COLLOQUIUM MATHEMATICUM

VOL. 101 2004 NO. 1

L^p BOUNDS FOR SPECTRAL MULTIPLIERS ON RANK ONE NA-GROUPS WITH ROOTS NOT ALL POSITIVE

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Abstract. We consider a family of non-unimodular rank one NA-groups with roots not all positive, and we show that on these groups there exists a distinguished left invariant sub-Laplacian which admits a differentiable L^p functional calculus for every $p \geq 1$.

0. Introduction. Let G be a real connected Lie group, let X_j, $j = 1, \ldots d$, be some left invariant vector fields on G which generate the Lie algebra of G, and consider the left invariant sub-Laplacian $\Delta = -\sum_{j=1}^{d} X_j^2$. On the space $L^2(G)$ relative to the right invariant Haar measure on G, the operator Δ is formally self-adjoint and non-negative. Then, from the spectral theorem, every Borel function m bounded on $\mathbb{R}^+$ determines a bounded operator on $L^2(G)$ via the formula $m(\Delta) = \int_{\mathbb{R}^+} m(\lambda)\, dE_\lambda$, where $\Delta = \int_{\mathbb{R}^+} \lambda\, dE_\lambda$ is the spectral resolution of Δ. A question which arises naturally is the following (see Hörmander [12] for $\mathbb{R}^n$): is it possible, under certain conditions regarding the function m, to extend $m(\Delta)$ to a bounded operator on $L^p(G)$ for some $p \neq 2$?

We concentrate on the case where G is a solvable Lie group with exponential volume growth (for Lie groups with polynomial volume growth, see Christ [2], and Alexopoulos [1]). Two classes of solvable Lie groups with exponential volume growth and invariant sub-Laplacians emerge in the miscellaneous works on that problem: Lie groups with sub-Laplacians which admit a differentiable L^p functional calculus (see e.g. Hebisch [8, 9, 10], Cowling, Giulini, Hulanicki and Mauceri [4], Mustapha [14], Gnewuch [7]); and Lie groups with sub-Laplacians of holomorphic L^p type (see Christ and Müller [3], Ludwig and Müller [13], and Hebisch, Ludwig and Müller [11]). In this paper, we prove a multiplier theorem for groups and sub-Laplacians belonging to the first class.

2000 *Mathematics Subject Classification*: Primary 22E30; Secondary 22E25, 43A15, 43A80, 47A60, 47D05.

Key words and phrases: sub-Laplacian, solvable Lie group, stratified group, exponential volume growth, multiplier theorem, differentiable functional calculus.

We consider a family of Lie groups G such that each G is a semidirect product of a real nilpotent Lie group N (not necessarily Euclidean) with the real line $\mathbb{R}$, and the action is semisimple and has nonzero eigenvalues all positive but one. We show that on each G there is a left invariant sub-Laplacian with differentiable functional calculus on $L^p(G)$ for all $p \geq 1$. This result is new when N is non-Euclidean (for N Euclidean, see Hebisch [10]); in the case of eigenvalues not all positive with N non-Euclidean, previous multiplier theorems concern exclusively invariant sub-Laplacians of holomorphic L^p type (see Christ and Müller [3], Ludwig and Müller [13], and Hebisch, Ludwig and Müller [11]).

1. Results. Let us begin by introducing some notations, and recalling some basic notions about stratified groups (those can be found in the book of Folland and Stein [6]).

Let H be a stratified group, that is, a connected simply connected nilpotent Lie group whose Lie algebra $\mathfrak{h}$ has a vector space decomposition

$$\mathfrak{h} = \bigoplus_{j=1}^{n} V_j,$$

where the subspaces V_j satisfy

$$[V_1, V_j] = V_{j+1}, \qquad j = 1, \dots, n-1.$$

This structure of dilations on $\mathfrak{h}$ corresponds to a structure of dilations on H, which is given by the one-parameter group of automorphisms of the group H defined by

$$\sigma_t = \exp_H \circ\, \sigma_t \circ \exp_H^{-1}, \qquad t \in \mathbb{R},$$

where $\exp_H$ denotes the exponential map from $\mathfrak{h}$ to H. Endowed with this structure of dilations, the nilpotent Lie group H is said to be a *homogeneous group* of *homogeneous dimension*

$$Q = \sum_{j=1}^{n} j \dim V_j.$$

Let α be real negative, and let $G = H \times \mathbb{R} \times \mathbb{R}$ be the Lie group with product

$$g_1 \cdot g_2 = (h_1, a_1, t_1) \cdot (h_2, a_2, t_2) = (h_1 \cdot \sigma_{t_1} h_2, a_1 + e^{\alpha t_1} a_2, t_1 + t_2),$$

where $g_i = (h_i, a_i, t_i) \in G$, $i = 1, 2$. Observe that the Lie group G is solvable with exponential volume growth, and that it is non-unimodular whenever $\alpha \neq -Q$. We endow G with the right invariant Haar measure

$$dg = dh\, da\, dt,$$

dh being a bi-invariant Haar measure on H, and da and dt being the Lebesgue measures on $\mathbb{R}$ corresponding respectively to the variables a and t.

In what follows, we identify $\mathfrak{h}$ with an ideal of the Lie algebra $\mathfrak{g}$ of G. Fix a basis $\{e_1, \ldots, e_d\}$ of the vector space V_1. To each vector e_j, $j = 1, \ldots, d$, we associate a left invariant vector field $\widetilde{X}_j$ on H by setting

$$\widetilde{X}_j\phi(h) = \partial_s\phi(h \cdot \exp_H(se_j))|_{s=0}, \qquad h \in H,\ \phi \in C^1(H),$$

and a left invariant vector field X_j on G by setting

$$X_j\phi(g) = \partial_s\phi(g \cdot \exp(se_j))|_{s=0}, \qquad g \in G,\ \phi \in C^1(G),$$

where exp denotes the exponential map from $\mathfrak{g}$ to G. It is easy to see that $X_j = e^t\widetilde{X}_j$, $j = 1, \ldots, d$. We define two more left invariant vector fields on G,

$$X_0 = \partial_t, \qquad X_{d+1} = e^{\alpha t}\partial_a.$$

Note that the system

$$\chi = \{X_0, \ldots, X_{d+1}\}$$

satisfies Hörmander's condition on G.

We now consider the operator $-\sum_{j=0}^{d+1} X_j^2$ defined on the set $C_0^\infty(G)$ of smooth functions compactly supported in G. Let Δ denote the Friedrichs extension of this operator on $L^2(G)$ (i.e. the smallest self-adjoint extension),

$$\Delta = -\sum_{j=0}^{d+1} X_j^2.$$

The operator Δ so defined is a left invariant sub-Laplacian on G.

The aim of this paper is to prove the following multiplier theorem.

THEOREM 1.1. *Let G and Δ be as above. Suppose that m is a real function compactly supported in $]0, \infty[$ which belongs to the Sobolev space $H^{Q+5+\varepsilon}(\mathbb{R}^+)$ for some $\varepsilon > 0$. Then the operator $m(\Delta)$ extends to an operator bounded on $L^p(G)$ for all $p \geq 1$.*

REMARK 1.1. The degree of regularity of our L^p multipliers is not sharp. Indeed for $H = \mathbb{R}^n$ we need $m \in H^{n+5+\varepsilon}(\mathbb{R}^+)$, whereas Hebisch proves in [10] that $m \in H^{n+9/2+\varepsilon}(\mathbb{R}^+)$ is sufficient. The interest of our result is not quantitative but qualitative: we give the first example of a sub-Laplacian which admits a differentiable L^p functional calculus, $p \neq 2$, on an NA-group with roots not all positive and for which $N \neq \mathbb{R}^n$.

To prove Theorem 1.1, we estimate the heat kernel $\{p_z\}_{\Re z>0}$ associated with the sub-Laplacian Δ,

$$e^{-z\Delta}\phi = \phi *_l p_z, \qquad \phi \in C_0^\infty(G),\ \Re z > 0,$$

where $*_l$ denotes the convolution product in the space $L^2(G, d^lg)$ relative to $d^lg = e^{-(Q+\alpha)t}dg$, the left invariant Haar measure on G. We show that p_{1+is}

is uniformly bounded in $L^1(G)$ by a polynomial in $s \in \mathbb{R}$. It was proved by Hebisch [9] that $m(\Delta)$ is then bounded on $L^p(G)$ for all $p \geq 1$. Following Hebisch, Theorem 1.1 derives from the result below.

THEOREM 1.2. *Let G and p_z be as above. There is a constant $C > 0$ such that*

$$\|p_{1+is}\|_{L^1(G)} \leq C(1+|s|)^{Q+9/2}, \quad s \in \mathbb{R}.$$

Our purpose is thus to establish Theorem 1.2. In order to do that, we estimate the norm of p_{1+is} in $L^1(G)$ by its norm in the space $L^2(G)$ with some weight ω. In Section 2, we define ω, we give an estimate of p_{1+is} in $L^2(G, \omega dg)$ (Theorem 2.1), and we show that this implies Theorem 1.2. The remainder of the paper is devoted to the proof of Theorem 2.1.

We use the variable constant convention, which means that in a sequence of equations, identical names will possibly be applied to different constants (whose dependence on the parameters of the equations is clear). The notations introduced in Sections 1 and 2 hold throughout the paper, except in Section 4.1 and the appendix, Section 7; these two sections contain general results on Lie groups and have their own local notations.

