{Z}^{(I)}$ is a group isomorphism, the products $\mathfrak{C}_I^{\mathrm{tr}} \cdot \mathfrak{C}_I^{-1}$ and $(\mathfrak{C}_I^{\mathrm{tr}} \cdot \mathfrak{C}_I^{-1}) \cdot \mathfrak{C}_I$ are defined, and

$$\mathfrak{C}_I^{\mathrm{tr}} = (\mathfrak{C}_I^{\mathrm{tr}} \cdot \mathfrak{C}_I^{-1}) \cdot \mathfrak{C}_I = \mathfrak{C}_I^{\mathrm{tr}} \cdot (\mathfrak{C}_I^{-1} \cdot \mathfrak{C}_I).$$

Hence, for any $x, y \in \mathbb{Z}^{(I)}$, we get

$$\begin{aligned} b_C(\sigma_I^{\mathrm{tr}}(x), \sigma_I^{\mathrm{tr}}(y)) &= (x \cdot \mathfrak{C}_I^{\mathrm{tr}}) \cdot [\mathfrak{C}_I^{-1} \cdot (y \cdot \mathfrak{C}_I^{\mathrm{tr}})^{\mathrm{tr}}] \\ &= (x \cdot \mathfrak{C}_I^{\mathrm{tr}}) \cdot [\mathfrak{C}_I^{-1} \cdot \mathfrak{C}_I \cdot y^{\mathrm{tr}}] \\ &= x \cdot \mathfrak{C}_I^{\mathrm{tr}} \cdot y^{\mathrm{tr}} = b_I^{\mathrm{tr}}(x,y), \end{aligned}$$

that is, the right hand diagram in (3.8) is commutative.

Assume that $a^{\trianglerighteq}$ is finite for each $a \in I$. It follows that each row of $\mathfrak{C}_I$ has a finite number of non-zero entries, and therefore, for every $x \in \mathbb{Z}^{(I)}$, the vector $\sigma_I^{\bullet}(x) = x \cdot \mathfrak{C}_I^{\bullet} \in \mathbb{Z}^I$ lies in $\mathbb{Z}^{(I)}$. Hence, $\sigma_I^{\bullet} : \mathbb{Z}^{(I)} \to \mathbb{Z}^{(I)}$ is a group isomorphism. Moreover, the products $(\mathfrak{C}_I^{\bullet} \cdot \mathfrak{C}_I^{-1}) \cdot (\mathfrak{C}_I^{\bullet})^{\mathrm{tr}}$, $\mathfrak{C}_I^{\bullet} \cdot [\mathfrak{C}_I^{-1} \cdot (\mathfrak{C}_I^{\bullet})^{\mathrm{tr}}]$ are defined and the commutativity of the left hand diagram in (3.8) is a consequence of the equalities

$$\widehat{\mathfrak{C}}_I = (\mathfrak{C}_I^\bullet \cdot \mathfrak{C}_I^{-1}) \cdot (\mathfrak{C}_I^\bullet)^{\mathrm{tr}} = \mathfrak{C}_I^\bullet \cdot [\mathfrak{C}_I^{-1} \cdot (\mathfrak{C}_I^\bullet)^{\mathrm{tr}}]. \tag{3.9}$$

To prove the first equality of (3.9), we note that

$$(\mathfrak{C}_I^\bullet)^{\mathrm{tr}} = \left[\begin{array}{c|c} \mathfrak{C}_{I^-}^{\mathrm{tr}} & 0 \\ \hline 0 & E \end{array}\right], \qquad \mathfrak{C}_I^{-1} = \left[\begin{array}{c|c} \mathfrak{C}_{I^-}^{-1} & -\mathfrak{C}_{I^-}^{-1} \cdot U \\ \hline 0 & E \end{array}\right],$$

$$\mathfrak{C}_I^\bullet - \mathfrak{C}_I = \left[\begin{array}{c|c} 0 & -U \\ \hline 0 & 0 \end{array}\right].$$

Then we get

$$\begin{aligned}
&(\mathfrak{C}_I^\bullet \cdot \mathfrak{C}_I^{-1}) \cdot (\mathfrak{C}_I^\bullet)^{\mathrm{tr}} \\
&\quad = \left(\left[\begin{array}{c|c} \mathfrak{C}_{I^-} & 0 \\ \hline 0 & E \end{array}\right] \cdot \left[\begin{array}{c|c} \mathfrak{C}_{I^-}^{-1} & -\mathfrak{C}_{I^-}^{-1} \cdot U \\ \hline 0 & E \end{array}\right]\right) \cdot \left[\begin{array}{c|c} \mathfrak{C}_{I^-}^{\mathrm{tr}} & 0 \\ \hline 0 & E \end{array}\right] \\
&\quad = \left[\begin{array}{c|c} \mathfrak{C}_{I^-} \cdot \mathfrak{C}_{I^-}^{-1} & \mathfrak{C}_{I^-} \cdot (-\mathfrak{C}_{I^-}^{-1} \cdot U) \\ \hline 0 & E \end{array}\right] \cdot \left[\begin{array}{c|c} \mathfrak{C}_{I^-}^{\mathrm{tr}} & 0 \\ \hline 0 & E \end{array}\right] \\
&\quad = \left[\begin{array}{c|c} E & -U \\ \hline 0 & E \end{array}\right] \cdot \left[\begin{array}{c|c} \mathfrak{C}_{I^-}^{\mathrm{tr}} & 0 \\ \hline 0 & E \end{array}\right] = \left[\begin{array}{c|c} \mathfrak{C}_{I^-}^{\mathrm{tr}} & -U \\ \hline 0 & E \end{array}\right] = \widehat{\mathfrak{C}}_I.
\end{aligned}$$

The equality $\widehat{\mathfrak{C}}_I = \mathfrak{C}_I^\bullet \cdot [\mathfrak{C}_I^{-1} \cdot (\mathfrak{C}_I^\bullet)^{\mathrm{tr}}]$ follows in a similar way. ■

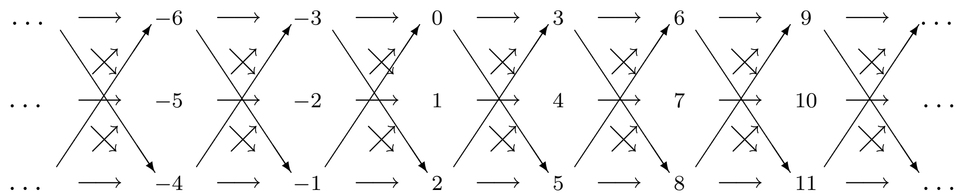
Corollary 3.10. *Let I be an intervally finite connected poset and let $C = K^\square I$. Then the quadratic forms $q_C, q_{C^{\mathrm{op}}}, \overline{q}_I : \mathbb{Z}^{(I)} \to \mathbb{Z}$ coincide, and*

$$q_C(x) = q_{C^{\mathrm{op}}}(x) = \overline{q}_I(x) = \sum_{j\in I} x_j^2 + \sum_{p\prec q\in I} c_{pq}^- x_p x_q,$$

for any $x \in \mathbb{Z}^{(I)}$, where c_{pq}^- is given by the formulae (2.11) *and* (2.13).

Proof. Since, according to Lemma 2.4, there is a coalgebra isomorphism $C^{\mathrm{op}} \cong K^\square I^*$, the corollary follows from Propositions 2.12 and 3.7(a). ■

Example 3.11. Let $I = \mathbb{Z}$ be the poset with Hasse quiver Q_I of the form



The matrices $\mathfrak{C}_I \in \mathbb{M}_{\mathbb{Z}}^{\preceq}(\mathbb{Z})$ and $\mathfrak{C}_I^{-1}$ have the forms

$$
\begin{array}{c}
\begin{array}{ccccccccccccccccc}
 & & & & & -4 & -3 & -2 & -1 & 0 & & & & & 9 & 10 & \\
 & & & & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & & & & \downarrow & \downarrow &
\end{array}
\\
\mathfrak{C}_I = \left[\begin{array}{ccccccccccccccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\ldots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\ldots & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\ldots & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\ldots & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & \ldots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & &
\end{array}\right]
\begin{array}{l}
 \\ \leftarrow -4 \\ \leftarrow -3 \\ \leftarrow -2 \\ \leftarrow -1 \\ \leftarrow 0 \\ \leftarrow 1 \\ \leftarrow 2 \\ \leftarrow 3 \\ \vdots \\ \\ \\ \\
\end{array}
\end{array}
$$

$$
\mathfrak{C}_I^{-1} =
$$

$$
\left[\begin{array}{ccccccccccccccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\ldots & 1 & -1 & -1 & -1 & 2 & 2 & 2 & -4 & -4 & -4 & 8 & 8 & 8 & -16 & -16 & \ldots \\
\ldots & 0 & 1 & 0 & 0 & -1 & -1 & -1 & 2 & 2 & 2 & -4 & -4 & -4 & 8 & 8 & \ldots \\
\ldots & 0 & 0 & 1 & 0 & -1 & -1 & -1 & 2 & 2 & 2 & -4 & -4 & -4 & 8 & 8 & \ldots \\
\ldots & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 2 & 2 & 2 & -4 & -4 & -4 & 8 & 8 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & -1 & 2 & 2 & 2 & -4 & -4 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 & 2 & 2 & 2 & -4 & -4 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 2 & 2 & 2 & -4 & -4 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & -1 & 2 & 2 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 & 2 & 2 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 2 & 2 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & \ldots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & &
\end{array}\right]
\begin{array}{l}
 \\ \leftarrow -4 \\ \leftarrow -3 \\ \leftarrow -2 \\ \leftarrow -1 \\ \leftarrow 0 \\ \leftarrow 1 \\ \leftarrow 2 \\ \leftarrow 3 \\ \vdots \\ \\ \\ \\
\end{array}
$$

Now we illustrate the use of Proposition 3.7 in computing the entry $c^-_{09} = -4$ of $\mathbf{C}_I^{-1}$. By (2.13) applied to $a = 0$ and $b = 9$, we get $\ell(0,9) = 3$ and $c^-_{09} = -\widehat{c}^{(1)}_{09} + \widehat{c}^{(2)}_{09} - \widehat{c}^{(3)}_{09} = -1 + 6 - 9 = -4$, because

$$\begin{aligned}
\widehat{c}^{(1)}_{09} &= c_{09} = 1,\\
\widehat{c}^{(2)}_{09} &= c_{03}c_{39} + c_{04}c_{49} + c_{05}c_{59} + c_{06}c_{69} + c_{07}c_{79} + c_{08}c_{89} = 6,\\
\widehat{c}^{(3)}_{09} &= c_{03}c_{36}c_{69} + c_{03}c_{37}c_{79} + c_{03}c_{38}c_{89} + c_{04}c_{46}c_{69} + c_{04}c_{47}c_{79} + c_{04}c_{48}c_{89}\\
&\quad + c_{05}c_{56}c_{69} + c_{05}c_{57}c_{79} + c_{05}c_{58}c_{89} = 9.
\end{aligned}$$

It follows that the Euler $\mathbb{Z}$-bilinear form $b_C : \mathbb{Z}^{(\mathbb{Z})} \times \mathbb{Z}^{(\mathbb{Z})} \to \mathbb{Z}$ and the ordinary $\mathbb{Z}$-bilinear form $b_I^{\mathrm{tr}} = \widehat{b}_I : \mathbb{Z}^{(\mathbb{Z})} \times \mathbb{Z}^{(\mathbb{Z})} \to \mathbb{Z}$ are given by the formulae

$$\begin{aligned}
b_C(x,y) &= \sum_{p\in\mathbb{Z}} x_p y_p + \sum_{p\in\mathbb{Z}} \sum_{q=p+3}^{\infty} (-1)^{q-p-2} 2^{q-p-3}\\
&\qquad\qquad \cdot (x_p + x_{p+1} + x_{p+2}) \cdot (y_q + y_{q+1} + y_{q+2}),\\
\widehat{b}_I(x,y) &= \sum_{p\in\mathbb{Z}} x_p y_p + \sum_{p\in\mathbb{Z}} \sum_{q=p+3}^{\infty} (x_p + x_{p+1} + x_{p+2}) \cdot (y_q + y_{q+1} + y_{q+2}),
\end{aligned}$$

for all $x = (x_j)_{j\in\mathbb{Z}}$ and $y = (y_j)_{j\in\mathbb{Z}}$ in the free abelian group $\mathbb{Z}^{(I)} = \mathbb{Z}^{(\mathbb{Z})}$.

