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A $3G$-THEOREM FOR JORDAN DOMAINS IN $\mathbb{R}^2$

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Abstract. We prove a new $3G$-Theorem for the Laplace Green function G on an arbitrary Jordan domain D in $\mathbb{R}^2$. This theorem extends the recent one proved on a Dini-smooth Jordan domain.

1. Introduction. In this paper we prove a new $3G$-Theorem for the Laplace Green function G on an arbitrary Jordan domain D in $\mathbb{R}^2$ which extends the one recently proved by Selmi in [15] on a Dini-smooth Jordan domain and improves the ones due to Chung and Zhao [4]. The $3G$-Theorem is of independent interest in potential theory. In particular, it is the key in proving the existence and comparison of the continuous perturbed Green functions associated with $\Delta - \mu$, when μ is in a general class of signed Radon measures (see [13] and [14]). We also derive other inequalities for the Laplace Green function G on the Jordan domain D. In particular, we provide new and simple proofs of the ones proved in the Dini-smooth case.

In Section 2, we give some notations and we state some known results that will be used in this work. In Section 3, we prove the inequalities for the Laplace Green function G on the Jordan domain D.

2. Notations and known results. A domain D in $\mathbb{R}^2$ is called a *Jordan domain* if D is bounded and ∂D consists of finitely many disjoint closed Jordan curves.

If D_1 and D_2 are two domains in $\mathbb{R}^2$, then we say that a function Φ from D_1 onto D_2 is an *extended conformal mapping* if Φ is a 1-1 conformal mapping from D_1 onto D_2 which can be extended to a homeomorphism from $\overline{D}_1$ onto $\overline{D}_2$.

In this paper we consider a Jordan domain D in $\mathbb{R}^2$ and we denote by G the Δ-Green function on D, where Δ is the Laplace operator. We recall

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that for all $x, y \in D$,

$$G(x,y) = \log \frac{1}{|x-y|} + w(x,y),$$

where $w(\cdot, y)$ is harmonic in D and $G(z,y) = 0$ for $z \in \partial D$ (see [17]).

From Lemma 6.17 in [4], we know the following.

LEMMA 2.1. *There exists an extended conformal mapping Φ from D onto a bounded C^∞-domain Ω.*

REMARK 2.2. In the special case when D is Dini-smooth we know by Theorem 3.5 in [12] that the derivative Φ' has a continuous extension from $\overline{D}$ onto $\overline{\Omega}$.

In the following Ω denotes a bounded C^∞-domain in $\mathbb{R}^2$ and Φ an extended conformal mapping from D onto Ω. We denote by G_Ω the Δ-Green function on Ω. For $x \in \Omega$, let $\delta(x) = d(x, \partial\Omega)$, the distance from x to the boundary of Ω. We also denote by $d(\Omega)$ the diameter of Ω. From Lemma 6.18 in [4] we have

LEMMA 2.3. *For all $x, y \in D$, we have*

$$G(x,y) = G_\Omega(\Phi(x), \Phi(y)).$$

From Theorem 6.23 in [4], we also have the following estimates.

THEOREM 2.4. *There exists a constant $C = C(\Omega) > 0$ such that for all $x, y \in \Omega$,*

$$\frac{1}{C} \log\left(1 + \frac{\delta(x)\delta(y)}{|x-y|^2}\right) \le G_\Omega(x,y) \le C \log\left(1 + \frac{\delta(x)\delta(y)}{|x-y|^2}\right).$$

In this paper we also fix a point x_0 in D and we set $\varphi(x) = 1 \wedge G(x, x_0)$ for all $x \in D$. We put $z_0 = \Phi(x_0)$. For $u \in \mathbb{R}^2$, let $g(u) = (\log \frac{1}{|u|}) \vee 1$. We will also use the inequalities

$$\begin{aligned} &\text{(i)} \quad \log(1+t) \le t \qquad \text{for all } t \ge 0,\\ &\text{(ii)} \quad \log(1+t) \ge t/2 \qquad \text{for all } t \in [0,1]. \end{aligned}$$