2. Estimates on the heat kernel: L^1 through L^2. Let ρ be the Carnot–Carathéodory distance on G associated with the Hörmander system χ. We denote by $|g| = \rho(e, g)$ the distance from an element g in G to the unit e of G, and by $B_R = \{g \in G : |g| < R\}$, $R > 0$, the ball centered at e of radius R.

PROPOSITION 2.1. *If there exists a non-negative function ω on G such that*

$$\|(1+\omega)^{-1/2}\|_{L^2(B_R)} \leq C(1+R)^{3/2}, \quad R > 0,$$
$$\|\omega^{1/2} p_{1+is}\|_{L^2(G)} \leq C(1+|s|)^{Q+3/2}, \quad s \in \mathbb{R},$$

then the conclusion of Theorem 1.2 *is true.*

Proof. This is a rewriting of a result proved by Hebisch [9]. ∎

To prove Theorem 1.2, we show that there exists a function ω on G which has the properties required by Proposition 2.1. Let $|\cdot|_H$ be a homogeneous norm on the homogeneous group H, that is, a function continuous and non-negative on H, smooth away from the unit e_H of H, and which satisfies for all h in H: $|h|_H = |h^{-1}|_H$; $|\sigma_t h|_H = e^t |h|_H$ for every t in $\mathbb{R}$; $|h|_H = 0$ if and only if $h = e_H$. We put

$$\omega(g) = \omega(h, a, t) = |h|_H^Q |a|, \quad g = (h, a, t) \in G.$$

LEMMA 2.1. *There is a constant $C > 0$ such that*

$$\|(1+\omega)^{-1/2}\|_{L^2(B_R)} \leq C(1+R)^{3/2}, \quad R > 0.$$

Proof. By the geometry of the Lie group G, there exists $C > 0$ such that

$$B_R \subset \{g = (h, a, t) : |h|_H \leq Ce^{CR},\ |a| \leq Ce^{CR},\ |t| \leq C(R+1)\}, \qquad R > 0.$$

Now the assertion follows from easy computations. ∎

THEOREM 2.1. *There is a constant* $C > 0$ *such that*

$$\|\omega^{1/2} p_{1+is}\|_{L^2(G)} \leq C(1+|s|)^{Q+3/2}, \qquad s \in \mathbb{R}.$$

Lemma 2.1 and Theorem 2.1 prove Theorem 1.2, by Proposition 2.1. So the point is now to demonstrate Theorem 2.1.

To do this, we show that the function $s \mapsto \|\omega^{1/2} p_{1+is}\|^2_{L^2(G)}$ satisfies a certain differential inequality which, once integrated, implies the expected estimate on $\|\omega^{1/2} p_{1+is}\|^2_{L^2(G)}$. The remainder of the paper is organized as follows. In Section 3, we consider the functions

$$f_{k,\xi}(s) = \|\, |h|_H^{k/2} |a|^{1/2} p_{\xi+is}\|^2_{L^2(G)}, \qquad s \in \mathbb{R},\ \xi \in D_{Q-k},\ k = 0, \ldots, Q,$$

where D_n, $n \in \mathbb{N}$, denotes the disc $\{z \in \mathbb{C} : |z-1| \leq 1 - 1/2^n\}$. Observe that

$$f_{Q,\xi}(s) = f_Q(s) = \|\omega^{1/2} p_{1+is}\|^2_{L^2(G)}, \qquad s \in \mathbb{R},\ \xi \in D_0.$$

We fix $k \in [0, Q]$ and $\xi \in D_{Q-k}$, and we show that $\partial_s f_{k,\xi}$ is bounded by a certain quantity. We evaluate directly one part of this quantity (Sections 4 and 5), and we estimate the other part by $f_{0,\xi_0}, \ldots, f_{k-1,\xi_{k-1}}$ with $\xi_j \in D_{Q-j}$ (Section 6). In Section 6, we insert these estimates in the estimate of $\partial_s f_{k,\xi}$ to obtain a certain differential inequality that can be integrated using an induction argument. This proves, modulo a pointwise estimate on the heat kernel, that $f_{k,\xi}$ is bounded by a polynomial in s; in the particular case where $k = Q$, it proves Theorem 2.1. In the appendix, Section 7, we establish a pointwise estimate on the heat kernel with complex time on a general non-unimodular Lie group, which completes the proof of Theorem 2.1.

REMARK. In what follows, we shall denote by $\langle \cdot\,, \cdot \rangle$ the scalar product in $L^2(G)$, and when no ambiguity is possible, by $\|\cdot\|_{L^2}$ the norm $\|\cdot\|_{L^2(G)}$.

3. First step towards a differential inequality. In this section, we prove the following proposition, which provides estimates of the derivatives $\partial_s f_{k,\xi}$ of the functions $f_{k,\xi}$ defined above.

PROPOSITION 3.1. *Let* k *be an integer in* $[0, Q]$. *There exists a constant* $C_k > 0$ *such that, for any* ξ *in* D_{Q-k} *and* s *in* $\mathbb{R}$,

$$|\partial_s \| \, |h|_H^{k/2} |a|^{1/2} p_{\xi+is} \|_{L^2}^2|$$
$$\le \begin{cases} C_k \displaystyle\sum_{\varepsilon=0}^{1} \sum_{l=1-\varepsilon}^{k} \sup_{\zeta \in D_{Q-k+1}} \|e^{(l+\alpha)t/2} |h|_H^{(k-l)/2} |a|^{(1-\varepsilon)/2} p_{\zeta+is}\|_{L^2}^2 & \\ & \textit{when } k = 1, \ldots, Q, \\ C_0 \displaystyle\sup_{\zeta \in D_{Q+1}} \|e^{\alpha t/2} p_{\zeta+is}\|_{L^2}^2 & \textit{when } k = 0. \end{cases}$$

First we show some auxiliary results (Lemmata 3.1–3.3). The proof of Proposition 3.1 is given at the end of the section.

LEMMA 3.1. *Let X be a left invariant vector field of the Hörmander system χ, let λ be real, and n_1, n_2 be integers. Let ϕ be the function defined by $\phi(g) = |h|_H^{n_1} |a|^{n_2}$, where $g = (h, a, t) \in G$. Then there exists a constant $C > 0$ such that, for any $\Re z > 1/2^{Q+1}$,*

$$\|e^{\lambda t/2} \phi^{1/2} X p_z\|_{L^2}^2 \le C \sup_{|\zeta - z| < 1/2^{Q+1}} \|e^{\lambda t/2} \phi^{1/2} p_\zeta\|_{L^2}^2 + \sum_{j=1}^{d+1} |\langle X_j p_z, e^{\lambda t} X_j(\phi) p_z \rangle|.$$

Proof. The proof is a slight modification of an argument of Hebisch [9]. We shall assume that $X \neq X_0$; the proof for $X = X_0$ is similar. By integration by parts,

$$\begin{aligned} &\langle \Delta p_z, e^{\lambda t} \phi p_z \rangle \\ &= \sum_{j=0}^{d+1} \|e^{\lambda t/2} \phi^{1/2} X_j p_z\|_{L^2}^2 + \lambda \langle X_0 p_z, e^{\lambda t} \phi p_z \rangle + \sum_{j=1}^{d+1} \langle X_j p_z, e^{\lambda t} X_j(\phi) p_z \rangle \\ &\ge \|e^{\lambda t/2} \phi^{1/2} X_0 p_z\|_{L^2}^2 + \|e^{\lambda t/2} \phi^{1/2} X p_z\|_{L^2}^2 \\ &\quad - |\lambda| \cdot \|e^{\lambda t/2} \phi^{1/2} X_0 p_z\|_{L^2} \cdot \|e^{\lambda t/2} \phi^{1/2} p_z\|_{L^2} - \sum_{j=1}^{d+1} |\langle X_j p_z, e^{\lambda t} X_j(\phi) p_z \rangle| \\ &\ge \|e^{\lambda t/2} \phi^{1/2} X p_z\|_{L^2}^2 - \frac{\lambda^2}{4} \|e^{\lambda t/2} \phi^{1/2} p_z\|_{L^2}^2 - \sum_{j=1}^{d+1} |\langle X_j p_z, e^{\lambda t} X_j(\phi) p_z \rangle|. \end{aligned}$$

Since p_z depends analytically on z, by the Cauchy formula there exists a positive constant C such that, for all $x > 1/2^{Q+1}$ and $y \in \mathbb{R}$,

$$\begin{aligned} \|e^{\lambda t/2} \phi^{1/2} \Delta p_{x+iy}\|_{L^2}^2 &= \|e^{\lambda t/2} \phi^{1/2} \partial_x p_{x+iy}\|_{L^2}^2 \\ &\le C \sup_{|\zeta - (x+iy)| < 1/2^{Q+1}} \|e^{\lambda t/2} \phi^{1/2} p_\zeta\|_{L^2}^2. \end{aligned}$$

The desired inequality follows. ∎

For integers k, l, and $X \in \chi$, we set

$$I_{k,l,X}(\xi, s) = \int_G e^{(l+\alpha)t} |h|_H^{k-l} |p_{\xi+is}(h,a,t)|\, |Xp_{\xi+is}(h,a,t)|\, dh\, da\, dt,$$

where ξ is positive and s is real.