We give in [32] a characterisation of the incidence coalgebras $K^\square I$ of intervally finite posets I such that the Euler form $b_{K^\square I} = \overline{b}_I : \mathbb{Z}^{(I)} \times \mathbb{Z}^{(I)} \to \mathbb{Z}$ is weakly positive, i.e. $b_{K^\square I}(v) > 0$ for every non-zero vector $v \in \mathbb{Z}^{(I)}$ with non-negative coordinates. They are just the representation-directed coalgebras in the sense of [30, Section 6]. We also show in [32] that every such coalgebra $K^\square I$ is tame of discrete comodule type [27] and $\mathrm{gl.dim}\, K^\square I \le 2$. Moreover, we present there a complete list of all connected and intervally finite posets I such that $b_{K^\square I}$ is weakly positive.

4. Comodule categories over incidence coalgebras of intervally finite posets. For any poset I, we denote by Q_I its Hasse quiver. The K-algebra homomorphism $KQ_I \to KI$ associating to any arrow $p \to q$ of Q_I the matrix unit $e_{pq} \in KI$ induces a K-algebra isomorphism

$$KQ_I/\Omega_I \cong KI, \tag{4.1}$$

where Ω_I is the two-sided ideal of KQ_I generated by all commutativity relations, that is, by all differences $w' - w'' \in KQ_I$ of paths w', w'' of length $m \ge 2$ with a common source and common target (see [1, Chapter II] and [25, Chapter 14]).

We denote by $K^\square Q_I$ the path K-coalgebra of the quiver Q_I, and by

$$K^\square(Q_I, \Omega_I) = \Omega_I^\perp = \{\psi \in K^\square Q_I;\ \langle \psi, \Omega_I\rangle = 0\} \tag{4.2}$$

the path K-coalgebra of the bound quiver (Q_I, Ω_I), viewed as a subcoalgebra of $K^{\square}Q_I$ (see [27], [28], and [31]).

One of the main aims of this section is to study the comodule category $K^{\square}I$-Comod by means of K-linear representations of I. We recall that a *K-linear representation* of a poset I is a system $X = (X_p, {}_q\varphi_p)_{p\prec q}$, where X_p is a K-vector space for each $p \in I$, ${}_q\varphi_p : X_p \to X_q$ is a K-linear map for all $p \prec q$, and ${}_s\varphi_q \circ {}_q\varphi_p = {}_s\varphi_p$ for all $p \prec q \prec s$. A *morphism* $f : X \to X'$ of representations is a system $f = (f_p)_{p\in I}$ of K-linear maps $f_p : X_p \to X'_p$ such that ${}_q\varphi'_p \circ f_p = f_q \circ {}_q\varphi_p$ for $p \prec q$ [23].

We denote by $\mathrm{Rep}_K(I)$ the Grothendieck K-category of K-linear representations of I, and by $\mathrm{rep}_K(I) \supseteq \mathrm{rep}^{\ell f}_K(I)$ the abelian full subcategories of $\mathrm{Rep}_K(I)$ formed by the finitely generated objects and by the finitely generated representations of finite length, respectively. Finally, we denote by $\mathrm{Rep}^{\ell f}_K(I)$ the full Grothendieck subcategory of $\mathrm{Rep}_K(I)$ formed by the locally finite representations, that is, directed unions of objects from $\mathrm{rep}^{\ell f}_K(I)$.

We say that $X = (X_p, {}_q\varphi_p)_{p\preceq q}$ is *locally nilpotent* if for any $p \in I$ and $x_p \in X_p$ there exists an integer $m \geq 1$ such that ${}_{i_m}\varphi_{i_{m-1}} \circ \cdots \circ {}_{i_2}\varphi_{i_1} \circ {}_{i_1}\varphi_{i_0}(x_p) = 0$ for all paths $i_0 \to i_1 \to \cdots \to i_m$ of length m in the Hasse quiver Q_I (see [15], [27], [9, Section 7.4]). The representation X is said to be *nilpotent* if there exists an $m \geq 1$ such that ${}_{i_m}\varphi_{i_{m-1}} \circ \cdots \circ {}_{i_2}\varphi_{i_1} \circ {}_{i_1}\varphi_{i_0} = 0$ for all paths $i_0 \to i_1 \to \cdots \to i_m$ in Q_I. We denote by $\mathrm{nilrep}^{\ell f}_K(I)$ the full subcategory of $\mathrm{rep}^{\ell f}_K(I)$ formed by all nilpotent representations of finite length, and by $\mathrm{Rep}^{\ell n\ell f}_K(I)$ the full subcategory of $\mathrm{Rep}^{\ell f}_K(I)$ formed by all locally nilpotent representations. Any representation of I of finite length is nilpotent, that is, $\mathrm{nilrep}^{\ell f}_K(I) = \mathrm{rep}^{\ell f}_K(I)$, and hence $\mathrm{Rep}^{\ell n\ell f}_K(I) = \mathrm{Rep}^{\ell f}_K(I)$.

PROPOSITION 4.3. *Let I be a connected intervally finite poset, and $C = K^{\square}I$.*

(a) *There exists a K-coalgebra isomorphism $K^{\square}I \cong K^{\square}(Q_I, \Omega_I)$ (see (4.5)).*
(b) *The pseudocompact K-algebra $C^* = (K^{\square}I)^*$ dual to $K^{\square}I$ is isomorphic to the completion $\widehat{KI} = \mathbb{M}^{\preceq}_I(K)$ of KI in the cofinite topology. In particular, there is a K-algebra isomorphism $(K^{\square}I)^* \cong KI = \mathbb{M}^{\preceq}_I(K)$ if I is finite.*
(c) *The functor* (4.8) *constructed below defines K-linear category equivalences*

$$\begin{aligned} K^{\square}I\text{-comod} &\cong \mathrm{nilrep}^{\ell f}_K(I) = \mathrm{rep}^{\ell f}_K(I), \\ K^{\square}I\text{-Comod} &\cong \mathrm{Rep}^{\ell n\ell f}_K(I) = \mathrm{Rep}^{\ell f}_K(I). \end{aligned} \tag{4.4}$$

(d) *Under the identification $K^{\square}I$-Comod $\cong \mathrm{Rep}^{\ell f}_K(I)$, for each $p \in I$, the simple comodule $S_I(p)$ is identified with the representation $S_I(p) =$*

$(\overline{K}_q^{(p)}, {}_sO_q^{(p)})$, *where* $\overline{K}_q^{(p)} = K$ *if* $q = p$, $\overline{K}_q^{(p)} = 0$ *if* $q \neq p$, *and* ${}_sO_q^{(p)} = 0$ *for all* $s \prec q$.

(e) *Under the identification* $K^{\Box}I$-Comod $\cong \mathrm{Rep}_K^{\ell f}(I)$, *for each* $p \in I$, *the injective comodule* $E_I(p)$ *is identified with the representation* $E_I(p) = (K_q^{(p)}, {}_s\psi_q^{(p)})$, *where* $K_q^{(p)} = K$ *if* $q \preceq p$, $K_q^{(p)} = 0$ *if* $q \npreceq p$, ${}_s\psi_q^{(p)} : K_q^{(p)} \to K_s^{(p)}$ *is the identity map if* $s \prec q \prec p$, *and* ${}_s\psi_q^{(p)} = 0$ *otherwise.*

Proof. (a) For each $p, q \in I$, set

$$\Omega_I^{\perp}(p,q) = \{\psi \in KQ^I(p,q);\ \langle \psi, \Omega_I \rangle = 0\}.$$

It is easy to check that $\Omega_I^{\perp}(p,q) = K\widehat{e}_{pq}$, where $\widehat{e}_{pq}$ is the sum of all oriented paths ω in Q_I starting in p and ending with q (see [28, 3.12]). It follows that the K-linear map

$$\theta : K^{\Box}I \to K^{\Box}(Q_I, \Omega_I), \qquad q_{pq} \mapsto \widehat{e}_{pq}, \tag{4.5}$$

is a K-coalgebra isomorphism.

(b) Because of the isomorphism (4.5) of coalgebras, (b) reduces to the corresponding statement for $K^{\Box}(Q_I, \Omega_I)$, where the arguments in the proof of Theorem 3.14 in [28] apply. The details are left to the reader.

(c) It is shown in [20], [27], [28], and [30, Proposition 3.3] that the K-linear category equivalences

$$K^{\Box}Q_I\text{-Comod} \cong \mathrm{Rep}_K^{\ell f}(Q_I) \quad \text{and} \quad K^{\Box}Q_I\text{-comod} \cong \mathrm{rep}_K^{\ell f}(Q_I)$$

established in [27] restrict to the category equivalences

$$\begin{aligned} K^{\Box}(Q_I, \Omega_I)\text{-Comod} &\cong \mathrm{Rep}_K^{\ell n \ell f}(Q_I, \Omega_I) = \mathrm{Rep}_K^{\ell f}(Q_I, \Omega_I) \cong \mathrm{Rep}_K^{\ell f}(I), \\ K^{\Box}(Q_I, \Omega_I)\text{-comod} &\cong \mathrm{nilrep}_K^{\ell f}(Q_I, \Omega_I) = \mathrm{rep}_K^{\ell f}(Q_I, \Omega_I) \cong \mathrm{rep}_K^{\ell f}(I). \end{aligned}$$

Hence, in view of (4.5), we get (4.4).

For the convenience of the reader, we give a direct construction of an equivalence

$$\Phi : K^{\Box}I\text{-Comod} \xrightarrow{\simeq} \mathrm{Rep}_K^{\ell f}(I). \tag{4.6}$$

Recall that the Yoneda map $\widetilde{\varepsilon}_M : \mathrm{Hom}_C(M, C) \to \mathrm{Hom}_K(M, K) = M^*$ [27, 4.9] is an isomorphism for any left C-comodule M. Moreover, there is a natural isomorphism $M \cong \mathrm{hom}_K(M^*, K) = (M^*)^{\circ}$ of left C-comodules, where $(M^*)^{\circ} = \mathrm{hom}_K(M^*, K)$ is the set of all continuous K-linear maps from the pseudocompact K-vector space M^* to K (see [27, Sections 2–4]).