3. Inequalities for the Green function G. In this section we prove a new $3G$-Theorem on the Jordan domain D which extends the recent one proved in the Dini-smooth case in [15] and the ones due to Chung and Zhao in [4]. We also derive other inequalities on the Green function. We have

THEOREM 3.1 ($3G$-Theorem). *There exists a constant $C = C(D) > 0$ such that for all $x, y, z \in D$, we have*

$$\frac{G(x,z)G(z,y)}{G(x,y)} \le C\left[\frac{\varphi(z)}{\varphi(x)}\, G(x,z) + \frac{\varphi(z)}{\varphi(y)}\, G(z,y)\right].$$

To prove Theorem 3.1, we need the following lemma.

Lemma 3.2. *There exists a constant $C = C(D) > 0$ such that for all $x \in D$, we have*

$$C^{-1}\delta(\Phi(x)) \leq \varphi(x) \leq C\delta(\Phi(x)).$$

Proof. In view of Lemma 2.3, it suffices to prove

$$C^{-1}\delta(z) \leq 1 \wedge G_\Omega(z, z_0) \leq C\delta(z) \quad \text{for all } z \in \Omega.$$

From Theorem 2.4, we have

$$C^{-1}\left(1\wedge\log\left(1+\frac{\delta(z)\delta(z_0)}{|z-z_0|^2}\right)\right) \leq 1\wedge G_\Omega(z, z_0) \leq C\left(1\wedge\log\left(1+\frac{\delta(z)\delta(z_0)}{|z-z_0|^2}\right)\right).$$

If $\delta(z) \geq \delta(z_0)/2$, then

$$1 \wedge G_\Omega(z, z_0) \leq \frac{2}{\delta(z_0)}\,\delta(z). \tag{1}$$

If $\delta(z) \leq \delta(z_0)/2$, then by using (i) and the inequality $|\delta(x)-\delta(y)| \leq |x-y|$, for all $x, y \in \Omega$, we have

$$1 \wedge G_\Omega(z, z_0) \leq C\,\frac{\delta(z)\delta(z_0)}{|z-z_0|^2} \leq C\,\frac{\delta(z)\delta(z_0)}{|\delta(z)-\delta(z_0)|^2} \leq \frac{4C}{\delta(z_0)}\,\delta(z). \tag{2}$$

Combining (1) and (2), we obtain

$$1 \wedge G_\Omega(z, z_0) \leq \frac{4C}{\delta(z_0)}\,\delta(z) \quad \text{for all } z \in \Omega.$$

Now we prove the lower bound. If $\delta(z)\delta(z_0)/|z-z_0|^2 \geq 1$, then

$$1 \wedge G_\Omega(z, z_0) \geq \frac{\log 2}{C} \geq \frac{\log 2}{Cd(\Omega)}\,\delta(z). \tag{3}$$

If $\delta(z)\delta(z_0)/|z-z_0|^2 \leq 1$, then by using (ii), it follows that

$$1 \wedge G_\Omega(z, z_0) \geq \frac{1}{2C}\,\frac{\delta(z)\delta(z_0)}{|z-z_0|^2} \geq \frac{\delta(z_0)}{2Cd(\Omega)^2}\,\delta(z). \tag{4}$$

Combining (3) and (4), we obtain

$$1 \wedge G_\Omega(z, z_0) \geq C'\delta(z) \quad \text{for all } z \in \Omega,$$

with $C' = \delta(z_0)/2Cd(\Omega)^2$.

Corollary 3.3. *If D is Dini-smooth, then there exists a constant $C = C(D) > 0$ such that for all $x \in D$, we have*

$$C^{-1}d(x) \leq \varphi(x) \leq Cd(x),$$

where $d(x)$ means the distance from x to the boundary of D.