LEMMA 3.2. *Let $k \in [0, Q]$ and $l \in [0, k]$ be integers, and let $X \in \chi$. There exists a constant $C > 0$ such that for any ξ in D_{Q-k} and any real s,*

$$I_{k,l,X}(\xi, s) \le C \sum_{m=l}^{k} \sup_{\zeta \in D_{Q-k+1}} \|e^{(m+\alpha)t/2} |h|_H^{(k-m)/2} p_{\zeta+is}\|_{L^2}^2.$$

Proof. Fix k in $[0, Q]$ and $X \in \chi$. Let us start by estimating $I_{k,l,X}$ for $l = k$. For any $\xi \in D_{Q-k}$ and $s \in \mathbb{R}$, $\Re(\xi + is) \ge 1/2^{Q-k} > 1/2^{Q+1}$. Consequently, by Lemma 3.1,

$$\begin{aligned} I_{k,l,X}(\xi, s) &= \int_G e^{(k+\alpha)t} |p_{\xi+is}(h,a,t)|\, |Xp_{\xi+is}(h,a,t)|\, dh\, da\, dt \\ &\le \|e^{(k+\alpha)t/2} p_{\xi+is}\|_{L^2} \cdot \|e^{(k+\alpha)t/2} X p_{\xi+is}\|_{L^2} \\ &\le C \sup_{|\zeta-(\xi+is)|<1/2^{Q+1}} \|e^{(k+\alpha)t/2} p_\zeta\|_{L^2}^2 \\ &= C \sup_{|\zeta-\xi|<1/2^{Q+1}} \|e^{(k+\alpha)t/2} p_{\zeta+is}\|_{L^2}^2, \qquad \xi \in D_{Q-k},\ s \in \mathbb{R}. \end{aligned}$$

For any ζ such that $|\zeta - \xi| < 1/2^{Q+1}$, one has $|\zeta - 1| \le 1 - 1/2^{Q-k+1}$; thus

$$I_{k,l,X}(\xi, s) \le C \sup_{\zeta \in D_{Q-k+1}} \|e^{(k+\alpha)t/2} p_{\zeta+is}\|_{L^2}^2, \qquad \xi \in D_{Q-k},\ s \in \mathbb{R},$$

which is the expected estimate for $l = k$.

Let us now estimate $I_{k,l,X}$ for $l \in [0, k-1]$. Assume that there is $l \in [1, k]$ such that, for some $C > 0$,

$$I_{k,l,X}(\xi, s) \le C \sum_{m=l}^{k} \sup_{\zeta \in D_{Q-k+1}} \|e^{(m+\alpha)t/2} |h|_H^{(k-m)/2} p_{\zeta+is}\|_{L^2}^2, \qquad \xi \in D_{Q-k},\ s \in \mathbb{R},$$

and let us estimate $I_{k,l-1,X}$. Again by Lemma 3.1,

$$\begin{aligned} &I_{k,l-1,X}(\xi, s) \\ &\quad= \int_G e^{(l-1+\alpha)t} |h|_H^{k-l+1} |p_{\xi+is}(h,a,t)|\, |Xp_{\xi+is}(h,a,t)|\, dh\, da\, dt \\ &\quad\le \|e^{(l-1+\alpha)t/2} |h|_H^{(k-l+1)/2} p_{\xi+is}\|_{L^2} \cdot \|e^{(l-1+\alpha)t/2} |h|_H^{(k-l+1)/2} X p_{\xi+is}\|_{L^2} \end{aligned}$$

$$\begin{aligned}
&\leq \|e^{(l-1+\alpha)t/2}|h|_H^{(k-l+1)/2} p_{\xi+is}\|_{L^2}^2 \\
&\quad + \|e^{(l-1+\alpha)t/2}|h|_H^{(k-l+1)/2} X p_{\xi+is}\|_{L^2}^2 \\
&\leq C \sup_{|\zeta-(\xi+is)|<1/2^{Q+1}} \|e^{(l-1+\alpha)t/2}|h|_H^{(k-l+1)/2} p_\zeta\|_{L^2}^2 \\
&\quad + \sum_{j=1}^{d+1} |\langle X_j p_{\xi+is}, e^{(l-1+\alpha)t} X_j(|h|_H^{k-l+1}) p_{\xi+is}\rangle|, \qquad \xi \in D_{Q-k},\ s \in \mathbb{R}.
\end{aligned}$$

We treat each term of the right-hand side separately. First we observe that

$$\begin{aligned}
\sup_{|\zeta-(\xi+is)|<1/2^{Q+1}} & \|e^{(l-1+\alpha)t/2}|h|_H^{(k-l+1)/2} p_\zeta\|_{L^2}^2 \\
&= \sup_{|\zeta-\xi|<1/2^{Q+1}} \|e^{(l-1+\alpha)t/2}|h|_H^{(k-l+1)/2} p_{\zeta+is}\|_{L^2}^2 \\
&\leq \sup_{\zeta\in D_{Q-k+1}} \|e^{(l-1+\alpha)t/2}|h|_H^{(k-l+1)/2} p_{\zeta+is}\|_{L^2}^2.
\end{aligned}$$

Next, using $X_{d+1}(|h|_H^{k-l+1}) = 0$, we find that

$$|\langle X_{d+1} p_{\xi+is}, e^{(l-1+\alpha)t} X_{d+1}(|h|_H^{k-l+1}) p_{\xi+is}\rangle| = 0.$$

Now we estimate

$$|\langle X_j p_{\xi+is}, e^{(l-1+\alpha)t} X_j(|h|_H^{k-l+1}) p_{\xi+is}\rangle|$$

for $1 \leq j \leq d$, remembering that $X_j = e^t \widetilde{X}_j$. An argument using the homogeneity of the norm $|\cdot|_H$ proves that for all $n \geq 1$ there is $C > 0$ which satisfies $\widetilde{X}_j(|h|_H^n) \leq C|h|_H^{n-1}$. Then

$$\begin{aligned}
(1)\qquad & |\langle X_j p_{\xi+is}, e^{(l-1+\alpha)t} X_j(|h|_H^{k-l+1}) p_{\xi+is}\rangle| \\
&\qquad \leq C\langle |X_j p_{\xi+is}|, e^{(l+\alpha)t}|h|_H^{k-l}|p_{\xi+is}|\rangle, \qquad \xi \in D_{Q-k},\ s \in \mathbb{R}.
\end{aligned}$$

We recognize $I_{k,l,X_j}(\xi, s)$, bounded by assumption on l. Thus

$$\begin{aligned}
&|\langle X_j p_{\xi+is}, e^{(l-1+\alpha)t} X_j(|h|_H^{k-l+1}) p_{\xi+is}\rangle| \\
&\quad \leq C \sum_{m=l}^{k} \sup_{\zeta\in D_{Q-k+1}} \|e^{(m+\alpha)t/2}|h|_H^{(k-m)/2} p_{\zeta+is}\|_{L^2}^2, \qquad \xi \in D_{Q-k},\ s \in \mathbb{R}.
\end{aligned}$$

Finally, this implies the required estimate on $I_{k,l-1,X}$:

$$I_{k,l-1,X}(\xi, s) \leq C \sum_{m=l-1}^{k} \sup_{\zeta\in D_{Q-k+1}} \|e^{(m+\alpha)t/2}|h|_H^{(k-m)/2} p_{\zeta+is}\|_{L^2}^2, \qquad \xi \in D_{Q-k},\ s \in \mathbb{R}. \ \blacksquare$$

For integers k, l, and $X \in \chi$, we set

$$J_{k,l,X}(\xi,s) = \int_G e^{lt}|h|_H^{k-l}|a|\,|p_{\xi+is}(h,a,t)|\,|Xp_{\xi+is}(h,a,t)|\,dh\,da\,dt,$$

where ξ is positive and s is real.

LEMMA 3.3. *Let $k \in [0,Q]$ and $l \in [0,k]$ be integers, and let $X \in \chi$. There exists a constant $C > 0$ such that, for any ξ in D_{Q-k} and any real s,*

$$J_{k,l,X}(\xi,s) \le C \sum_{\varepsilon=0}^{1}\sum_{m=l}^{k} \sup_{\zeta\in D_{Q-k+1}} \|e^{(m+\varepsilon\alpha)t/2}|h|_H^{(k-m)/2}|a|^{(1-\varepsilon)/2}p_{\zeta+is}\|_{L^2}^2.$$

Proof. The proof is similar to that of Lemma 3.2, modulo the fact that the integrals $J_{k,l-1,X}$ are estimated via Lemma 3.1 not only by the integrals $J_{k,l,Y}$ with $Y \in \chi$, but also by $I_{k,l-1,Y}$. Lemma 3.2 is used to conclude the proof. ■

Proof of Proposition 3.1. For all integers k in $[0,Q]$,

$$\begin{aligned}
\partial_s\|\,|h|_H^{k/2}|a|^{1/2}p_{\xi+is}\|_{L^2}^2 &= 2\Re\langle -i\Delta p_{\xi+is}, |h|_H^k|a|p_{\xi+is}\rangle \\
&= -2\Im\sum_{j=0}^{d+1}\langle X_j^2 p_{\xi+is}, |h|_H^k|a|p_{\xi+is}\rangle \\
&= 2\Im\sum_{j=0}^{d+1}(\|\,|h|_H^{k/2}|a|^{1/2}X_j p_{\xi+is}\|_{L^2}^2 + \langle X_j p_{\xi+is}, X_j(|h|_H^k|a|)p_{\xi+is}\rangle) \\
&= 2\sum_{j=1}^{d+1}\Im\langle X_j p_{\xi+is}, X_j(|h|_H^k|a|)p_{\xi+is}\rangle.
\end{aligned}$$

Then for $k=0$, $\xi \in D_Q$ and $s \in \mathbb{R}$,

$$\begin{aligned}
|\partial_s\|\,|a|^{1/2}p_{\xi+is}\|_{L^2}^2| &\le 2\sum_{j=1}^{d+1}|\langle X_j p_{\xi+is}, X_j(|a|)p_{\xi+is}\rangle| \\
&\le 2\langle |X_{d+1}p_{\xi+is}|, e^{\alpha t}|p_{\xi+is}|\rangle \\
&= 2I_{0,0,X_{d+1}}(\xi,s).
\end{aligned}$$

Hence by Lemma 3.2, there exists $C_0 > 0$ such that

$$|\partial_s\|\,|a|^{1/2}p_{\xi+is}\|_{L^2}^2| \le C_0 \sup_{\zeta\in D_{Q+1}}\|e^{\alpha t/2}p_{\zeta+is}\|_{L^2}^2, \qquad \xi \in D_Q,\ s\in\mathbb{R},$$

which proves the assertion of Proposition 3.1 for $k=0$.