It follows that there are natural isomorphisms of K-vector spaces

$$M \cong (M^*)^\circ = [\mathrm{Hom}_C(M,C)]^\circ \cong \Big[\mathrm{Hom}_C\Big(M, \bigoplus_{p\in I} E_I(p)\Big)\Big]^\circ$$
$$\cong \Big[\check{\prod_{p\in I}} \mathrm{Hom}_C(M, E_I(p))\Big]^\circ \cong \bigoplus_{p\in I} [\mathrm{Hom}_C(M, E_I(p))]^\circ = \bigoplus_{p\in I} M_p,$$

where

$$M_p = [\mathrm{Hom}_C(M, E_I(p))]^\circ$$

is viewed as a K-vector space, $\mathrm{Hom}_C(M, E_I(p))$ is viewed as a pseudocompact K-vector space in a natural way (see [27, Sections 2–4]), and the vector subspace

$$\check{\prod_{p\in I}} \mathrm{Hom}_C(M, E_I(p)) = \Big\{(\psi_p) \in \prod_{p\in I} \mathrm{Hom}_C(M, E_I(p));\ \text{for each } m \in M,$$
$$\psi_p(m) = 0 \text{ for almost all } p\Big\}$$

of $\prod_{p\in I} \mathrm{Hom}_C(M, E_I(p))$ is viewed as the product in the category of pseudocompact K-vector spaces. One can show (see [5, p. 870] and [31, Section 3]) that

$$(4.7)\quad M_p = [\mathrm{Hom}_C(M, E_I(p))]^\circ = \{m \in M;\ \varrho_M^-(m) = e_p \otimes m\} = M \cdot e_p,$$

where ϱ_M^- is the composite map $M \xrightarrow{\varrho_M} (K^\square I) \otimes M \xrightarrow{\pi_0} (K^\square I)_0 \otimes M$ and $\pi_0 : K^\square I \to (K^\square I)_0 = \mathrm{soc}\, K^\square I$ is the canonical K-linear projection (see also [31, (3.2) and (4.6)] and [5, p. 870]).

We define the functor Φ of (4.6) by setting

$$(4.8)\qquad \Phi(M) = (M_p, {}_q\varphi_p)_{p\prec q},$$

where ${}_q\varphi_p : M_p \to M_q$, for $p \prec q$, is the K-linear map defined as follows. Let ${}_p\kappa_q : E_I(q) \to E_I(p)$ be the C-comodule homomorphism such that $\xi_{qp}({}_p\kappa_q) = e_{pq}$, where $\xi_{qp} : \mathrm{Hom}_C(E_I(q), E_I(p)) \xrightarrow{\simeq} Ke_{pq}$ is the isomorphism (2.6). We take for ${}_q\varphi_p$ the induced K-linear map $[\mathrm{Hom}_C(M, {}_p\kappa_q)]^\circ : [\mathrm{Hom}_C(M, E_I(p))]^\circ \to [\mathrm{Hom}_C(M, E_I(q))]^\circ$.

Given $f \in \mathrm{Hom}_C(M, M')$, we set $\Phi(f) = (f_p)_{p\in I}$, where the map $f_p = [\mathrm{Hom}_C(f, E_I(p))]^\circ : M_p \to M'_p$ is induced by f. It is clear that we have thus defined an exact faithful K-linear functor $\Phi : K^\square I\text{-Comod} \to \mathrm{Rep}_K(I)$.

Now we show that, for any M in C-Comod, the representation $\Phi(M) = (M_p, {}_q\varphi_p)_{p\prec q}$ is locally nilpotent and locally finite. First we assume that $\dim_K M$ is finite. In view of the isomorphism $M \cong \bigoplus_{p\in I} M_p$, each vector space M_p is of finite K-dimension, and $M_p = 0$ for all but a finite number of $p \in I$. It follows that the representation $\Phi(M) = (M_p, {}_q\varphi_p)_{p\prec q}$ is nilpotent of finite length.

Next we assume that M is arbitrary. Note that if M' is a C-subcomodule of M then the inclusion $M' \subseteq M$ induces an embedding $\Phi(M') \hookrightarrow \Phi(M)$ of representations of I. Moreover, if M_β is a directed family of finite-dimensional C-subcomodules of M such that $M = \bigcup_\beta M_\beta$, then the embeddings $\Phi(M_\beta) \hookrightarrow \Phi(M)$ induce the equality $\Phi(M) = \bigcup_\beta \Phi(M_\beta)$ (see [27, Sections 3–4]). Since each representation $\Phi(M_\beta)$ is nilpotent of finite length, it follows that $\Phi(M)$ is locally nilpotent and locally finite. Consequently, $\Phi(M)$ lies in the category $\mathrm{Rep}_K^{\ell f}(I) = \mathrm{Rep}_K^{\ell n\ell f}(I)$.

It is clear that the functor Φ is fully faithful and exact. To show that it is dense, we note that if $(M_p, {}_q\varphi_p)_{p\prec q}$ is a nilpotent representation of I of finite length then each M_p is of finite K-dimension, and $M_p = 0$ for all but finitely many $p \in I$. Then $M = \bigoplus_{p\in I} M_p$ is of finite K-dimension and one easily shows that the linear maps ${}_q\varphi_p$ induce a left C-comultiplication $\delta_M : M \to C \otimes M$ on M such that M is a comodule and $\Phi(M) = (M_p, {}_q\varphi_p)_{p\prec q}$. Hence, by simple limit arguments, Φ is dense, and consequently it is an equivalence of categories (see also [30, Proposition 3.3]).

(d) By applying (4.8) to the simple comodule $M = S_I(p)$, we get $M_q = [\mathrm{Hom}_C(M, E_I(p))]^\circ = [\mathrm{Hom}_C(S_I(q), E_I(p))]^\circ \cong K$ if $p = q$, and $M_q = 0$ if $p \neq q$, because $S_I(q)$ is the unique simple subcomodule of the injective comodule $E_I(q)$ and $S_I(p) \ncong S_I(q)$ for $p \neq q$.

(e) Applying (4.8) to the injective comodule $M = E_I(p)$, the isomorphism (2.6) yields $M_q = \mathrm{Hom}_C(M, E_I(p))^\circ = \mathrm{Hom}_C(E_I(q), E_I(p))^* \cong Ke_{pq} \cong K$ if $p \preceq q$, and $M_q = 0$ if $p \npreceq q$. Hence (e) follows.

5. Minimal injective resolutions of simple comodules. We recall that a coalgebra C is said to be *right semiperfect* if every indecomposable injective left C-comodule is finite-dimensional. The semiperfect incidence coalgebras are characterised as follows.

LEMMA 5.1. *Let I be an intervally finite poset, and $C = K^\square I$.*

(a) *The coalgebra C is right semiperfect if and only if, for each $b \in I$, the left cone ${}^{\trianglelefteq}b$ is finite.*
(b) *The coalgebra C is left semiperfect if and only if, for each $a \in I$, the right cone $a^{\trianglerighteq}$ is finite.*

Proof. (a) We recall from Proposition 3.7(a) that ${}_CF^{\mathrm{tr}} = \mathfrak{C}_I$ and the bth column of $\mathfrak{C}_I$ is the vector $\mathbf{lgth}\, E_I(b)$. Hence, $E_I(b)$ is finite-dimensional if and only if $\mathbf{lgth}\, E_I(b)$ has a finite number of non-zero coordinates, or equivalently, ${}^{\trianglelefteq}b$ is finite (apply Proposition 4.3(e)).

(b) Since C is left semiperfect if and only if C^{op} is right semiperfect, and since by Lemma 2.4, there is a coalgebra isomorphism $C^{\mathrm{op}} \cong K^\square I^*$, where I^* is the poset opposite to I, it follows that (b) is a consequence of (a). ■

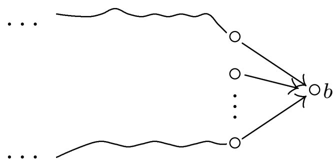
PROPOSITION 5.2. *Let I be an intervally finite poset, and $C = K^{\square}I$.*

(a) *If $b \in I$ is such that the left cone ${}^{\trianglelefteq}b$ is of finite width then the left C-comodule $E_I(b)$ is artinian.*

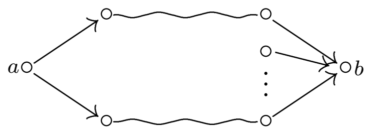
(b) *If I is left locally bounded then the coalgebra C is locally left artinian and left cocoherent.*

Proof. (a) Assume that $b \in I$ and ${}^{\trianglelefteq}b$ is of finite width $\mathbf{w}({}^{\trianglelefteq}b)$, that is, ${}^{\trianglelefteq}b$ contains $\mathbf{w}({}^{\trianglelefteq}b)$ pairwise incomparable elements, and $\mathbf{w}({}^{\trianglelefteq}b)$ is maximal with this property. We visualise the cone ${}^{\trianglelefteq}b$ and the interval $[a, b] = a^{\trianglerighteq} \cap {}^{\trianglelefteq}b$, with $a \prec b$, as follows:

${}^{\trianglelefteq}b$:



$[a, b]$:



By Proposition 4.3(e), the left C-comodule $E_I(b)$ is identified with the representation $E_I(b) = (K_q^{(b)}, {}_s\psi_q^{(b)})$ of I that is constant over ${}^{\trianglelefteq}b$, that is, $K_q^{(b)} = K$ if $q \in {}^{\trianglelefteq}b$, $K_q^{(b)} = 0$ if $q \notin {}^{\trianglelefteq}b$, ${}_s\psi_q^{(b)} : K_q^{(b)} \to K_s^{(b)}$ is the identity map if $s \prec q \prec b$, and ${}_s\psi_q^{(b)} = 0$ otherwise.

Let $X = (X_q, {}_s\widetilde{\psi}_q^{(b)})$ be a subrepresentation of $E_I(b)$. Then X_q is either K or zero, and ${}_s\psi_q^{(b)} : X_q \to X_s$ is the identity map on K or the zero map. The assumption that X is a subrepresentation of $E_I(b)$ implies that:

- if $X_a = K$, then $X_p = K$ for all $p \in [a, b]$,
- if $X_a = 0$, then $X_p = 0$ for all $p \preceq a$.

Consider the support

$$\mathcal{S}(X) = \mathbf{supp}(X) = \{a \in I;\ X_a \neq 0\}$$

of X as a suposet of ${}^{\trianglelefteq}b \subseteq I$; and set

$$\mathcal{S}^{-}(X) = \{p \in {}^{\trianglelefteq}b;\ X_p = 0\}.$$

Obviously, ${}^{\trianglelefteq}b = \mathcal{S}(X) \cup \mathcal{S}^{-}(X)$. The previous observations yield:

- if $a \in \mathcal{S}(X)$, then $[a, b] \subseteq \mathcal{S}(X)$,
- if $a \in \mathcal{S}^{-}(X)$, then ${}^{\trianglelefteq}a \subseteq \mathcal{S}^{-}(X)$.

Since I is assumed to be intervally finite, the set $\max \mathcal{S}^{-}(X)$ of all maximal elements of $\mathcal{S}^{-}(X)$ is finite and the subposet

$$\mathcal{S}^{+}(X) = \{b\} \cup \{p \in {}^{\trianglelefteq}b;\ q \prec p \text{ for some } q \in \max \mathcal{S}^{-}(X)\}$$

of ${}^{\trianglelefteq}b$ is also finite. It is easy to see that, given a subrepresentation $Y \subseteq X$ of $X \subseteq E_I(b)$, we have $\mathcal{S}(Y) \subseteq \mathcal{S}(X)$, $\mathcal{S}^{-}(Y) \supseteq \mathcal{S}^{-}(X)$, and $\mathcal{S}(Y) = \mathcal{S}(X)$

if and only if $X = Y$. It follows that if $\mathcal{S}(X)$ is finite then the representation X is artinian.