Proof. Since D is Dini-smooth, then as pointed out in Remark 2.2, Φ' has an extended continuous extension from $\overline{D}$ onto $\overline{\Omega}$. Let $x \in D$ and $z \in \partial D$ be such that $d(x) = |x-z|$. We have $z \in \overline{B}(x, d(x)) \subset \overline{D}$ and then

$$|\Phi(x) - \Phi(z)| \leq |x-z| \sup_{\overline{D}} |\nabla\Phi| \leq Cd(x).$$

Since $\Phi(z) \in \partial\Omega$ it follows that

$$\delta(\Phi(x)) \leq |\Phi(x) - \Phi(z)| \leq Cd(x).$$

In the same way by considering $\sup_{\overline{\Omega}} |\nabla\Phi^{-1}| \leq C$, we obtain

$$d(x) \leq C\delta(\Phi(x)).$$

Proof of Theorem 3.1. By Lemmas 2.3 and 3.2, the inequality in Theorem 3.1 is equivalent to

$$\frac{G_\Omega(x,z)G_\Omega(z,y)}{G_\Omega(x,y)} \leq C\left[\frac{\delta(z)}{\delta(x)}\, G_\Omega(x,z) + \frac{\delta(z)}{\delta(y)}\, G_\Omega(z,y)\right]$$

for all $x, y, z \in \Omega$. In view of Theorem 2.4, it suffices to prove

$$N(x,y) \leq C[N(x,z) + N(z,y)], \tag{5}$$

where

$$N(x,y) = \frac{\delta(x)\delta(y)}{\log\left(1 + \frac{\delta(x)\delta(y)}{|x-y|^2}\right)}.$$

By using (i), we obtain

$$N(x,z) + N(z,y) \geq |x-z|^2 + |z-y|^2. \tag{6}$$

Case 1: $\delta(x)\delta(y)/|x-y|^2 \leq 1$. From (ii), we have

$$N(x,y) \leq 2|x-y|^2 \leq 4(|x-z|^2 + |z-y|^2). \tag{7}$$

Hence the inequality (5) holds from (6) and (7) with $C = 4$.

Case 2: $\delta(x)\delta(y)/|x-y|^2 \geq 1$. By symmetry, we may assume $|x-z| \geq |z-y|$.

Subcase 1: $|x-z| \geq \frac{1}{2}[\delta(x) \wedge \delta(y)]$. We have

$$\delta(x) \vee \delta(y) \leq \delta(x) \wedge \delta(y) + |x-y| \leq \delta(x) \wedge \delta(y) + |x-z| + |z-y| \leq 4|x-z|,$$

and then

$$\delta(x)\delta(y) \leq 8|x-z|^2.$$

Hence

$$N(x,y) \leq \frac{\delta(x)\delta(y)}{\log 2} \leq \frac{8}{\log 2}\, |x-z|^2. \tag{8}$$

The inequality (5) holds from (6) and (8) with $C = 8/\log 2$.

Subcase 2: $|x-z| \leq \frac{1}{2}[\delta(x) \wedge \delta(y)]$. We have

$$2|x-z| \geq |x-z| + |z-y| \geq |x-y|, \tag{9}$$

and

$$|\delta(z) - \delta(y)| \leq |z-y| \leq \tfrac{1}{2}\delta(y),$$

which yields

$$\tfrac{1}{2}\delta(y) \leq \delta(z) \leq \tfrac{3}{2}\delta(y). \tag{10}$$

From (9) and (10), we obtain

$$\frac{\delta(x)\delta(z)}{|x-z|^2} \le 6\,\frac{\delta(x)\delta(y)}{|x-y|^2},$$

which implies

$$\begin{aligned}\log\left(1+\frac{\delta(x)\delta(z)}{|x-z|^2}\right) &\le \log 6 + \log\left(1+\frac{\delta(x)\delta(y)}{|x-y|^2}\right)\\ &\le \left(\frac{\log 6}{\log 2}+1\right)\log\left(1+\frac{\delta(x)\delta(y)}{|x-y|^2}\right).\end{aligned} \tag{11}$$

Hence from (10) and (11), we get

$$N(x,y) \le 2\left(\frac{\log 6}{\log 2}+1\right)N(x,z),$$

which implies (5) with $C = 2\left(\frac{\log 6}{\log 2}+1\right)$. ∎

Remark 3.4. In the special case when D is Dini-smooth, Theorem 3.1 and Corollary 3.3 imply Theorem 1 in [15]. The idea in [15] to prove Theorem 1 is based on the result for the unit disk and the extended conformal mapping Riemann theorem when D is simply connected, and by repeating the technique finitely many times, the result holds when D is multiply connected. Here a new and a short proof based on Theorem 2.4 is given.