Now assume that k is in $[1,Q]$. The argument used to deduce the estimate (1) in the proof of Lemma 3.2 implies that there is $C_k > 0$ such that, for every $\xi \in D_{Q-k}$ and $s \in \mathbb{R}$,

$$|\partial_s \| \, |h|_H^{k/2} |a|^{1/2} p_{\xi+is} \|_{L^2}^2 | \le 2 \sum_{j=1}^{d+1} |\langle X_j p_{\xi+is}, X_j(|h|_H^k |a|) p_{\xi+is} \rangle|$$

$$\le C_k \sum_{j=1}^{d} \langle |X_j p_{\xi+is}|, e^t |h|_H^{k-1} |a| |p_{\xi+is}| \rangle + 2 \langle |X_{d+1} p_{\xi+is}|, e^{\alpha t} |h|_H^k |p_{\xi+is}| \rangle$$

$$= C_k \sum_{j=1}^{d} J_{k,1,X_j}(\xi, s) + 2 I_{k,0,X_{d+1}}(\xi, s).$$

By Lemmata 3.2 and 3.3, for all $k \in [1, Q]$ we have

$$|\partial_s \| \, |h|_H^{k/2} |a|^{1/2} p_{\xi+is} \|_{L^2}^2 |$$
$$\le C_k \Big(\sum_{\varepsilon=0}^{1} \sum_{m=1}^{k} \sup_{\zeta \in D_{Q-k+1}} \| e^{(m+\varepsilon\alpha)t/2} |h|_H^{(k-m)/2} |a|^{(1-\varepsilon)/2} p_{\zeta+is} \|_{L^2}^2$$
$$+ \sum_{m=0}^{k} \sup_{\zeta \in D_{Q-k+1}} \| e^{(m+\alpha)t/2} |h|_H^{(k-m)/2} p_{\zeta+is} \|_{L^2}^2 \Big)$$
$$\le C_k \sum_{\varepsilon=0}^{1} \sum_{m=0}^{k} \sup_{\zeta \in D_{Q-k+1}} \| e^{(m+\varepsilon\alpha)t/2} |h|_H^{(k-m)/2} |a|^{(1-\varepsilon)/2} p_{\zeta+is} \|_{L^2}^2, \qquad \xi \in D_{Q-k},\ s \in \mathbb{R},$$

which ends the proof of Proposition 3.1. ∎

4. Estimates on the heat kernel in $L^2(G)$ weighted by exponentials in t. In this section, we establish *explicit* weighted estimates on the heat kernel (Proposition 4.1, Corollary 4.1, Proposition 4.2, Corollary 4.2). Those estimates will allow us to initialize a process of successive integration of the differential inequality (Sections 5 and 6), which will lead to Theorem 2.1 (Section 6).

PROPOSITION 4.1. *There is a positive constant C such that*

$$\| e^{\alpha t/2} p_{\xi+is} \|_{L^2}^2 \le C(1+|s|)^2, \qquad \xi \in D_{Q+1},\ s \in \mathbb{R}.$$

Proof. This follows trivially from an estimate established by Hebisch [9, p. 203]. ∎

An easy consequence of Proposition 4.1 is the following.

COROLLARY 4.1. *There is a positive constant C such that, for any integer k in $[\alpha, Q]$,*

$$\| e^{kt/2} p_{\xi+is} \|_{L^2}^2 \le C(1+|s|)^2, \qquad \xi \in D_{Q+1},\ s \in \mathbb{R}.$$

Proof. Let k be an integer in $[\alpha, Q]$. For every $\xi \in D_{Q+1}$ and $s \in \mathbb{R}$,

$$\begin{aligned}\|e^{kt/2}p_{\xi+is}\|^2_{L^2} &\le \int_{-\infty}^{0} \Big(\int_{H\times\mathbb{R}} e^{\alpha t}|p_{\xi+is}(h,a,t)|^2\,dh\,da\Big)\,dt \\ &\quad + \int_{0}^{\infty} \Big(\int_{H\times\mathbb{R}} e^{Qt}|p_{\xi+is}(h,a,t)|^2\,dh\,da\Big)\,dt \\ &\le \|e^{\alpha t/2}p_{\xi+is}\|^2 + \int_G e^{Qt}|p_{\xi+is}(g)|^2\,dg.\end{aligned}$$

Now

$$\begin{aligned}\int_G e^{Qt}|p_{\xi+is}(g)|^2\,dg &= \int_G e^{-Qt}|p_{\xi+is}(g^{-1})|^2\,d(g^{-1}) \\ &= \int_G e^{-Qt}|\delta(g)p_{\xi+is}(g)|^2\delta(g^{-1})\,dg \\ &= \int_G e^{-Qt}e^{(Q+\alpha)t}|p_{\xi+is}(g)|^2\,dg = \|e^{\alpha t/2}p_{\xi+is}\|^2_{L^2}.\end{aligned}$$

Thus by Proposition 4.1, there exists $C > 0$ such that

$$\|e^{kt/2}p_{\xi+is}\|^2_{L^2} \le 2\|e^{\alpha t/2}p_{\xi+is}\|^2 \le C(1+|s|)^2, \quad \xi \in D_{Q+1},\ s\in\mathbb{R}. \ ■$$

A refinement of the estimate of Hebisch [9], more sophisticated than Proposition 4.1, is given by the following proposition.

PROPOSITION 4.2. *There is a positive constant C such that*

$$\|e^{\alpha t/2}p_{x/2+\eta s+i(y+s)}\|^2_{L^2} \le C\,\frac{s}{\eta^2}, \quad x+iy\in D_Q,\ \eta\in\,]0,1[,\ s\ge 1.$$

The proof of Proposition 4.2 uses tools related to Schrödinger operators. In Section 4.1 below, we introduce the results on Schrödinger operators we need; then we prove Proposition 4.2 in Section 4.2. But before that, we complete the list of our explicit weighted estimates on the heat kernel by the following corollary of Proposition 4.2.

COROLLARY 4.2. *There is a positive constant C such that, for any integer k in* $[1, Q]$,

$$\|e^{kt/2}p_{x/2+\eta s+i(y+s)}\|^2_{L^2} \le C\,\frac{s}{\eta^2}, \quad x+iy\in D_Q,\ \eta\in\,]0,1[,\ s\ge 1.$$

Proof. The proof is analogous to that of Corollary 4.1. ■

4.1. *On Schrödinger operators.* Note that the results presented here are general, and are not specific to the groups introduced in Section 1.

Let G_0 be a real connected simply connected Lie group, $\mathfrak{g}_0$ be its Lie algebra, and dg_0 be a right invariant Haar measure on G_0. We consider a family $\chi_0 = \{Y_1, \dots, Y_n\}$ of left invariant vector fields on G_0 satisfying

Hörmander's condition. Let ρ_0 denote the Carnot–Carathéodory distance relative to χ_0, and τ the corresponding distance to the unit 0 of G_0,

$$\tau(g_0) = \rho_0(0, g_0), \qquad g_0 \in G_0.$$

Let $\{f_1, \dots, f_n\}$ be a family of real functions in $C^1(G_0)$. We define the operators

$$U_j\phi = (Y_j + if_j)\phi, \qquad \phi \in C_0^\infty(G_0),\ j = 1, \dots, n,$$

and their adjoint operators U_j^*, $j = 1, \dots, n$, in $L^2(G_0)$. We consider the operator $\sum_{j=1}^n U_j^* U_j$ defined on $C_0^\infty(G_0)$. That operator is symmetric and non-negative on $L^2(G_0)$, and thus it admits a Friedrichs extension. Let H denote this extension,

$$H = \sum_{j=1}^n U_j^* U_j.$$

The operator H is a Schrödinger operator on $L^2(G_0)$. The semigroup e^{-zH} is well defined for $\Re z > 0$, and so is the kernel of the semigroup that we shall denote by $e^{-zH}\delta_0$.