Suppose that $X \subseteq E_I(b)$ with $\mathcal{S}(X)$ infinite. Then $\mathcal{S}(X)$ has a cofinite poset disjoint union presentation

$$(*) \qquad \mathcal{S}(X) = \mathcal{S}_X \cup I_X,$$

where $\mathcal{S}_X$ is a finite poset containing b and there is no relation $p \preceq q$ with $p \in \mathcal{S}_X$ and $q \in I_X$. If, in addition, I_X is infinite with no minimal elements, then we call $(*)$ a *fin-infinite presentation* of $\mathcal{S}(X)$. One such presentation is given by setting $\mathcal{S}_X = \mathcal{S}^+(X)$ and $I_X = \mathcal{S}(X) \setminus \mathcal{S}^+(X)$.

Now we prove that, given a subrepresentation $X \subseteq E_I(b)$, every proper subrepresentation $Y \subset X$ of X is artinian. We proceed by induction on the width $\mathbf{w}(I_X)$ of the poset I_X in a fin-infinite presentation $(*)$ of $\mathcal{S}(X)$.

Assume that $\mathbf{w}(I_X) = 1$, that is, I_X is of the form $\cdots \to \bullet \to \bullet \to \cdots \to \bullet \to \bullet$. Assume that $Y \subset X$ is a proper subrepresentation of X. Then $Y_p = 0$ for some $p \in \mathcal{S}(X)$. If $p \in I_X$, then $Y_q = 0$ for all $q \preceq p$, and $\mathcal{S}(Y)$ is finite. It follows that Y is artinian, and we are done. Assume that $p \in \mathcal{S}_X$. It follows that $\mathcal{S}(Y) = \mathcal{S}_Y \cup I_Y$ with $\mathcal{S}_Y = \mathcal{S}_X \setminus \mathcal{S}^-(Y)$ is a fin-infinite presentation such that $\mathbf{w}(I_Y) = 1$. Note also that $|\mathcal{S}_Y| < |\mathcal{S}_X|$.

If $\mathcal{S}_Y = \{b\}$ then, by the preceding arguments, every proper subrepresentation Y' of Y is artinian. If $\mathcal{S}_Y \neq \{b\}$ and Y' is a proper subrepresentation of Y, then $\mathcal{S}(Y')$ is finite (and Y' is artinian) or $\mathcal{S}(Y')$ has a fin-infinite presentation $\mathcal{S}(Y') = \mathcal{S}_{Y'} \cup I_{Y'}$, where $|\mathcal{S}_{Y'}| < |\mathcal{S}_Y|$ and $\mathbf{w}(I_{Y'}) = 1$. Hence, Y' is artinian, by an obvious induction on $|\mathcal{S}_{Y'}|$. It follows that Y is also artinian, and this finishes the proof in case $\mathbf{w}(I_X) = 1$.

Assume that $\mathbf{w}(I_X) = r \geq 2$ and the claim is proved for all $Z \subseteq E_I(b)$ such that $\mathbf{w}(I_Z) \leq r-1$ for some fin-infinite presentation of $\mathcal{S}(Z)$. Fix a fin-infinite presentation $(*)$ for X and take a proper subrepresentation $Y \subset X$. Then $Y_p = 0$ for some $p \in \mathcal{S}(X)$, that is, $p \in \mathcal{S}^-(Y)$ and ${}^{\trianglelefteq}p \subseteq \mathcal{S}^-(Y)$.

Assume that $p \in I_X$, and let $p_1, \ldots, p_s \in I_X$ be all maximal elements in $\mathcal{S}^-(Y) \cap I_X$. It follows that $1 \leq s \leq r$, the set $\mathcal{S}' = [p_1, b] \cup \cdots \cup [p_s, b]$ is finite, and the poset $I_Y = I_X \setminus \mathcal{S}'$ has no minimal elements, because I_X has none. Now we show that $I_Y = I_X \setminus \mathcal{S}'$ has width smaller than r. To prove this, we note that every $p' \in I_Y \subset I_X$ is incomparable with each p_j, because obviously the relations $p_1 \preceq p', \ldots, p_s \preceq p'$ do not hold, and the relation $p' \preceq p_j \in S^-(Y)$ for some j would yield $p' \in S^-(Y) \cap I_Y = \emptyset$, a contradiction. Since $s \geq 1$, we have $\mathbf{w}(I_Y) \leq r - s \leq r - 1$, as claimed.

Consequently, we get a cofinite presentation $\mathcal{S}(Y) = \mathcal{S}_Y \cup I_Y$, where $\mathcal{S}_Y = \mathcal{S}'$. If I_Y is empty, then $\mathcal{S}(Y)$ is finite and Y is artinian. If I_Y is not empty, it is infinite of width smaller than r and the presentation $\mathcal{S}(Y) = \mathcal{S}_Y \cup I_Y$ is fin-infinite. Then, by induction, every proper subrepresentation $Y' \subset Y$ of Y is artinian. It follows that Y is artinian, and we are done. In particular, this shows that any proper subrepresentation $Y \subset X$ such that $\mathcal{S}(Y)$ has a cofinite presentation $\mathcal{S}(Y) = \mathcal{S}_Y \cup I_Y$ with $\mathcal{S}_Y = \{b\}$ is artinian.

Assume that $p \notin I_X$, that is, $p \in \mathcal{S}_X$. It follows that $\mathcal{S}(Y) = \mathcal{S}_Y \cup I_Y$ with $\mathcal{S}_Y = \mathcal{S}_X \setminus \mathcal{S}^-(Y)$ is a fin-infinite presentation such that $\mathbf{w}(I_Y) \le r$. Note that $|\mathcal{S}_Y| < |\mathcal{S}_X|$. If $\mathcal{S}_Y = \{b\}$, then Y is artinian. If $\mathcal{S}_Y \neq \{b\}$ and Y' is a proper subrepresentation of Y, then $\mathcal{S}(Y')$ is finite (and Y' is artinian) or $\mathcal{S}(Y')$ has a fin-infinite presentation $\mathcal{S}(Y') = \mathcal{S}_{Y'} \cup I_{Y'}$, where $|\mathcal{S}_{Y'}| < |\mathcal{S}_Y|$ and $\mathbf{w}(I_{Y'}) \le r$. Hence, Y' is artinian, by an obvious induction on $|\mathcal{S}_{Y'}|$. It follows that Y is also artinian. This finishes the proof of our claim in case $\mathbf{w}(I_X) = r$.

To finish the proof of (a), assume that

$$E_I(b) \supseteq X^{(1)} \supseteq X^{(2)} \supseteq \cdots$$

is a chain of subrepresentations of $E_I(b)$. It terminates, because otherwise some of the inclusions is proper; then some $X^{(m)}$ is a proper subrepresentation of $E_I(b)$, and since by our claim, $X^{(m)}$ is artinian, we get a contradiction.

(b) First we consider the special case when each cone ${}^{\trianglelefteq}b$ has finite width. It follows from (a) that the C-comodule $E_I(b)$ is artinian, that is, C is left locally artinian. Hence it follows easily that C is left cocoherent (see [13, Proposition 1.3]). The proof in the general case when I is left locally bounded (that is, no ${}^{\trianglelefteq}b$ contains infinitely many pairwise incomparable elements) is analogous. It depends on (a) extended to the case of ${}^{\trianglelefteq}b$ without infinitely many pairwise incomparable elements. The argument given above adapts to this situation. ■

To formulate the main result of this section, we need some notation. Given two finite subsets I_1, I_2 of a poset I, we write

$$I_2 \trianglelefteq I_1 \quad \text{or} \quad I_1 \trianglerighteq I_2$$

if $I_1 \cap \max I_2 \neq \emptyset$ and for any $i_2 \in I_2$ there is an $i_1 \in \max I_1$ such that $i_2 \prec i_1$. We write

$$I_2 \prec I_1$$

if $I_1 \cap I_2 = \emptyset$, $i_1 \not\preceq i_2$ for all $i_1 \in I_1$ and $i_2 \in I_2$, and for any $i_2 \in I_2$ there is an $i_1 \in I_1$ such that $i_2 \prec i_1$.

Given M in C-Comod, the subposet

$$\mathbf{supp}(M) = \mathbf{supp}(\mathbf{lgth}\, M) = \{p \in I;\ (\mathbf{lgth}\, M)_p = \ell_p(M) \neq 0\}$$

of I is called the *support* of M, where $\ell_p(M) = \dim_K \operatorname{Hom}_C(M, E_I(p))$ (see [30, Proposition 2.6(b)]).

We recall from [30, Definition 4.15] that a basic coalgebra C with indecomposable left C-comodule decompositions $\operatorname{soc} C = \bigoplus_{j \in I_C} S(j)$ and $C = \bigoplus_{j \in I_C} E(j)$, is a *left Euler coalgebra* if $\dim_K \operatorname{Hom}_C(E(i), E(j))$ is finite for all $i, j \in I_C$, $\operatorname{Ext}^m_C(S(i), S(j)) = 0$ for m sufficiently large, and every $S(j)$

admits an injective resolution $0 \to S(j) \to E_0^{(j)} \to E_1^{(j)} \to \cdots$ such that $E_m^{(j)}$ is socle-finite for $m \geq 0$, and for each $i \in I_C$ there exists $m_i \geq 0$ such that $\mathrm{Hom}_C(E_r^{(j)}, E(i)) = 0$ for all $r \geq m_i$.

It is shown in [30] that for any finite-dimensional comodules M and N over a left Euler coalgebra C,

$$b_C(\mathbf{lgth}\, M, \mathbf{lgth}\, N) = \chi_C(M, N) + \partial_C(M, N),$$

where b_C is the Euler form and $\partial_C(M, N) \in \mathbb{Z}$ is the defect in the sense of [30, Definition 4.12].

Now we show that $C = K^{\square} I$ is a left Euler coalgebra, by describing the structure of the mininimal injective resolution of any artinian left C-comodule.

THEOREM 5.3. *Let I be a left locally bounded and intervally finite poset, and let $C = K^{\square} I$ with the decompositions* (2.7).

(a) *C is a locally left artinian and left Euler coalgebra, the left Cartan matrix ${}_C F = \boldsymbol{\mathfrak{C}}_I^{\mathrm{tr}} \in \mathbb{M}_I^{\preceq}(\mathbb{Z})$ has a right and a left inverse, the Euler defect $\partial_C : \mathbb{Z}^{(I)} \times \mathbb{Z}^{(I)} \to \mathbb{Z}$* [30, (4.23)] *is zero, for any M and N in C-*comod *the Euler characteristic $\chi_C(M, N)$ is an integer, and*

$$b_C(\mathbf{lgth}\, M, \mathbf{lgth}\, N) = \chi_C(M, N).$$

(b) *Assume that N is an artinian left C-comodule and*

$$0 \to N \xrightarrow{h_0^N} E_0^N \xrightarrow{h_1^N} E_1^N \xrightarrow{h_2^N} \cdots \tag{5.4}$$

is a minimal injective resolution of N. Given $m \geq 0$, we set $\Omega_m^N = \mathrm{Im}\, h_m^N \subseteq E_m^N$ and $I_m^N = \mathbf{supp}(\mathrm{soc}\, \Omega_m^N)$. Then

(b1) *$I_0^N = \mathbf{supp}(\mathrm{soc}\, N)$ and, for any $m \geq 0$, I_m^N is a finite subset of $\mathbf{supp}(E_0^N) \subseteq I$, the injective comodule E_m^N is socle-finite artinian, and E_m^N has the decomposition*

$$E_m^N = \bigoplus_{a \in I_m^N} E_I(a)^{d_{ma}^N}, \tag{5.5}$$

where d_{mp}^N, with $p \in I_m^N$, is a non-zero integer if $I_m^N \neq \emptyset$,

(b2) *the following relations hold:*

$$\begin{array}{ccccccc}
\mathbf{supp}(E_0^N) & \supseteq & \mathbf{supp}(E_1^N) & \supseteq & \mathbf{supp}(E_2^N) & \supseteq & \cdots \\
\| & & \cup\!\uparrow & & \cup\!\uparrow & & \cdots \\
I_0^N = \mathbf{supp}(\mathrm{soc}\, N) & \trianglerighteq & I_1^N & \trianglerighteq & I_2^N & \trianglerighteq & \cdots
\end{array}$$

(b3) *for each $a \in I$ such that $a \prec b$ for some $b \in I_0^N$, there exists $m_a \geq 0$ such that*

- *$a \not\preceq q$ for all $q \in I_m^N$ and $m \geq m_a$,*

- $\mathrm{Hom}_C(E_m^N, E_I(a)) = 0$ *for all* $m \geq m_a$,
- $\mathrm{Ext}_C^m(S_I(a), N) = 0$ *for all* $m \geq m_a$.