The following lemma extends Proposition 7 in [15] to arbitrary Jordan domains. Here also a different and simple proof is given.

Lemma 3.5. *There exists a constant $C = C(D) > 0$ such that for all $x, y \in D$, we have*

$$\frac{\varphi(y)}{\varphi(x)}\,G(x,y) \le C(1+G(x,y)).$$

Proof. In view of Lemma 2.3, Lemma 3.2, and Theorem 2.4, it suffices to prove

$$\frac{\delta(y)}{\delta(x)}\log\left(1+\frac{\delta(x)\delta(y)}{|x-y|^2}\right) \le C\left[1+\log\left(1+\frac{\delta(x)\delta(y)}{|x-y|^2}\right)\right] \tag{12}$$

for all $x, y \in \Omega$.

If $\delta(y) \le 2\delta(x)$, then (12) holds with $C = 2$.

If $\delta(y) \ge 2\delta(x)$, then $|x-y| \ge \delta(y)-\delta(x) \ge \frac{1}{2}\delta(y)$. Hence by using (i), we obtain

$$\frac{\delta(y)}{\delta(x)}\log\left(1+\frac{\delta(x)\delta(y)}{|x-y|^2}\right) \le \frac{\delta(y)^2}{|x-y|^2} \le 4,$$

and (12) holds with $C = 4$. ∎

LEMMA 3.6. *There exists a constant $C = C(D) > 0$ such that for all $x, y \in D$, we have*

$$1 + G(x, y) \le Cg(x - y).$$

Proof. For all $x, y \in D$, we have

$$G(x, y) = \log \frac{1}{|x - y|} + w(x, y),$$

where $w(\cdot, y)$ is harmonic in D and $G(z, y) = 0$ for $z \in \partial D$. Hence, for $z \in \partial D$ we have

$$w(z, y) = \log |z - y| \le \log(d(D)),$$

and by the maximum principle, we obtain $w(x, y) \le \log(d(D))$ for all $x, y \in D$. This yields

$$G(x, y) \le \log \frac{1}{|x - y|} + \log(d(D)),$$

and the assertion follows. ■

From Theorem 3.1 and Lemmas 3.5 and 3.6, we derive Theorems 6.24 and 6.15 ($3G$-Theorems) proved by Chung and Zhao in [4].

COROLLARY 3.7. *There exists a constant $C = C(D) > 0$ such that for all $x, y, z \in D$, we have*

$$\frac{G(x, z)G(z, y)}{G(x, y)} \le C[1 + G(x, z) + G(z, y)].$$

COROLLARY 3.8. *There exists a constant $C = C(D) > 0$ such that for all $x, y, z \in D$, we have*

$$\frac{G(x, z)G(z, y)}{G(x, y)} \le C[g(x - z) + g(z - y)].$$

Consequences. As shown for domains in $\mathbb{R}^n$ with $n \ge 3$ (see [13] and [14]), the $3G$-inequality allows us to prove the existence and uniqueness of the continuous Green function G_μ for the Schrödinger operator $\Delta - \mu$ on D and its comparability to G when μ is in a class of signed Radon measures more general than the well known Kato class introduced in [2] and later used by several authors to study the potential theory of $\Delta - V(x) = 0$ (see [3] and [9]). The comparability result also covers the bounded and small perturbations studied in [11], [10], and [1], and as an important consequence it implies that $\Delta - \mu$ and Δ have the same potential theory when μ is in a large class of signed Radon measures (see also [5]–[8] and [16]).

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