LEMMA 4.1. *Let K be a compact set in the half-plane $\{z \in \mathbb{C} : \Re z > 0\}$. There is a positive constant C independent of the family of functions $\{f_1, \dots, f_n\}$ such that for every real c, $x + iy$ in K, η in $]0,1[$ and $s \geq 1$,*

$$\|e^{-(x/2+\eta s+i(y+s))H}\delta_0\|_{L^2(G_0, e^{2c\tau(g_0)}\,dg_0)} \leq C \exp\Big(C\,\frac{s}{\eta^2}\,c^2\Big).$$

Proof. It is analogous to the proof of Lemma 1.4 in Hebisch [9]. ∎

4.2. *Proof of Proposition 4.2.* Let the notations be those of Sections 1 and 2 again. Let G_0 denote the Lie group $G_0 = H \times \mathbb{R}$, with H as in Section 1 and with product

$$g_0 \cdot \widetilde{g}_0 = (h, t) \cdot (\widetilde{h}, \widetilde{t}\,) = (h \cdot \sigma_t \widetilde{h}, t + \widetilde{t}\,), \qquad g_0 = (h, t),\ \widetilde{g}_0 = (\widetilde{h}, \widetilde{t}\,) \in G_0.$$

We equip G_0 with the right invariant Haar measure $dg_0 = dh\,dt$, and we identify the left invariant vector fields $\{X_0, \dots, X_d\}$ on G with left invariant vector fields on G_0. It is then easy to show that $\chi_0 = \{X_0, \dots, X_d\}$ is a Hörmander system on G_0. Let ρ_0 denote the Carnot–Carathéodory distance on G_0 related to χ_0, and τ the corresponding distance from an element in G_0 to the unit 0 of G_0.

Now consider the operator $-\sum_{j=0}^d X_j^2 + e^{2\alpha t}a^2$, where a is a real parameter, defined on $C_0^\infty(G_0)$. That operator has the form $\sum_{j=1}^n U_j^* U_j$ described above in Section 4.1. Thus it admits a Friedrichs extension on $L^2(G_0)$, denoted by H_a,

$$H_a = -\sum_{j=0}^d X_j^2 + e^{2\alpha t}a^2.$$

Fix $x+iy \in D_Q$, $\eta \in]0,1[$ and $s \geq 1$. From the Plancherel formula,

$$\begin{aligned}\|e^{\alpha t/2} p_{x/2+\eta s+i(y+s)}\|^2_{L^2(G)} &= \int_{\mathbb{R}} \|e^{\alpha t/2} p_{x/2+\eta s+i(y+s)}\|^2_{L^2(G_0)}\, da \\ &= \int_{\mathbb{R}} \|e^{\alpha t/2} e^{-(x/2+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)}\, da \\ &= \int_{|a|<e^{-cs/\eta^2}} + \int_{e^{-cs/\eta^2}\leq |a|<1} + \int_{1\leq |a|},\end{aligned}$$

for any positive constant c. Let us estimate each integral separately.

Easy geometrical considerations on the Lie group G_0 show that there is a constant $C>0$ for which

$$|t| \leq C(\tau(g_0)+1), \qquad g_0=(h,t)\in G_0.$$

Hence there is $C>0$ such that, for every $a\in\mathbb{R}$,

$$\begin{aligned}&\|e^{\alpha t/2} e^{-(x/2+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)} \\ &\qquad\qquad \leq C\|e^{C\tau(g_0)} e^{-(x/2+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)}.\end{aligned}$$

From this we deduce, by Lemma 4.1, that there is $C_1>0$ such that, for every $a\in\mathbb{R}$,

$$\begin{aligned}&\|e^{\alpha t/2} e^{-(x/2+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)} \\ &\qquad \leq C_1 \exp\biggl(C_1\,\frac{s}{\eta^2}\biggr), \qquad x+iy\in D_Q,\ \eta\in]0,1[,\ s\geq 1.\end{aligned}$$

Let us now choose the constant c equal to C_1 in the integrals to estimate. One then has for any $x+iy\in D_Q$, $\eta\in]0,1[$ and $s\geq 1$,

$$\int_{|a|<e^{-C_1 s/\eta^2}} \|e^{\alpha t/2} e^{-(x/2+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)}\, da \leq 2C_1. \tag{2}$$

For every $a\neq 0$ and every smooth function ϕ,

$$\begin{aligned}\|e^{\alpha t/2}\phi\|^2_{L^2(G_0)} &\leq \frac{1}{|a|}\,\|e^{\alpha t/2}|a|^{1/2}\phi\|^2_{L^2(G_0)} \\ &\leq \frac{1}{|a|}\,(\|e^{\alpha t} a\phi\|^2_{L^2(G_0)} + \|\phi\|^2_{L^2(G_0)}).\end{aligned}$$

The operator $-\sum_{j=0}^d X_j^2$ is non-negative on $L^2(G_0)$, thus for every $a\neq 0$ and for every function ϕ in the domain of H_a,

$$\begin{aligned}\|e^{\alpha t} a\phi\|^2_{L^2(G_0)} &= \langle e^{2\alpha t}a^2\phi,\phi\rangle_{L^2(G_0)} \leq \langle H_a\phi,\phi\rangle_{L^2(G_0)} \\ &\leq \|H_a\phi\|^2_{L^2(G_0)} + \|\phi\|^2_{L^2(G_0)},\end{aligned}$$

which implies

$$\|e^{\alpha t/2}\phi\|^2_{L^2(G_0)} \leq \frac{2}{|a|}\,(\|H_a\phi\|^2_{L^2(G_0)} + \|\phi\|^2_{L^2(G_0)}).$$

Taking $\phi = e^{-(x/2+\eta s+i(y+s))H_a}\delta_0$ in the above inequality, we obtain

$$\begin{aligned}
&\|e^{\alpha t/2}e^{-(x/2+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)} \\
&\quad\leq \frac{2}{|a|}\,(\|H_a e^{-(x/2+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)} + \|e^{-(x/2+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)}).
\end{aligned}$$

It is easy to check that the functions $s \mapsto \|H_a e^{-(x/2+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)}$ and $s \mapsto \|e^{-(x/2+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)}$ have non-positive derivatives on $\mathbb{R}^+$. Therefore they decrease on $\mathbb{R}^+$, and we have

$$\begin{aligned}
&\|e^{\alpha t/2}e^{-(x/2+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)} \\
&\quad\leq \frac{2}{|a|}\,(\|H_a e^{-(x/2+iy)H_a}\delta_0\|^2_{L^2(G_0)} + \|e^{-(x/2+iy)H_a}\delta_0\|^2_{L^2(G_0)}) \\
&\quad\leq \frac{2}{|a|}\,(\|H_a e^{-(x/4+iy)H_a}\|^2_{\mathcal{L}(L^2(G_0),L^2(G_0))} + \|e^{-(x/4+iy)H_a}\|^2_{\mathcal{L}(L^2(G_0),L^2(G_0))}) \\
&\qquad\times \|e^{-(x/4)H_a}\delta_0\|^2_{L^2(G_0)}.
\end{aligned}$$

We know from spectral theory that the operators $H_a e^{-(x/4+iy)H_a}$ and $e^{-(x/4+iy)H_a}$ are bounded on $L^2(G_0)$ uniformly in x, y for $x+iy \in D_Q$, and that the bound is independent of the parameter $a \in \mathbb{R}$. We deduce easily from the boundedness of $e^{-(x/4+iy)H_a}$ that $\|e^{-(x/4)H_a}\delta_0\|^2_{L^2(G_0)}$ is bounded uniformly in x for $x+iy \in D_Q$, and independently of a. As a consequence, the second integral is such that, for any $x+iy \in D_Q$, $\eta \in]0,1[$, and $s \geq 1$,

$$\begin{aligned}
(3)\qquad &\int_{e^{-C_1 s/\eta^2}\leq |a|<1} \|e^{\alpha t/2}e^{-(x/2+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)}\,da \\
&\qquad\qquad \leq C \int_{e^{-C_1 s/\eta^2}\leq |a|<1} \frac{1}{|a|}\,da \leq C\frac{s}{\eta^2}.
\end{aligned}$$

And we also have

$$\begin{aligned}
&\int_{1\leq |a|} \|e^{\alpha t/2}e^{-(x/2+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)}\,da \\
&\qquad\leq C \int_{1\leq |a|} \frac{1}{|a|}\,\|e^{-(x/4)H_a}\delta_0\|^2_{L^2(G_0)}\,da \\
&\qquad\leq C \int_{\mathbb{R}} \frac{1}{|a|}\,\|e^{-(x/4)H_a}\delta_0\|^2_{L^2(G_0)}\,da = C\|p_{x/4}\|^2_{L^2(G)}.
\end{aligned}$$

It is a straightforward application of Theorem 7.1 (see Appendix, Section 7) that $\|p_{x/4}\|_{L^2(G)}$ is bounded uniformly in x for $x+iy \in D_Q$. Then for any

$x+iy \in D_Q$, $\eta \in]0,1[$, and $s \geq 1$,

$$\int\limits_{1\leq|a|} \|e^{\alpha t/2} e^{-(x/2+\eta s+i(y+s))H_a} \delta_0\|^2_{L^2(G_0)}\, da \leq C. \tag{4}$$

Using the estimates (2)–(4), we find that there is $C > 0$ such that

$$\|e^{\alpha t/2} p_{x/2+\eta s+i(y+s)}\|^2_{L^2(G)} \leq C\biggl(1+\frac{s}{\eta^2}\biggr) \leq C\,\frac{s}{\eta^2}, \quad x+iy \in D_Q,\ \eta \in]0,1[,\ s \geq 1,$$

which proves Proposition 4.2. ∎

5. Estimates on the heat kernel in $L^2(G)$ weighted by polynomials in $|h|_H$. In this section, we derive new estimates on the heat kernel in $L^2(G)$ from those of Section 4. The main result is given by the following proposition; we shall use it to estimate, by polynomials in s, some of the terms in the inequality of Proposition 3.1.

PROPOSITION 5.1. *Let k be an integer in $[1,Q]$. There exists a positive constant C_k such that*

$$\|\,|h|_H^{k/2} p_{\xi+is}\|^2_{L^2} \leq C_k(1+|s|)^{3+k}, \quad \xi \in D_Q,\ s \in \mathbb{R}.$$

We start by showing two technical lemmata before proving Proposition 5.1 at the end of the section.