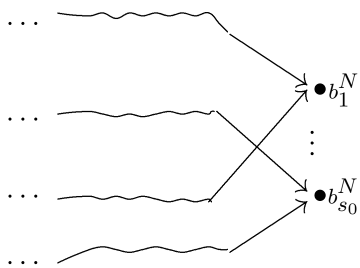
Proof. First we prove (b). Since N is artinian, $\mathrm{soc}\, N$ is finite-dimensional. Hence, the set $I_0^N = \mathbf{supp}(\mathrm{soc}\, \Omega_0^N) = \mathbf{supp}(\mathrm{soc}\, N)$ is finite and $d_{0p}^N = \dim_K \mathrm{Hom}_C(S_I(p), \mathrm{soc}\, N)$ is finite for all $p \in I_0^N$. It follows that the injective envelope E_0^N of N has the form (5.5) with $m = 0$. Since the coalgebra C is locally left artinian (Proposition 5.2(b)), each C-comodule $E(p)$ is artinian, and hence so is E_0^N. It follows that the C-comodule $\Omega_1^N = \mathrm{Im}\, h_1^N$ is artinian, and hence the comodule $\mathrm{soc}\, \Omega_1^N$ is finite-dimensional, the set $I_1^N = \mathbf{supp}(\mathrm{soc}\, \Omega_1^N)$ is finite and $d_{1p}^N = \dim_K \mathrm{Hom}_C(S_I(p), \mathrm{soc}\, \Omega_1^N)$ is finite for all $p \in I_0^N$. It follows that the injective envelope E_1^N of Ω_1^N has the form (5.5) with $m = 1$, and is an artinian C-comodule. Continuing this procedure, we show that the resolution (5.4) consists of socle-finite artinian comodules of the form (5.5), the condition (b1) is satisfied, and $\mathbf{supp}(E_0^N) \supseteq \mathbf{supp}(E_1^N) \supseteq \cdots$.

To prove (b2), for any $m \geq 0$, we consider the set

$$\max I_m^N = \{b_1^N, \ldots, b_{s_m}^N\}.$$

According to Proposition 4.3, the C-comodules Ω_m^N and E_m^N can be viewed as K-linear representations $\Omega_m^N = (\Omega_{m,p}^N, {}_q\varphi_p^{N,m})$ and $E_m^N = (E_{m,p}^N, {}_q\psi_p^{N,m})$ of I. It follows from the form of the simple representations $S_I(p)$ and the indecomposable injective representations $E_I(p)$ that $N_p = E_{0,p}^N$ for any $p \in \max I_0^N$, $\mathbf{supp}(N) \subseteq \mathbf{supp}(E_0^N)$, and $\mathbf{supp}(E_0^N) = {}^{\trianglelefteq}b_1^N \cup \cdots \cup {}^{\trianglelefteq}b_{s_0}^N$ has the form

$\mathbf{supp}(E_0^N)$:



Since $\Omega_1^N \cong E_0^N/N$, we have $\Omega_{1,p}^N = 0$ for all $p \in \max I_m^N = \{b_1^N, \ldots, b_{s_m}^N\}$, and hence

$$\begin{aligned}\mathbf{supp}(\Omega_1^N) \subseteq \mathbf{supp}(E_1^N) \subseteq \mathbf{supp}(E_0^N) \setminus \max I_m^N \\ = ({}^{\trianglelefteq}b_1^N \cup \cdots \cup {}^{\trianglelefteq}b_{s_0}^N) \setminus \{b_1^N, \ldots, b_{s_m}^N\}.\end{aligned}$$

It follows that $I_1^N = \mathbf{supp}(\mathrm{soc}\, \Omega_1^N) \subseteq ({}^{\trianglelefteq}b_1^N \cup \cdots \cup {}^{\trianglelefteq}b_{s_0}^N) \setminus \{b_1^N, \ldots, b_{s_m}^N\}$, and so $I_1^N \trianglelefteq I_0^N$. By applying similar arguments to the artinian C-comodules $\Omega_1^N \subseteq E_1^N$, $\Omega_2^N \subseteq E_2^N, \ldots$, we get the relations $\cdots \trianglelefteq I_2^N \trianglelefteq I_1^N$ required in (b2).

To prove (b3), fix $a \in I$ such that $a \prec b$ for some $b \in I_0^N$. Since, by our assumption, $[a,b]$ is finite, we can choose $b \in \max I_0^N$ such that $[a,b]$ is of maximal length; say $b = b_1^N$ and $\ell = \ell[a,b]$. Assume, to the contrary, that for each $s \geq 1$ there are $m \geq s$ and $q \in I_m^N$ such that $a \preceq q$. By (b2), for each $m \geq 1 + \ell$, there is a path $a \to a_1 \to \cdots \to a_m \to b_j$ for some $b_j \in \max I_0^N$, and we get a contradiction with the maximality of the length of $[a,b]$. Hence, the first statement of (b3) follows. To prove the second, assume to the contrary that $\mathrm{Hom}_C(E_m^N, E(a)) \neq 0$ for some $m \geq m_a$. It follows that there is a direct summand $E(q)$ of E_m^N, with $q \in I_m^N$, such that $\mathrm{Hom}_C(E(p), E(a)) \neq 0$. Thus, the formula (2.6) yields $a \preceq q$, contrary to the first statement of (b3). To prove the third statement in (b3), we note that, by the minimality of the injective resolution (5.4) of N, the following four conditions are equivalent:

- $\mathrm{Ext}_C^m(S_I(a), N) = 0$,
- $E_I(a)$ is a direct summand of E_m^N,
- there is a monomorphism $S_I(a) \hookrightarrow \Omega_m^N$,
- $a \in \mathbf{supp}(\mathrm{soc}\, \Omega_m^N) = I_m^N$.

It follows that, for $m \geq m_a$, we have $\mathrm{Ext}_C^m(S_I(a), N) = 0$, because $a \notin I_m^N$, by the first statement in (b3).

To prove (a), we recall from Propositions 2.5, 5.2, and Corollary 2.9, that the coalgebra C is Hom-computable, locally left artinian, left cocoherent, the left Cartan matrix ${}_CF = \mathfrak{C}_I^{\mathrm{tr}} \in \mathbb{M}_I^{\preceq}(\mathbb{Z})$ has a right and a left inverse, and the Euler form $b_C : \mathbb{Z}^{(I)} \times \mathbb{Z}^{(I)} \to \mathbb{Z}$ is defined. By (b) applied to $N = S_I(b)$, the minimal injective resolution of each $S_I(b)$ is socle-finite, left artinian, has the form (5.4) and the conditions (b1)–(b3) are satisfied. To prove that C is a left Euler coalgebra it is sufficient to show that $\mathrm{Hom}_C(E_m^{S_I(b)}, E_I(a)) = 0$ and $\mathrm{Ext}_C^m(S_I(a), S_I(b)) = 0$ for m sufficiently large. This is a consequence of (b3) if $a \preceq b$. Assume now that $a \not\preceq b$. It follows from Proposition 4.3 that $\mathbf{supp}(E_I(b)) = {}^{\trianglelefteq}b$. Hence, in view of (2.6), we get

$$\mathrm{Hom}_C(E_m^{S_I(b)}, E_I(a)) \cong \bigoplus_{q \in I_m^{S_I(b)}} \mathrm{Hom}_C(E_I(q), E_I(a))^{d_{mq}^{S_I(b)}} = 0$$

for any $m \geq 0$, because we assume that $a \not\preceq b$. Then (b1) yields

$$I_m^{S_I(b)} = \mathbf{supp}(\mathrm{soc}\, \Omega_m^{S_I(b)}) \subseteq \mathbf{supp}(E_m^{S_I(b)}) \subseteq \mathbf{supp}(E_0^{S_I(b)}) = {}^{\trianglelefteq}b,$$

and consequently $a \not\preceq q$ for all $q \in I_m^{S_I(b)}$.

Moreover, $\mathrm{Ext}_C^m(S_I(a), S_I(b)) = 0$ for $m \geq 0$, as $\mathrm{Ext}_C^m(S_I(a), S_I(b)) \neq 0$ would yield $S_I(a) \hookrightarrow \Omega_m^{S_I(b)} \subseteq E_m^{S_I(b)}$; hence $a \in \mathbf{supp}(\mathrm{soc}\, \Omega_m^{S_I(b)}) = I_m^{S_I(b)} \subseteq I_0^{S_I(b)} = {}^{\trianglelefteq}b$, a contradiction. This finishes the proof that C is a left Euler coalgebra.

Since C is a left Euler coalgebra and, by [30, Theorem 4.18],

$$b_C(\mathbf{lgth}\, M, \mathbf{lgth}\, N) = \chi_C(M,N) + \partial_C(M,N)$$

for all M and N in C-comod, it remains to show that $\partial_C(M,N) = 0$. Since $\partial_C : C\text{-comod} \times C\text{-comod} \to \mathbb{Z}$ is an additive function in each variable, it is sufficient to show that $\partial_C(S_I(a), N) = 0$ for any $a \in I$.

We recall that ${}_CF = \mathfrak{C}_I^{\mathrm{tr}} \in \mathbb{M}_I^{\preceq}(\mathbb{Z})$. Hence, in view of [30, Theorem 4.18] and its proof, if $m_0 \geq 0$ is the minimal integer such that $\mathrm{Ext}_C^j(S_I(a), N) = 0$ and $\mathrm{Hom}_C(S_I(a), E_j^N) = 0$ for all $j \geq m_0 + 1$, then, for $s = 2, 3, \ldots$,

$$\partial_C(S_I(a), N) = (-1)^{m_0+s} \cdot [\mathbf{lgth}\, S_I(a)] \cdot (\mathfrak{C}_I^{-1} \cdot [\mathbf{lgth}\, \Omega_{m_0+s}^N]^{\mathrm{tr}})$$

is a well defined integer. For this purpose, we recall from Proposition 2.5(d) that, given $b \in I$, the transpose of $\mathbf{lgth}\, E_I(b) = (c_{pb})_{p\in I} \in \mathbb{Z}^{(I)}$ is the bth column of $\mathfrak{C}_I$. Then the equality $\mathfrak{C}_I^{-1} \cdot \mathfrak{C}_I = E$ implies that $\mathfrak{C}_I^{-1} \cdot [\mathbf{lgth}\, E_I(b)]^{\mathrm{tr}}$ is defined. Hence, $\mathfrak{C}_I^{-1} \cdot [\mathbf{lgth}\, E_j^N]^{\mathrm{tr}}$ is defined for all $j \in I$, and consequently $\mathfrak{C}_I^{-1} \cdot [\mathbf{lgth}\, \Omega_{m_0+s}^N]^{\mathrm{tr}}$ is defined.

We recall from Proposition 2.12 that $\mathfrak{C}_I^{-1} = [c_{ij}^-]_{i,j\in I}$, with c_{ij}^- defined by (2.13). In particular, $c_{ij}^- = 0$ if $i \not\preceq j$. Given $m \geq 0$, we set $\mathbf{lgth}\, \Omega_m^N = [\ldots, \omega_p^N, \ldots]_{p\in I}$. Since $\omega_p^N = 0$ for $p \notin \mathbf{supp}(\Omega_m^N)$, we get

$$\begin{aligned}[]
[\mathbf{lgth}\, S_I(a)] \cdot (\mathfrak{C}_I^{-1} \cdot [\mathbf{lgth}\, \Omega_m^N]^{\mathrm{tr}}) = e_a \cdot (\mathfrak{C}_I^{-1} \cdot [\mathbf{lgth}\, \Omega_m^N]^{\mathrm{tr}}) &= \sum_{p\in I} c_{ap}^- \cdot \omega_p^N \\
&= \sum_{p\in \mathbf{supp}(\Omega_m^N)} c_{ap}^- \cdot \omega_p^N.
\end{aligned}$$

First we assume that $a \notin \mathbf{supp}(E_0^N) \supseteq \mathbf{supp}(\Omega_m^N)$. Then $a \not\preceq p$ and $c_{ap}^- = 0$ for all $p \in \mathbf{supp}(\Omega_m^N)$, and hence the last sum is zero.

Next we assume that $a \in \mathbf{supp}(E_0^N) \subseteq {}^{\trianglelefteq}b$, that is, $a \preceq b$, for some $b \in I_0^N = \mathbf{supp}(\mathrm{soc}\, E_0^N) = \mathbf{supp}(\mathrm{soc}\, N)$. By (b3), there exists $m_a \geq 0$ such that $a \not\preceq q$ for all $q \in I_m^N$ and $m \geq m_a$. It follows from (b1) and (b2) that, for any $m \geq m_a$, we have $a \notin \mathbf{supp}(E_m^N) \supseteq \mathbf{supp}(\Omega_m^N)$, that is, $a \not\preceq p$ and $c_{ap}^- = 0$ for all $p \in \mathbf{supp}(\Omega_m^N)$. Hence, the above sum is again zero. This shows that, for any $a \in I$, there exists $n_a \geq m_0 + 1$ such that

$$\begin{aligned}
\partial_C(S_I(a), N) &= (-1)^m [\mathbf{lgth}\, S_I(a)] \cdot (\mathfrak{C}_I^{-1} \cdot [\mathbf{lgth}\, \Omega_m^N]^{\mathrm{tr}}) \\
&= (-1)^m \sum_{p\in \mathbf{supp}(\Omega_m^N)} c_{ap}^- \cdot \omega_p^N = 0
\end{aligned}$$

for each $m \geq n_a$. This finishes the proof of the theorem. ∎

The following corollary is a consequence of Theorem 5.3 and its proof.

COROLLARY 5.6. *Assume that I is a left locally bounded and intervally finite poset, and $C = K^{\square}I$ with the decompositions* (2.7). *Let N be an artinian left C-comodule with minimal injective resolution* (5.4).

(a) *For any* $a \in I$, *there exists* $m_a \geq 0$ *such that* $\mathrm{Hom}_C(E_m^N, E_I(a)) = 0$ *for all* $m \geq m_a$.

(b) *For any left* C*-comodule* M *of finite* K*-dimension there exists* $m_{M,N} \geq 0$ *such that* $\mathrm{Ext}_C^m(M,N) = 0$ *and* $\mathrm{Hom}_C(M, E_m^N) = 0$ *for all* $m \geq m_{M,N}$.

(c) $\partial_C(M,N) = 0$ *and* $\widehat{b}_C(\mathbf{lgth}\, M, \mathbf{lgth}\, N) = \chi_C(M,N)$ *for any left* C*-comodule* M *of finite* K*-dimension, where* $\widehat{b}_C : K_0^+(C) \times \widehat{K}_0^+(C) \to \mathbb{Z}$ *is the* $\mathbb{Z}$*-bilinear form defined in* [30, (4.11)]. ∎

We finish this section by the structure theorem on finitely copresented left $K^\square I$-comodules and the Grothendieck group $\mathbf{K}_0(K^\square I\text{-Comod}_{\mathrm{fc}})$ of the category $K^\square I\text{-Comod}_{\mathrm{fc}}$ of finitely copresented left $K^\square I$-comodules defined in the usual way (see [1] and [27]).

THEOREM 5.7. *Let* I *be a left locally bounded and intervally finite poset.*

(a) *The category* $K^\square I\text{-Comod}_{\mathrm{fc}}$ *is abelian and coincides with the category of artinian left* $K^\square I$*-comodules. Moreover*, $K^\square I\text{-Comod}_{\mathrm{fc}}$ *is closed under taking extensions, contains* $K^\square I\text{-comod}$ *and* $K^\square I\text{-inj}$, *has enough injectives, and every comodule* N *in* $K^\square I\text{-Comod}_{\mathrm{fc}}$ *has an injective resolution in* $K^\square I\text{-Comod}_{\mathrm{fc}}$.

(b) *The Grothendieck group* $\mathbf{K}_0(K^\square I\text{-Comod}_{\mathrm{fc}})$ *contains the subgroup* $\mathbf{K}_0(K^\square I\text{-comod}) + \mathbf{K}_0(K^\square I\text{-inj})$. *The group* $\mathbf{K}_0(K^\square I\text{-inj})$ *is free abelian of rank* $|I|$, *the classes* $[E_I(a)]$, *with* $a \in I$, *form its free* $\mathbb{Z}$*-basis and the group homomorphism* $\mathbf{lgth} : \mathbf{K}_0(K^\square I\text{-Comod}_{\mathrm{fc}}) \to \mathbb{Z}^I$, $[X] \mapsto \mathbf{lgth}\, X$, *restricts to the group isomorphism*

$$\mathbf{lgth} : \mathbf{K}_0(K^\square I\text{-inj}) \xrightarrow{\simeq} \bigoplus_{a \in I} \mathbb{Z} \cdot \mathbf{e}(a) \subseteq \mathbb{Z}^I,$$

where $\mathbf{e}(a) = \mathbf{lgth}\, E_I(a) \in \mathbb{Z}^I$.

(c) *If* I *is of width at most two then*

$$\mathbf{K}_0(K^\square I\text{-Comod}_{\mathrm{fc}}) = \mathbf{K}_0(K^\square I\text{-comod}) + \mathbf{K}_0(K^\square I\text{-inj}),$$

that is, the group $\mathbf{K}_0(K^\square I\text{-Comod}_{\mathrm{fc}})$ *is generated by the classes* $[S_I(a)]$ *of the simple comodules* $S_I(a)$ *and the classes* $[E_I(a)]$ *of their injective covers* $E_I(a)$, *with* $a \in I$. *The homomorphism* $\mathbf{lgth} : \mathbf{K}_0(K^\square I\text{-Comod}_{\mathrm{fc}}) \to \mathbb{Z}^I$ *defines the epimorphism* (*cf.* [30, (4.8)])

$$\mathbf{lgth} : \mathbf{K}_0(K^\square I\text{-Comod}_{\mathrm{fc}}) \to \bigoplus_{a \in I} \mathbb{Z} \cdot e_a + \bigoplus_{a \in I} \mathbb{Z} \cdot \mathbf{e}(a) \subseteq \mathbb{Z}^I. \tag{5.8}$$

(d) *If* I *is of width at most two and every simple left* $K^\square I$*-comodule is of finite injective dimension then* $\mathbf{K}_0(K^\square I\text{-comod}) \subseteq \mathbf{K}_0(K^\square I\text{-inj})$ *and* $\mathbf{K}_0(K^\square I\text{-Comod}_{\mathrm{fc}}) = \mathbf{K}_0(K^\square I\text{-inj}) \cong \mathbb{Z}^{(I)}$.

Proof. (a) By Theorem 5.3, each $K^\square I$-comodule $E_I(a)$ is artinian. Since every finitely copresented left $K^\square I$-comodule M is socle-finite, M embeds in a direct sum $E_I(a_1) \oplus \cdots \oplus E_I(a_m)$ and hence is artinian. Conversely, every artinian left $K^\square I$-comodule M is socle-finite and therefore embeds in $E = E_I(a_1) \oplus \cdots \oplus E_I(a_m)$. Since E/M is artinian, it is socle-finite. This implies that $M \in K^\square I\text{-Comod}_{\mathrm{fc}}$. Consequently, $K^\square I\text{-Comod}_{\mathrm{fc}}$ is an abelian category, coincides with the category of artinian $K^\square I$-comodules, and contains $K^\square I$-comod and $K^\square I$-inj. Hence we also deduce the remaining statements in (a).

(b) The first statement follows from (a). For the second, we note that, according to Proposition 2.5(d), $K^\square I$ is Hom-computable and hence every artinian $K^\square I$-comodule N is computable, $\mathbf{lgth}\, N \in \mathbb{Z}^I$, and we have a group homomorphism $\mathbf{lgth} : \mathbf{K}_0(K^\square I\text{-Comod}_{\mathrm{fc}}) \to \mathbb{Z}^I$ (see [30, Section 2]) that restricts to the group isomorphisms $\mathbf{lgth} : \mathbf{K}_0(K^\square I\text{-comod}) \xrightarrow{\simeq} \mathbb{Z}^{(I)} \subseteq \mathbb{Z}^I$ and $\mathbf{lgth} : \mathbf{K}_0(K^\square I\text{-inj}) \xrightarrow{\simeq} \bigoplus_{a \in I} \mathbb{Z} \cdot \mathbf{e}(a) \subseteq \mathbb{Z}^I$. In view of Proposition 4.3(e), the subset $\{\mathbf{e}(a)\}_{a\in I}$ of $\mathbb{Z}^I$ is $\mathbb{Z}$-linearly independent.

(c) Assume that M is a finitely copresented $K^\square I$-comodule. Then M is socle-finite and there is an embedding $M \hookrightarrow E = E_I(a_1) \oplus \cdots \oplus E_I(a_m)$ for some $a_1, \dots, a_m \in I$.

We show by induction on $m \geq 1$ that

$$[M] \in \mathbf{K}_0(K^\square I\text{-comod}) + \mathbf{K}_0(K^\square I\text{-inj}).$$

Assume that $m = 1$ and let M be a non-zero subcomodule of $E_I(a)$, where $a = a_1$. Following the notation in Proposition 4.3(e) and in the proof of Proposition 5.2(c), we note that the support of $E_I(a)$ is the left cone ${}^{\trianglelefteq}a$, and $E_I(a)$ viewed as a representation of I is a constant diagram over ${}^{\trianglelefteq}a$, with K over all $j \in {}^{\trianglelefteq}a$, and zero over all $j \in I \setminus {}^{\trianglelefteq}a$. Since $M \neq 0$ we have $M_a \neq 0$, $\dim_K M_j \leq 1$ for all $j \in I \setminus {}^{\trianglelefteq}d$, and if $M_d = 0$ then $M_j = 0$ for all $j \in I \setminus {}^{\trianglelefteq}d$.