LEMMA 5.1. *Let k be an integer in $[2,Q]$. There exists a positive constant C_k such that, for any $x+iy$ in D_Q, η in $]0,1[$, and $s \geq 1$,*

$$\partial_s \|\,|h|_H^{k/2} p_{x/2+iy+(i+\eta)s}\|^2_{L^2} \leq \begin{cases} C_k\,\dfrac{s^{2/k}}{\eta^{1+4/k}}\,\|\,|h|_H^{k/2} p_{x/2+iy+(i+\eta)s}\|_{L^2}^{(2k-4)/k} & \textit{when } k \in [3,Q], \\[2ex] C_2\,\dfrac{s}{\eta^3} & \textit{when } k=2. \end{cases}$$

Proof. For every $x+iy \in D_Q$ and $\eta \in]0,1[$,

$$\partial_s \|\,|h|_H^{k/2} p_{x/2+iy+(i+\eta)s}\|^2_{L^2} = 2\Re \sum_{j=0}^{d+1} (i+\eta)\langle X_j^2 p_{x/2+iy+(i+\eta)s}, |h|_H^k p_{x/2+iy+(i+\eta)s}\rangle$$

$$
\begin{aligned}
&= -2\eta\sum_{j=0}^{d+1}\| \,|h|_H^{k/2}X_j p_{x/2+iy+(i+\eta)s}\|_{L^2}^2 \\
&\quad -2\Re\sum_{j=0}^{d+1}(i+\eta)\langle X_j p_{x/2+iy+(i+\eta)s}, X_j(|h|_H^k)p_{x/2+iy+(i+\eta)s}\rangle \\
&\leq -2\eta\sum_{j=0}^{d+1}\| \,|h|_H^{k/2}X_j p_{x/2+iy+(i+\eta)s}\|_{L^2}^2 \\
&\quad +2|i+\eta|\sum_{j=1}^{d}|\langle X_j p_{x/2+iy+(i+\eta)s}, X_j(|h|_H^k)p_{x/2+iy+(i+\eta)s}\rangle| \\
&\leq -2\eta\sum_{j=0}^{d+1}\| \,|h|_H^{k/2}X_j p_{x/2+iy+(i+\eta)s}\|_{L^2}^2 \\
&\quad +2\sqrt{2}\sum_{j=1}^{d}\langle |X_j p_{x/2+iy+(i+\eta)s}|, |X_j(|h|_H^k)|\,|p_{x/2+iy+(i+\eta)s}|\rangle, \qquad s\geq 1.
\end{aligned}
$$

Then by the argument used to prove estimate (1) in Lemma 3.2, there is $C_k > 0$ such that, for every $x+iy \in D_Q$ and $\eta \in]0,1[$,

$$
\begin{aligned}
&\partial_s \| \,|h|_H^{k/2} p_{x/2+iy+(i+\eta)s}\|_{L^2}^2 \\
&\qquad \leq -2\eta\sum_{j=0}^{d+1}\| \,|h|_H^{k/2}X_j p_{x/2+iy+(i+\eta)s}\|_{L^2}^2 \\
&\qquad\quad + C_k\sum_{j=1}^{d}\langle |X_j p_{x/2+iy+(i+\eta)s}|, e^t|h|_H^{k-1}|p_{x/2+iy+(i+\eta)s}|\rangle \\
&\qquad \leq -2\eta\sum_{j=1}^{d}\| \,|h|_H^{k/2}X_j p_{x/2+iy+(i+\eta)s}\|_{L^2}^2 \\
&\qquad\quad + C_k\sum_{j=1}^{d}\| \,|h|_H^{k/2}X_j p_{x/2+iy+(i+\eta)s}\|_{L^2}\cdot\|e^t|h|_H^{k/2-1}p_{x/2+iy+(i+\eta)s}\|_{L^2} \\
&\qquad \leq \frac{C_k}{\eta}\,\|e^t|h|_H^{k/2-1}p_{x/2+iy+(i+\eta)s}\|_{L^2}^2, \qquad s\geq 1.
\end{aligned}
$$

For $k=2$, this implies by Corollary 4.2 that

$$\begin{aligned}\partial_s \| \, |h|_H p_{x/2+iy+(i+\eta)s} \|_{L^2}^2 &\leq \frac{C_2}{\eta} \, \|e^t p_{x/2+iy+(i+\eta)s}\|_{L^2}^2 \\ &\leq C_2 \, \frac{s}{\eta^3}, \qquad x+iy \in D_Q, \ \eta \in \,]0,1[, \ s \geq 1,\end{aligned}$$

which proves the assertion for $k = 2$.

For $k \in [3, Q]$,

$$\begin{aligned}\partial_s \| |h|_H^{k/2} p_{x/2+iy+(i+\eta)s} \|_{L^2}^2 &\leq \frac{C_k}{\eta} \, \|e^t |h|_H^{k/2-1} p_{x/2+iy+(i+\eta)s}\|_{L^2}^2 \\ &\leq \frac{C_k}{\eta} \, \|e^{2t} |p_{x/2+iy+(i+\eta)s}|^{4/k}\|_{L^{k/2}} \cdot \| \, |h|_H^{k-2} |p_{x/2+iy+(i+\eta)s}|^{(2k-4)/k} \|_{L^{k/(k-2)}} \\ &= \frac{C_k}{\eta} \, \|e^{(k/2)t} p_{x/2+iy+(i+\eta)s}\|_{L^2}^{4/k} \cdot \| \, |h|_H^{k/2} p_{x/2+iy+(i+\eta)s}\|_{L^2}^{(2k-4)/k}.\end{aligned}$$

Thus by Corollary 4.2, for every $x + iy \in D_Q$, $\eta \in \,]0, 1[$, $s \geq 1$,

$$\begin{aligned}\partial_s \| \, |h|_H^{k/2} p_{x/2+iy+(i+\eta)s} \|_{L^2}^2 &\leq \frac{C_k}{\eta} \left(\frac{s}{\eta^2} \right)^{2/k} \| \, |h|_H^{k/2} p_{x/2+iy+(i+\eta)s}\|_{L^2}^{(2k-4)/k} \\ &\leq C_k \, \frac{s^{2/k}}{\eta^{1+4/k}} \, \| \, |h|_H^{k/2} p_{x/2+iy+(i+\eta)s}\|_{L^2}^{(2k-4)/k},\end{aligned}$$

which proves the assertion for $k \in [3, Q]$. ■

LEMMA 5.2. *Let k be an integer in $[2, Q]$. There exists a positive constant C_k such that, for any $x + iy$ in D_Q, η in $]0, 1[$, and $s \geq 1$,*

$$\| \, |h|_H^{k/2} p_{x/2+iy+(i+\eta)s} \|_{L^2}^2 \leq C_k \, \frac{s^{1+k/2}}{\eta^{2+k/2}}.$$

Proof. We estimate $\| \, |h|_H^{k/2} p_{x/2+iy+(i+\eta)s} \|_{L^2}^2$ first for $k = 2$, then for k in $[3, Q]$.

By Lemma 5.1,

$$\partial_s \| \, |h|_H p_{x/2+iy+(i+\eta)s} \|_{L^2}^2 \leq C_2 \, \frac{s}{\eta^3}, \qquad s \geq 1.$$

Hence

$$\| \, |h|_H p_{x/2+iy+(i+\eta)s} \|_{L^2}^2 \leq \| \, |h|_H p_{x/2+\eta+i(y+1)} \|_{L^2}^2 + C_2 \, \frac{s^2}{\eta^3}, \qquad s \geq 1.$$

Now Theorem 7.1 implies that

$$\| \, |h|_H p_{x/2+\eta+i(y+1)} \|_{L^2}^2 \leq c, \qquad x + iy \in D_Q, \ \eta \in \,]0, 1[.$$

Then

$$\| \, |h|_H p_{x/2+iy+(i+\eta)s} \|_{L^2}^2 \leq c + C_2 \, \frac{s^2}{\eta^3} \leq C_2 \, \frac{s^2}{\eta^3}, \quad x+iy \in D_Q, \ \eta \in \,]0,1[, \ s \geq 1,$$

proving the assertion for $k = 2$.

Now fix $k \in [3, Q]$. Set $\psi(s) = \| \, |h|_H^{k/2} p_{x/2+iy+(i+\eta)s} \|_{L^2}^2$. Lemma 5.1 ensures that

$$\psi'(s) \leq C_k \, \frac{s^{2/k}}{\eta^{1+4/k}} \, \psi^{(k-2)/k}(s), \qquad s \geq 1.$$

Hence

$$\psi(s) \leq \left(\psi^{2/k}(1) + C_k \, \frac{s^{1+2/k}}{\eta^{1+4/k}} \right)^{k/2}, \qquad s \geq 1.$$

By Theorem 7.1,

$$\psi(1) = \| \, |h|_H^{k/2} p_{x/2+\eta+i(y+1)} \|_{L^2}^2 \leq c_k, \qquad x + iy \in D_Q, \ \eta \in \,]0, 1[.$$

In terms of the heat kernel, we have

$$\begin{aligned} \| \, |h|_H^{k/2} p_{x/2+iy+(i+\eta)s} \|_{L^2}^2 &\leq \left(c_k + C_k \, \frac{s^{1+2/k}}{\eta^{1+4/k}} \right)^{k/2} \\ &\leq C_k \, \frac{s^{1+k/2}}{\eta^{2+k/2}}, \qquad x + iy \in D_Q, \ \eta \in \,]0, 1[, \ s \geq 1, \end{aligned}$$

which proves the statement for $k \in [3, Q]$. ■

5.1. *Proof of Proposition 5.1.* Let us estimate $\| \, |h|_H^{k/2} p_{\xi+is} \|_{L^2}^2$ first for $k = 1$, and then for $k \in [2, Q]$.