Consider the subposet $\mathcal{I}_M = \{j \in I;\, M_j = 0\}$ of ${}^{\trianglelefteq}a$. If $\mathcal{I}_M$ is empty then $M = E(a)$ and we are done, because $[M] \in \mathbf{K}_0(K^\square I\text{-inj})$. Assume that $\mathcal{I}_M$ is not empty. If there is a unique maximal element $b \in \mathcal{I}_M \subseteq {}^{\trianglelefteq}a$ then there exists an exact sequence $0 \to M \to E_I(a) \to E_I(b) \to 0$ in $K^\square I\text{-Comod}_{\mathrm{fc}}$, and we are done. If there are more than one maximal element in $\mathcal{I}_M$ then there are precisely two, say b and b_1, because we assume that I has width at most two. It is easy to check that there exists an exact sequence

$$0 \to M \to E_I(a) \to K(b, b_1) \to 0$$

in $K^\square I\text{-Comod}_{\mathrm{fc}}$, where

$$K(b, b_1) = (K(b, b_1)_j, {}_j\varphi_q)_{q \prec j}$$

viewed as a representation of I is defined as the constant diagram over ${}^{\trianglelefteq}b \cup {}^{\trianglelefteq}b_1 \subseteq \mathcal{I}_M \subseteq {}^{\trianglelefteq}a$, with $K(b,b_1)_j = K$ for $j \in {}^{\trianglelefteq}b \cup {}^{\trianglelefteq}b_1$, and $K(b,b_1)_j = 0$ for $j \in I \setminus ({}^{\trianglelefteq}b \cup {}^{\trianglelefteq}b_1)$. Since I has width at most two, only the following three cases can arise.

Case 0: ${}^{\trianglelefteq}b \cap {}^{\trianglelefteq}b_1 = \emptyset$. Then $K(b,b_1)$ is a direct sum of $E_I(b)$ and $E_I(b_1)$. Hence $[M] = [E_I(a)] - [K(b,b_1)] = [E_I(a)] - [E_I(b)] - [E_I(b_1)] \in \mathbf{K}_0(K^\square I\text{-inj})$.

Case 1: There is a unique maximal element c in ${}^{\trianglelefteq}b \cap {}^{\trianglelefteq}b_1$. It is easy to check that there exists an exact sequence

$$0 \to K(b,b_1) \to E_I(b) \oplus E_I(b_1) \to E_I(c) \to 0$$

in $K^\square I\text{-Comod}_{\text{fc}}$. Then $[K(b,b_1)] = [E_I(b)] + [E_I(b_1)] - [E_I(c)]$ belongs to $\mathbf{K}_0(K^\square I\text{-inj})$ and hence so does $[M] = [E_I(a)] - [K(b,b_1)]$.

Case 2: There are precisely two maximal elements c, c_1 in ${}^{\trianglelefteq}b \cap {}^{\trianglelefteq}b_1$. By assumption, $[c,b] \cup [c_1,b]$ is finite, and hence so is ${}^{\trianglelefteq}b \setminus {}^{\trianglelefteq}b_1 \subset [c,b] \cup [c_1,b]$. Define the $K^\square I$-comodule $N = (N_j, {}_j\kappa_q)_{q \prec j}$ as the constant diagram over ${}^{\trianglelefteq}b \setminus {}^{\trianglelefteq}b_1 \subseteq I$, with $N_j = K$ for every $j \in {}^{\trianglelefteq}b \setminus {}^{\trianglelefteq}b_1$, and $N_j = 0$ for every $j \in I \setminus ({}^{\trianglelefteq}b \setminus {}^{\trianglelefteq}b_1)$. Obviously, N is a subcomodule of $K(b,b_1)$ lying in $K^\square I$-comod and there exists an exact sequence

$$0 \to N \to K(b,b_1) \to E_I(b_1) \to 0$$

in $K^\square I\text{-Comod}_{\text{fc}}$. Then $[K(b,b_1)] = [N] - [E_I(b_1)]$ belongs to the Grothendieck group $\mathbf{K}_0(K^\square I\text{-comod}) + \mathbf{K}_0(K^\square I\text{-inj})$, and hence so does $[M] = [E_I(a)] - [K(b,b_1)]$. This completes the proof of our claim for $m = 1$.

Assume that $m \geq 2$ and the claim is proved for $m-1$. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & E' & \xrightarrow{u} & E & \xrightarrow{\pi} & E'' & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & M' & \xrightarrow{u'} & M & \xrightarrow{\pi'} & M'' & \to & 0 \end{array}$$

with exact rows, where $E' = E_I(a_1)$, $E'' = E_I(a_2) \oplus \cdots \oplus E_I(a_m)$, u is the canonical injection, π is the canonical projection, $M' = M \cap E'$, $M'' = \pi(M) \subseteq E''$, and the vertical arrows are the canonical injections. It follows that $[M] = [M'] + [M'']$ and, by the induction hypothesis, we get $[M] \in \mathbf{K}_0(K^\square I\text{-comod}) + \mathbf{K}_0(K^\square I\text{-inj})$. This completes the proof of (b).

(d) By our assumption and (a), every comodule $S_I(b)$ has a finite injective resolution

$$0 \to S_I(b) \to E_0^{(b)} \to E_1^{(b)} \to \cdots \to E_m^{(b)} \to 0$$

and the injective comodules $E_0^{(b)}, E_1^{(b)}, \ldots, E_m^{(b)}$ are socle-finite and artinian, that is, they lie in $K^\square I$-inj. Hence, $[S_I(b)] \in \mathbf{K}_0(K^\square I\text{-inj})$ and, in view of (b), the statement (c) follows. ■

COROLLARY 5.9. *Let I be a poset of width at most two. If M is a subcomodule of $E_I(a)$ for some $a \in I$, then one of the following three statements hold*:

(i) $\operatorname{inj.dim} M \le 1$, *the first syzygy comodule $\Omega_1(M) = E_I(a)/M$ is injective and* $\operatorname{soc} \Omega_1(M)$ *is a direct sum of at most two simple comodules*,

(ii) $\operatorname{inj.dim} M = 2$ *and there exists an exact sequence*

$$0 \to \Omega_1(M) \to E_I(b_1) \oplus E_I(b_2) \to E_I(c) \to 0$$

in $K^{\square}I$-$\mathrm{Comod}_{\mathrm{fc}}$ *with $b_1, b_2, c \in I$*,

(iii) *there exists a finite-dimensional subcomodule N of $\Omega_1(M)$ such that $\Omega_1(M)/N$ is an indecomposable injective comodule.*

Proof. Apply the proof of Theorem 5.7(b). ∎

6. Bass numbers for pairs of simple $K^{\square}I$-comodules. Assume that I is a left locally bounded and intervally finite poset, and $C = K^{\square}I$. Then C is basic, locally left artinian, left cocoherent, and, by Theorem 5.3, every artinian left C-comodule N admits a socle-finite artinian minimal injective resolution (5.4).

Following [29, Section 4] (see also Bass [3] and [17]), given a simple $K^{\square}I$-comodule S, an artinian left $K^{\square}I$-comodule N, and $m \ge 0$, we define the *mth Bass number* (or Betti number) $\mu_m^I(S, N)$ of the pair (S, N) to be the number of indecomposable direct summands isomorphic to $E(S)$ (the injective envelope of S) in a fixed (finite) indecomposable decomposition (5.5) of the mth term E_m^N of the injective resolution (5.4). It is clear that $\mu_m^I(S, N)$ does not depend on the resolution (5.4), nor on the decomposition (5.5) of E_m^N, by the Krull–Remak–Schmidt–Azumaya theorem. It is easy to check that

$$\mu_m^I(S, N) = \frac{\dim_K \operatorname{Ext}^m_{K^{\square}I}(S, N)}{\dim_K \operatorname{End}_{K^{\square}I} S} = \dim_K \operatorname{Ext}^m_{K^{\square}I}(S, N), \tag{6.1}$$

because $\operatorname{End}_{K^{\square}I} S \cong K$ (see [29, (4.24)]). If N is a simple $K^{\square}I$-comodule then $\mu_m^I(S, N)$ does not depend on the choice of the field K, by Proposition 2.5 and the proof of Theorem 5.3.

It follows from Corollary 5.6(a) that for each pair (S, N) there exists an integer $m_0 \ge 0$ such that $\mu_m^I(S, N) = 0$ for all $m \ge m_0$.

Now we show that, for any $a, b \in I$, the Bass number $\mu_m(S_I(a), S_I(b))$ is non-zero for at most one $m \ge 0$, and then $(-1)^m \mu_m^I(S_I(a), S_I(b))$ is the entry c_{ab}^- in the ath row of the matrix $\mathfrak{C}_I^{-1}$.

THEOREM 6.2. *Let I be a connected intervally finite poset, let $C = K^{\square}I$, and let $a, b \in I$.*

(a) *If* $c^-_{ab} = 0$ *then* $\mu^I_m(S_I(a), S_I(b)) = 0$ *for every* $m \geq 0$.

(b) *If* $c^-_{ab} \neq 0$ *then* $a \preceq b$ *and there exists a unique integer* $m_{ab} \geq 0$ *such that*

$$\mu^I_m(S_I(a), S_I(b)) = \begin{cases} 0 & \text{for } m \neq m_{ab}, \\ (-1)^m c^-_{ab} & \text{for } m = m_{ab}, \end{cases} \tag{6.3}$$

and $m_{ab} \leq \ell(a, b)$ *(see* (2.13)*).*

(c) *If* $a \not\preceq b$ *then* $\mu^I_m(S_I(a), S_I(b)) = 0$ *for every* $m \geq 0$.

Proof. Fix $b \in I$. By Theorem 5.3 and its proof, the minimal injective resolution

$$0 \to S_I(b) \xrightarrow{h_0^{(b)}} E_0^{(b)} \xrightarrow{h_1^{(b)}} E_1^{(b)} \xrightarrow{h_2^{(b)}} \cdots \xrightarrow{h_m^{(b)}} E_m^{(b)} \xrightarrow{h_{m+1}^{(b)}} E_{m+1}^{(b)} \to \cdots \tag{6.4}$$

of $S_I(b)$ is socle-finite, artinian, there exist pairwise disjoint finite subsets $I_0^{(b)} = \{b\}$, $I_1^{(b)}, \ldots, I_m^{(b)}, \ldots$ of I and integers $d_{am}^{(b)} \geq 0$ such that

$$E_0^{(b)} = E_I(b), \qquad E_m^{(b)} = \bigoplus_{a \in I_m^{(b)}} E_I(a)^{d_{ma}^{(b)}} = \bigoplus_{a \in I} E_I(a)^{d_{ma}^{(b)}} \qquad \text{for } m \geq 1,$$

$d_{b0}^{(b)} = 1$, $d_{ma}^{(b)} = 0$ for $a \in I \setminus I_m^{(b)}$, and the following four conditions are satisfied:

(i) $d_{mp}^{(b)} \geq 1$ if $p \in I_m^{(b)}$,

(ii) $\cdots \trianglelefteq I_2^{(b)} \trianglelefteq I_1^{(b)} \trianglelefteq I_0^{(b)} = \{b\}$,

(iii) $\cdots \subseteq \mathbf{supp}(E_2^{(b)}) \subseteq \mathbf{supp}(E_1^{(b)}) \subseteq {}^{\trianglelefteq}b$,

(iv) for each $a \in I$ with $a \prec b$ there exists $m_a \geq 0$ such that $a \not\preceq q$ for all $q \in I_{m_a}^{(b)}$.

Note that $\mathbf{supp}(E_0^{(b)}) = {}^{\trianglelefteq}b$, by Proposition 4.3(e).

Since the finite sets $I_0^{(b)} = \{b\}$, $I_1^{(b)}, I_2^{(b)}, \ldots$ are pairwise disjoint, it follows that if $E_I(a)$, for some $a \in I$, is a direct summand of $E_m^{(b)}$, then $E_I(a)$ is not a direct summand of $E_n^{(b)}$ for any $n \neq m$. In other words, if $d_{ma}^{(b)} \geq 1$, then $d_{na}^{(b)} = 0$ for all $n \neq m$.

We recall that $\mu^I_m(S_I(a), S_I(b)) = d_{ma}^{(b)}$. By Theorem 5.3, C is a left Euler coalgebra, $\mathfrak{C}_I$ has a unique left inverse $\mathfrak{C}_I^{-1}$, and $\mathfrak{C}_I^{-1} = ({}_CF^{-1})^{\mathrm{tr}}$. Hence $\mathfrak{C}_I^{-1}$ is the transpose of the matrix ${}_CD = [d_a^{(b)}]_{a,b \in I}$ constructed in [30, (4.22)]. It follows that

$$c^-_{ab} = \sum_{m=0}^{\infty} (-1)^m d_{ma}^{(b)} = \sum_{m=0}^{\infty} (-1)^m \mu^I_m(S_I(a), S_I(b)). \tag{6.5}$$

(a) Assume that $c^-_{ab} = 0$. Hence, $d_{ma}^{(b)} = 0$ for all $m \geq 0$, because otherwise $d_{am}^{(b)} \geq 1$ for some m_0, and by the above remarks, $d_{an}^{(b)} = 0$ for all $n \neq m_0$.

But this yields $0 = c^-_{ab} = (-1)^{m_0} d^{(b)}_{m_0 a} \neq 0$, a contradiction. This shows that $\mu^I_m(S_I(a), S_I(b)) = d^{(b)}_{ma} = 0$ for all $m \geq 0$.

(b) Assume that $c^-_{ab} \neq 0$. Then $d^{(b)}_{m_{ab}a} \neq 0$ for a unique integer $m_{ab} \geq 0$, and $d^{(b)}_{an} = 0$ for all $n \neq m_{ab}$, by the above observation. Hence, in view of (6.4), $c^-_{ab} = (-1)^{m_{ab}} d^{(b)}_{m_{ab}a} = (-1)^{m_{ab}} \mu^I_{m_{ab}}(S_I(a), S_I(b))$.

It remains to show that $m_{ab} \leq \ell(a, b)$. Since $d^{(b)}_{m_{ab}a} \neq 0$, $E(a)$ is a direct summand of $E^{(b)}_{m_{ab}}$ and therefore $a \in I^{(b)}_{m_{ab}}$. Hence, in view of (ii), we have $a \in I^{(b)}_{m_{ab}} \trianglelefteq \cdots \trianglelefteq I^{(b)}_2 \trianglelefteq I^{(b)}_1 \trianglelefteq I^{(b)}_0 = \{b\}$ and so there exists a chain $a \prec a_{m_{ab}-1} \prec \cdots \prec a_2 \prec a_1 \prec b$ in $[a, b]$ with $a_j \in I^{(b)}_j$ for $j = 1, \ldots, m_{ab}-1$. This shows that $m_{ab} \leq \ell(a, b)$.

(c) If $a \not\preceq b$ then $c^-_{ab} = 0$, by (2.11); hence (c) is a consequence of (a). ∎

The preceding theorem suggests the following definition.

Definition 6.6. Let I be a connected intervally finite poset. The *reduced length* of the pair (a, b) of elements of I is the integer $r\ell_I(a, b) \geq -1$ defined by the formula

$$(6.7)\qquad r\ell_I(a, b) = \begin{cases} -1 & \text{if } c^-_{ab} = 0, \\ m_{ab} & \text{if } a \preceq b \text{ and } c^-_{ab} \neq 0, \end{cases}$$

where $m_{ab} \geq 0$ is the unique integer such that the equalities (6.3) hold.

By applying the definition, Proposition 4.3 and the proof of Theorem 5.3, we get:

- $r\ell_I(a, b) = 0$ if and only if $a = b$,
- $r\ell_I(a, b) = 1$ if and only if there is an arrow $a \to b$ in the Hasse poset of I,
- $r\ell_I(a, b) = 2$ if $a \prec b$, $\ell(a, b) = 2$ and $[a, b] \subseteq I$ is not a chain,
- $r\ell_I(a, b) = -1$ if either $a \not\preceq b$, or $a \preceq b$ and $[a, b] \subseteq I$ is a chain of length at least two,
- if ${}^{\trianglelefteq}b = {}^{\trianglelefteq}b_1 \cup [b_1, b]$ for some $b_1 \prec b$, then $r\ell_I(a, b) = -1$ for all $a \prec b_1$.

The following corollary is an immediate consequence of Theorem 6.2.

Corollary 6.8. *Let I be a connected intervally finite poset.*

(a) *For any $m \geq 0$ and $a, b \in I$,*

$$\begin{aligned} \mu_m(S_I(a), S_I(b)) &= \dim_K \operatorname{Ext}^m_C(S_I(a), S_I(b)) \\ &= \begin{cases} 0 & \text{if } m \neq r\ell_I(a, b), \\ (-1)^m c^-_{ab} & \text{if } m = r\ell_I(a, b), \end{cases} \end{aligned}$$

(b) *For any $a \in I$ and $m \geq 0$, the C-comodule $E_I(a)$ is a direct summand (with multiplicity $\mu_m(S_I(a), S_I(b))$) of the mth term $E^{(b)}_m$ of the resolution (6.5) if and only if $c^-_{ab} \neq 0$ and $m = r\ell_I(a, b)$.*

Proof. Apply Theorem 6.2, the formula (6.4) and the definition of $r\ell_I(a,b)$. ∎

EXAMPLE 6.9. Let I be the poset of Example 3.11. Then

$$\begin{aligned}
r\ell_I(0,0) &= 0, \quad r\ell_I(0,1) = r\ell_I(0,2) = -1,\\
r\ell_I(0,3) &= r\ell_I(0,4) = r\ell_I(0,5) = 1,\\
r\ell_I(0,6) &= r\ell_I(0,7) = r\ell_I(0,8) = 2,\\
r\ell_I(0,9) &= r\ell_I(0,10) = r\ell_I(0,11) = 3,\\
\mu_1^I(S_I(0),S_I(3)) &= \mu_1^I(S_I(0),S_I(4)) = \mu_1^I(S_I(0),S_I(5))\\
&= |c_{03}^-| = |c_{04}^-| = |c_{05}^-| = |{-1}| = 1,\\
\mu_2^I(S_I(0),S_I(6)) &= \mu_2^I(S_I(0),S_I(7)) = \mu_2^I(S_I(0),S_I(8))\\
&= c_{06}^- = c_{07}^- = c_{08}^- = 2,\\
\mu_3^I(S_I(0),S_I(9)) &= \mu_3^I(S_I(0),S_I(10)) = \mu_3^I(S_I(0),S_I(11))\\
&= |c_{09}^-| = |c_{0\,10}^-| = |c_{0\,11}^-| = |{-4}| = 4.
\end{aligned}$$

More generally, $\mu_1^I(S_I(0),S_I(b)) = |c_{0b}^-|$ for $b = 3,4,\dots$.

EXAMPLE 6.10. Let I be the infinite poset with Hasse quiver

$$Q_I:\quad \begin{tikzcd}
\cdots \arrow[r] \arrow[dr] & -4 \arrow[r] \arrow[dr] & -2 \arrow[r] \arrow[dr] & 0 \arrow[dr] & & 3 \arrow[r] \arrow[drr] & 5 \arrow[r] & 7 \arrow[dr] & \\
\cdots \arrow[r] \arrow[ur] & -3 \arrow[r] \arrow[ur] & -1 \arrow[r] \arrow[ur] & 1 \arrow[r] & 2 \arrow[r] \arrow[ur] & 4 \arrow[r] \arrow[urr] & 6 \arrow[r] & 8 \arrow[r] & 9
\end{tikzcd}$$

A direct calculation shows that

$$\begin{aligned}
&r\ell_I(9,9) = 0,\\
&r\ell_I(7,9) = r\ell_I(8,9) = 1,\\
&r\ell_I(5,9) = r\ell_I(6,9) = -1,\\
&r\ell_I(3,9) = r\ell_I(4,9) = 2,\\
&r\ell_I(2,9) = 3, \quad r\ell_I(a,9) = -1 \quad \text{for all } a \le 1.
\end{aligned}$$

REMARKS 6.11. (a) By applying Theorem 6.2 and Corollary 6.8, one can describe the minimal injective resolution (6.4) of any simple left $K^\square I$-comodule $S_I(b)$, because we easily compute the matrix $\mathfrak{C}_I^{-1} = [c_{ij}^-]_{i,j\in I}$ by applying the recursive rules (2.11) and (2.13), and hence we can read off the Bass numbers $\mu_m^I(S_I(a),S_I(b))$, by applying Corollary 6.8(a). However, to perform this procedure, we need to find a simple formula for the reduced length $r\ell_I(a,b)$ of any pair $a,b\in I$ such that $a \prec b$. We formulate this below as an open problem.

(b) The computation of $r\ell_I(a,b)$ in I reduces to a finite subposet $J = J_{ab}$ of I as follows. It follows from Theorem 6.2 that $r\ell_I(a,b) \le \ell(a,b)$ for $a \prec b$.

Hence, in view of (ii), a belongs to one of the finite sets of the chain

$$I_m^{(b)} \trianglelefteq \cdots \trianglelefteq I_2^{(b)} \trianglelefteq I_1^{(b)} \trianglelefteq I_0^{(b)} = \{b\}$$

of subsets of $\mathbf{supp}(E_0^{(b)}) = {}^{\trianglelefteq}b$, where $m = \ell(a,b)$. Since I is intervally finite, one can find a finite and intervally finite subposet $J = J_{ab}$ of ${}^{\trianglelefteq}b$ containing the finite set $I_m^{(b)} \cup \cdots \cup I_2^{(b)} \cup I_1^{(b)} \cup I_0^{(b)} = \{b\}$. Then the restriction functor $\mathrm{res}_J : K^{\Box}I\text{-Comod} \to K^{\Box}J\text{-Comod}$ is exact, carries $S_I(a), S_I(b)$ to $S_J(a), S_J(b)$, and the injective resolution of $S_I(b)$ to an injective resolution of $S_J(b)$, and $\mathrm{Hom}_{K^{\Box}I}(S_I(a), E_I(j)) \cong \mathrm{Hom}_{K^{\Box}J}(S_J(a), E_J(j))$ for $j \in J$, so res_J induces the isomorphisms

$$\begin{aligned}\mathrm{Ext}^n_{K^{\Box}I}(S_I(a), S_I(b)) &\cong \mathrm{Ext}^n_{K^{\Box}J}(\mathrm{res}_J S_I(a), \mathrm{res}_J(S_I(b)) \\ &\cong \mathrm{Ext}^n_{K^{\Box}J}(S_J(a), S_J(b))\end{aligned}$$

for $n \leq m$. Here we follow the localisation technique for coalgebras studied in [11], [19], [29], [37]. It follows that $\mu_n^I(S_I(a), S_I(b)) = \mu_n^J(S_J(a), S_J(b))$ for $1 \leq n \leq m$, and the computation of $r\ell_I(a,b)$ in I reduces to the computation of $r\ell_J(a,b)$ in the finite subposet $J = J_{ab}$ of ${}^{\trianglelefteq}b \subseteq I$.

Open problems 6.12. (a) Give a combinatorial description of the reduced length $r\ell_I(a,b)$ of elements $a \prec b$ of I in terms of the finite interval $[a,b]$ viewed as a subposet of I. Does the length $r\ell_I(a,b)$ depend only on $[a,b]$?

(b) Following Theorem 5.7 and Corollary 5.9, describe the structure of the Grothendieck group $\mathbf{K}_0(K^{\Box}I\text{-Comod}_{\mathrm{fc}})$, where I is a left locally bounded and intervally finite poset of width ≥ 3. Prove that the homomorphism (5.8) is an isomorphism.

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