Arguments similar to those used to prove Proposition 3.1 when $k = 0$ show that there is a constant $C_1 > 0$ for which

$$|\partial_s \| \, |h|_H^{1/2} p_{\xi+is} \|_{L^2}^2| \leq C_1 \sup_{\zeta \in D_{Q+1}} \| e^{t/2} p_{\zeta+is} \|_{L^2}^2, \qquad \xi \in D_Q, \ s \in \mathbb{R}.$$

Hence by Corollary 4.1,

$$|\partial_s \| \, |h|_H^{1/2} p_{\xi+is} \|_{L^2}^2| \leq C_1 (1 + |s|)^2, \qquad s \in \mathbb{R},$$

where C_1 is a positive constant independent of $\xi \in D_Q$. This implies that

$$\| \, |h|_H^{1/2} p_{\xi+is} \|_{L^2}^2 \leq \| \, |h|_H^{1/2} p_\xi \|_{L^2}^2 + C_1 (1 + |s|)^3, \qquad \xi \in D_Q, \ s \in \mathbb{R}.$$

By Theorem 7.1,

$$\| \, |h|_H^{1/2} p_{\xi+is} \|_{L^2}^2 \leq C_1 (1 + |s|)^3, \qquad \xi \in D_Q, \ s \in \mathbb{R},$$

which gives the assertion for $k = 1$.

Now we assume that k is in $[2, Q]$. On the one hand, Theorem 7.1 implies that there exists $c_k > 0$ such that

$$\| \, |h|_H^{k/2} p_{\xi+is} \|_{L^2}^2 \leq c_k, \qquad \xi \in D_Q, \ |s| \leq 1.$$

On the other hand, we know from Lemma 5.2 that there is $C_k > 0$ such that

$$\| \, |h|_H^{k/2} p_{x/2+iy+(i+\eta)s} \|_{L^2}^2 \leq C_k \, \frac{s^{1+k/2}}{\eta^{2+k/2}}, \qquad x + iy \in D_Q, \ \eta \in \,]0, 1[, \ s \geq 1.$$

For $\eta = x/2s$, this implies that

$$\| \, |h|_H^{k/2} p_{x+iy+is} \|_{L^2}^2 \le C_k s^{1+k/2} \left(\frac{2s}{x} \right)^{2+k/2} \le C_k s^{3+k}, \quad x+iy \in D_Q, \ s \ge 1.$$

Now by a process analogous to the one used to establish the above estimate, one proves

$$\| \, |h|_H^{k/2} p_{x+iy+is} \|_{L^2}^2 \le C_k |s|^{3+k}, \quad x+iy \in D_Q, \ s \le -1.$$

Combining the estimates for $|s| \le 1$, $s \ge 1$ and $s \le -1$, we obtain

$$\| \, |h|_H^{k/2} p_{\xi+is} \|_{L^2}^2 \le C_k (1+|s|)^{3+k}, \quad \xi \in D_Q, \ s \in \mathbb{R}. \ \blacksquare$$

6. Proof of Theorem 2.1. In this section, we establish the following proposition which proves Theorem 2.1.

PROPOSITION 6.1. *Let k be an integer in $[0, Q]$. There exists a positive constant C such that*

$$\| \, |h|_H^{k/2} |a|^{1/2} p_{\xi+is} \|_{L^2}^2 \le C (1+|s|)^{3+Q+k}, \quad \xi \in D_{Q-k}, \ s \in \mathbb{R}.$$

Denouement. Theorem 2.1 follows from Proposition 6.1 with $k = Q$.

Proof of Proposition 6.1. The proof is by induction on k. By arguments similar to those used to prove Proposition 5.1 when $k = 1$,

$$\| \, |a|^{1/2} p_{\xi+is} \|_{L^2}^2 \le C (1+|s|)^3, \quad \xi \in D_Q, \ s \in \mathbb{R}.$$

This yields the assertion for $k = 0$.

Let us now fix an integer k_0 in $[1, Q]$, and assume that the assertion holds for every integer k in $[0, k_0 - 1]$. Proposition 3.1 ensures that there is $C > 0$ such that for any $\xi \in D_{Q-k_0}$,

$$\begin{aligned} |\partial_s \| \, |h|_H^{k_0/2} |a|^{1/2} p_{\xi+is} \|_{L^2}^2 | \\ \le C \Big(\sum_{l=0}^{k_0-1} \sup_{\zeta \in D_{Q-k_0+1}} \| e^{(l+\alpha)t/2} |h|_H^{(k_0-l)/2} p_{\zeta+is} \|_{L^2}^2 \\ + \sup_{\zeta \in D_{Q-k_0+1}} \| e^{(k_0+\alpha)t/2} p_{\zeta+is} \|_{L^2}^2 \\ + \sum_{l=1}^{k_0} \sup_{\zeta \in D_{Q-k_0+1}} \| e^{lt/2} |h|_H^{(k_0-l)/2} |a|^{1/2} p_{\zeta+is} \|_{L^2}^2 \Big), \quad s \in \mathbb{R}. \end{aligned}$$

This gives an estimate of $|\partial_s \| \, |h|_H^{k_0/2} |a|^{1/2} p_{\xi+is} \|_{L^2}^2|$ in terms of norms of the heat kernel in weighted L^2. We shall evaluate the norms separately, depending on whether the weight is purely exponential in t, polynomial in $|h|_H$ and $|a|$, or polynomial in $|h|_H$ only.

There is only one term with weight purely exponential in t: it is the supremum of $\|e^{(k_0+\alpha)t/2}p_{\zeta+is}\|_{L^2}^2$ for $\zeta \in D_{Q-k_0+1}$. By Corollary 4.1, we have

$$\sup_{\zeta\in D_{Q-k_0+1}} \|e^{(k_0+\alpha)t/2}p_{\zeta+is}\|_{L^2}^2 \le C(1+|s|)^2, \qquad s\in\mathbb{R}.$$

Concerning terms with weights polynomial in $|h|_H$ and $|a|$, they are the suprema of

$$\|e^{lt/2}|h|_H^{(k_0-l)/2}|a|^{1/2}p_{\zeta+is}\|_{L^2}^2 \qquad \text{for } \zeta\in D_{Q-k_0+1}$$

with $l=1,\dots,k_0$. Fix l in $[1,k_0]$. We have

$$\begin{aligned}
&\|e^{lt/2}|h|_H^{(k_0-l)/2}|a|^{1/2}p_{\zeta+is}\|_{L^2}^2\\
&\quad\le \int_{-\infty}^{0}\Big(\int_{H\times\mathbb{R}} |h|_H^{k_0-l}|a|\,|p_{\zeta+is}(h,a,t)|^2\,dh\,da\Big)\,dt\\
&\qquad+\int_{0}^{\infty}\Big(\int_{H\times\mathbb{R}} e^{(Q-k_0+l)t}|h|_H^{k_0-l}|a|\,|p_{\zeta+is}(h,a,t)|^2\,dh\,da\Big)\,dt\\
&\quad\le \|\,|h|_H^{(k_0-l)/2}|a|^{1/2}p_{\zeta+is}\|_{L^2}^2+\int_G e^{(Q-k_0+l)t}|h|_H^{k_0-l}|a|\,|p_{\zeta+is}(g)|^2\,dg.
\end{aligned}$$

Now

$$\begin{aligned}
&\int_G e^{(Q-k_0+l)t}|h|_H^{k_0-l}|a|\,|p_{\zeta+is}(g)|^2\,dg\\
&\qquad=\int_G e^{-(Q-k_0+l)t}|\sigma_{-t}(h^{-1})|_H^{k_0-l}|-e^{-\alpha t}a|\,|p_{\zeta+is}(g^{-1})|^2\,d(g^{-1})\\
&\qquad=\int_G e^{-(Q-k_0+l)t}e^{-(k_0-l)t}|h|_H^{k_0-l}e^{-\alpha t}|a|\,|p_{\zeta+is}(g)|^2e^{(Q+\alpha)t}\,dg\\
&\qquad=\|\,|h|_H^{(k_0-l)/2}|a|^{1/2}p_{\zeta+is}\|_{L^2}^2.
\end{aligned}$$

Thus, by assumption on k_0, there is $C>0$ such that

$$\begin{aligned}
\sup_{\zeta\in D_{Q-k_0+1}}\|e^{lt/2}|h|_H^{(k_0-l)/2}|a|^{1/2}p_{\zeta+is}\|_{L^2}^2 &\le 2\sup_{\zeta\in D_{Q-k_0+1}}\|\,|h|_H^{(k_0-l)/2}|a|^{1/2}p_{\zeta+is}\|_{L^2}^2\\
&\le C(1+|s|)^{3+Q+k_0-l}, \qquad s\in\mathbb{R}.
\end{aligned}$$

Concerning terms with weights polynomial in $|h|_H$ only, they are the suprema of

$$\|e^{(l+\alpha)t/2}|h|_H^{(k_0-l)/2}p_{\zeta+is}\|_{L^2}^2 \qquad \text{for } \zeta\in D_{Q-k_0+1}$$

with $l=0,\dots,k_0-1$. Fix l in $[0,k_0-1]$. We have

$$\|e^{(l+\alpha)t/2}|h|_H^{(k_0-l)/2}p_{\zeta+is}\|_{L^2}^2 \le \int\limits_{|h|_H<1}\Big(\int\limits_{\mathbb{R}\times\mathbb{R}} e^{(l+\alpha)t}|p_{\zeta+is}(h,a,t)|^2\,da\,dt\Big)\,dh + \int\limits_{|h|_H\ge 1}\Big(\int\limits_{\mathbb{R}\times\mathbb{R}} e^{(l+\alpha)t}|h|_H^{Q-l}|p_{\zeta+is}(h,a,t)|^2\,da\,dt\Big)\,dh$$
$$\le \|e^{(l+\alpha)t/2}p_{\zeta+is}\|_{L^2}^2 + \int\limits_G e^{(l+\alpha)t}|h|_H^{Q-l}|p_{\zeta+is}(g)|^2\,dg.$$

On the one hand, by Corollary 4.1,

$$\sup_{\zeta\in D_{Q-k_0+1}}\|e^{(l+\alpha)t/2}p_{\zeta+is}\|_{L^2}^2 \le C(1+|s|)^2,\qquad s\in\mathbb{R}.$$

On the other hand,

$$\int\limits_G e^{(l+\alpha)t}|h|_H^{Q-l}|p_{\zeta+is}(g)|^2\,dg = \int\limits_G e^{-(l+\alpha)t}|\sigma_{-t}(h^{-1})|_H^{Q-l}|p_{\zeta+is}(g^{-1})|^2\,d(g^{-1})$$
$$= \int\limits_G e^{-(l+\alpha)t}e^{-(Q-l)t}|h|_H^{Q-l}|p_{\zeta+is}(g)|^2e^{(Q+\alpha)t}\,dg$$
$$= \|\,|h|_H^{(Q-l)/2}p_{\zeta+is}\|_{L^2}^2.$$

Thus by Lemma 5.1, there is $C>0$ such that

$$\int\limits_G e^{(l+\alpha)t}|h|_H^{Q-l}|p_{\zeta+is}(g)|^2\,dg \le C(1+|s|)^{3+Q-l},\qquad s\in\mathbb{R}.$$

Finally,

$$|\partial_s\|\,|h|_H^{k_0/2}|a|^{1/2}p_{\xi+is}\|_{L^2}^2|$$
$$\le C\Big(\sum_{l=0}^{k_0-1}(1+|s|)^{3+Q-l}+(1+|s|)^2+\sum_{l=1}^{k_0}(1+|s|)^{3+Q+k_0-l}\Big)$$
$$\le C(1+|s|)^{3+Q+k_0-1},\quad \xi\in D_{Q-k_0},\ s\in\mathbb{R}.$$

Then by Theorem 7.1, there exists $C>0$ such that

$$\|\,|h|_H^{k_0/2}|a|^{1/2}p_{\xi+is}\|_{L^2}^2| \le \|\,|h|_H^{k_0/2}|a|^{1/2}p_{\xi}\|_{L^2}^2| + C(1+|s|)^{3+Q+k_0}$$
$$\le C(1+|s|)^{3+Q+k_0},\qquad \xi\in D_{Q-k_0},\ s\in\mathbb{R},$$

and Proposition 6.1 follows. ■

7. Appendix: Heat kernel with complex time on non-unimodular Lie groups. In this section, we prove a pointwise estimate on the heat kernel with complex time (Theorem 7.1) used in the previous sections. This estimate holds on general non-unimodular Lie groups, which seems to be new when the time is complex.

Let G be a real connected Lie group equipped with a right invariant Haar measure dg. We denote by $d^l g$ the corresponding left Haar measure and by δ the modular function of G,

$$dg = \delta(g) d^l g.$$

From now on, we shall assume that the Lie group G is non-unimodular, that is, $\delta \not\equiv 1$.

Let $\chi = \{X_1, \dots, X_n\}$ be a system of left invariant vector fields on G that satisfies Hörmander's condition. We form the left invariant sub-Laplacian on G,

$$\Delta = \sum_{j=1}^{n} X_j^2.$$

We associate with Δ the semigroup of operators $T^z = e^{-z\Delta}$, $\Re z > 0$, and we denote by p_z the heat kernel

$$e^{-z\Delta}\phi = p_z *_l \phi, \qquad \phi \in C_0^\infty(G),\ \Re z > 0,$$

where $*_l$ denotes the convolution product in $L^2(G, d^l g)$.

Theorem 7.1. *Let G, δ, p_z be as above, and λ be real positive. For any ε in $]0,1[$, there is a positive constant C such that*

$$|p_z(g)| \le C(\Re z)^{-n/2} e^{\lambda \Re(z)} \delta^{-1/2}(g) \exp\left(-\Re\left(\frac{|g|^2}{(4+\varepsilon)z}\right)\right), \quad g \in G,\ \Re(z) > 0,$$

where n is the local dimension of the Lie group G.

Proof. This is a straightforward consequence of Proposition 7.1 below. ∎

To state Proposition 7.1, we need to introduce additional notations. Set

$$A = \delta^{1/2} \Delta \delta^{-1/2} + \lambda\, \mathrm{Id}.$$

The operator A is self-adjoint and non-negative on $L^2(G, d^l g)$. We associate with A the semigroup of operators $S^z = e^{-zA}$, $\Re z > 0$. Elementary computations show that

$$S^z = e^{-\lambda z} \delta^{1/2} T^z \delta^{-1/2}, \qquad \Re z > 0.$$

It follows that, for every $\Re z > 0$,

$$q_z = e^{-\lambda z} \delta^{1/2} p_z \tag{5}$$

satisfies

$$S^z \phi = \phi *_l q_z, \qquad \phi \in C_0^\infty(G).$$

Proposition 7.1. *For any ε in $]0,1[$, there is a positive constant C such that*

$$|q_z(g)| \le C(\Re z)^{-n/2} \exp\left(-\Re\left(\frac{|g|^2}{(4+\varepsilon)z}\right)\right), \qquad g \in G,\ \Re(z) > 0.$$

Proof. By Davies [5, Theorem 3.4.8], Lemmata 7.1 and 7.2 below yield the conclusion. ∎

LEMMA 7.1. *For any ε in $]0,1[$, there is a positive constant C such that*

$$q_t(g) \le Ce^{-\lambda t/2} t^{-n/2} \exp\left(-\frac{|g|^2}{(4+\varepsilon)t}\right), \qquad g \in G,\ t > 0.$$

Proof. By the pointwise estimate on the heat kernel of Varopoulos [15, Theorem IX.1.2],

$$p_t(g) \le C\delta^{-1/2}(g)(\min(t,1))^{-n/2} \exp\left(-\frac{|g|^2}{(4+\varepsilon)t}\right), \qquad g \in G,\ t > 0.$$

Lemma 7.1 follows upon using (5) with $z = t \in]0,\infty[$. ∎

LEMMA 7.2. *There exists a positive constant C such that*

$$|q_z(g)| \le C(\Re z)^{-n/2}, \qquad g \in G,\ \Re(z) > 0.$$

Proof. For any real positive x and any real y,

$$q_{x+iy} = S^{x/3} S^{x/3+iy} q_{x/3}.$$

So

$$\|q_{x+iy}\|_{L^\infty(G,d^l g)} \le \|S^{x/3}\|_{2\to\infty} \|S^{x/3+iy}\|_{2\to 2} \|q_{x/3}\|_{L^2(G,d^l g)},$$

where

$$\|S^{x/3}\|_{2\to\infty} = \sup\{\|S^{x/3}\phi\|_{L^\infty(G,d^l g)} : \|\phi\|_{L^2(G,d^l g)} \le 1\},$$
$$\|S^{x/3+iy}\|_{2\to 2} = \sup\{\|S^{x/3+iy}\phi\|_{L^2(G,d^l g)} : \|\phi\|_{L^2(G,d^l g)} \le 1\}.$$

Let us estimate $\|S^{x/3}\|_{2\to\infty}$. For any $\phi \in C_0^\infty(G)$,

$$\|S^{x/3}\phi\|_{L^\infty(G,d^l g)} \le \|\phi *_l q_{x/3}\|_{L^\infty(G,d^l g)} \le \|\phi\|_{L^2(G,\,d^l g)} \|q_{x/3}\|_{L^2(G,d^l g)}.$$

Hence

$$\|S^{x/3}\|_{2\to\infty} \le \|q_{x/3}\|_{L^2(G,\,d^l g)},$$

which implies

$$\|q_{x+iy}\|_{L^\infty(G,d^l g)} \le \|q_{x/3}\|^2_{L^2(G,d^l g)},$$

We observe that, for any $t > 0$,

$$q_t(g^{-1}) = e^{-\lambda t}\delta^{1/2}(g^{-1}) p_t(g^{-1}) = e^{-\lambda t}\delta^{-1/2}(g) p_t(g)\delta(g) = q_t(g), \qquad g \in G.$$

Therefore

$$\|q_{x/3}\|^2_{L^2(G,d^l g)} = \int_G q_{x/3}(g) q_{x/3}(g^{-1})\, d^l g = q_{x/3} *_l q_{x/3}(e) = q_{2x/3}(e)$$

where e is the unit of the Lie group G. Then by Lemma 7.1, we have

$$\|q_{x+iy}\|_{L^\infty(G,d^l g)} \le Cx^{-n/2}, \qquad x > 0,\ y \in \mathbb{R}.$$

This completes the proof of Lemma 7.2. ∎

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Received 17 May 2004 (4457)