COLLOQUIUM MATHEMATICUM

VOL. 101 2004 NO. 1

# CYLINDER COCYCLE EXTENSIONS OF MINIMAL ROTATIONS ON MONOTHETIC GROUPS

BY

MIECZYSŁAW K. MENTZEN (Toruń) and ARTUR SIEMASZKO (Olsztyn)

**Abstract.** The main results of this paper are:

1. No topologically transitive cocycle $\mathbb{R}^m$-extension of minimal rotation on the unit circle by a continuous real-valued bounded variation $\mathbb{Z}$-cocycle admits minimal subsets.
2. A minimal rotation on a compact metric monothetic group does not admit a topologically transitive real-valued cocycle if and only if the group is finite.

**Introduction.** This paper is devoted to the problem of minimal subsets of cylinder transformations. Let $X$ be a compact metric space and $T : X \to X$ be a homeomorphism of $X$. Let $\varphi : X \to \mathbb{R}$ be a continuous function. By a *cylinder transformation* we mean a homeomorphism $T_\varphi : X \times \mathbb{R} \to X \times \mathbb{R}$ (or rather the $\mathbb{Z}$-action generated by it) given by the formula

$$T_\varphi(x, r) = (Tx, \varphi(x) + r).$$

We will also consider the case of $\mathbb{R}^m$ instead of $\mathbb{R}$. It was essentially proved by Besicovitch in [2] that the cylinder transformation cannot itself be minimal. We also mention a deep result of Le Calvez and Yoccoz saying that there is no minimal homeomorphism on the infinite annulus or more generally on the two-dimensional sphere with a finite set of points removed ([10]). This of course generalizes Besicovitch's result.

The problem of the minimal subsets of a cylinder transformation turns out to be related to the problem of possible forms of $\omega$-limit sets. H. Poincaré was the first to consider flows (generated by differential equations) on $\mathbb{R}^3$ that had time one homeomorphisms topologically isomorphic to cylinder cocycle extensions over irrational rotations ([15]). He made an attempt at classifying possible forms of the vertical sections of $\omega$-limit sets. His classification turned out to be partial and only Krygin gave the full classification in [6]. In [7] Krygin gave a full classification in the differentiable situation proving that

2000 *Mathematics Subject Classification*: Primary 54H20.

*Key words and phrases*: topological dynamics, cocycle, cocycle group extension, cylinder cascade, essential value, minimal set.

Research partly supported by KBN grant 1 P03A 038 26.

actually there are four possibilities: either $\{0\}$ (the case of coboundary), or $\mathbb{R}$ (the case of transitive point), or $\mathbb{R}^+$, or $\mathbb{R}^-$.

In Sections 2 and 3 of this paper we show that there are no minimal sets for any transitive cylinder transformation defined by bounded variation cocycles over an irrational rotation on the circle (Theorem 2.4) and over adding machines (Theorem 3.4). Moreover, the only compact monothetic groups that do not admit transitive cocycles are finite cyclic groups (Theorem 4.6).

The authors would like to express their gratitude to Mariusz Lemańczyk whose contribution to Section 2 is substantial. We also thank Bassam Fayad for a creative discussion on the subject and for calling our attention to Krygin's papers.

**1. Preliminaries.** Let $T$ be a homeomorphism of a locally compact metric space $(X,d)$. We will call the pair $(X,T)$ a (*locally compact*) *flow*. If $X$ is compact, then we call $(X,T)$ a *compact* flow. In what follows we will often refer to $(X,T)$ as either $X$ or $T$. When $(X,T)$ is a flow and $A\subset X$, we write $\mathrm{Orb}(A)=\{T^kx : x\in A,\ k\in\mathbb{Z}\}$ to denote the *orbit*, and $\overline{\mathrm{Orb}}(A)=\overline{\mathrm{Orb}(A)}$ to denote the *orbit closure* of the set $A$. We say that a set $M\subset X$ is $T$-*minimal* if $M$ is closed, non-empty and invariant (i.e. $T^{-1}(M)=M$) and $M$ has no proper subset with these properties. In particular, the flow $(X,T)$ is minimal iff $\overline{\mathrm{Orb}}(x)=X$ for each $x\in X$. A minimal subset is either discrete or perfect. It follows that a compact minimal set is either perfect or finite, while a locally compact and non-compact minimal set is either perfect or the orbit of any of its points. If $X$ itself is $T$-minimal then $(X,T)$ is called *minimal*. A point $x\in X$ is said to be *almost periodic* if $\overline{\mathrm{Orb}}(x)$ is a compact minimal set. Each compact flow admits a minimal subset; for general flows this is not true. If there exists a point $x\in X$ such that $\overline{\mathrm{Orb}}(x)=X$, then $(X,T)$ is called *topologically transitive*.

The simplest examples of minimal flows are minimal rotations on compact metrizable monothetic groups. If $X$ is such a group and $\overline{\{x_0^n : n\in\mathbb{Z}\}}=X$ for some $x_0\in X$, then the map $T(x)=x_0x$ is a minimal homeomorphism.

Following [4] we say that a flow $(X,T)$ is *uniformly rigid* if there exists an unbounded sequence $(n_t)_{t\geq1}$ of integers such that $T^{n_t}\to\mathrm{Id}_X$ uniformly as $t\to\infty$. In that case the sequence $(n_t)_{t\geq1}$ is called a *rigidity time* for $T$. Clearly all minimal rotations on compact metrizable monothetic groups are uniformly rigid.

Suppose that $(X,T)$ is a flow, and $G$ a locally compact metric group with unit element $e$. For a continuous map $\varphi : X\to G$ one can define a $\mathbb{Z}$-cocycle $\varphi^{(\cdot)}$ by

$$\varphi^{(n)}(x) = \begin{cases} \varphi(T^{n-1}x)\varphi(T^{n-2}x)\cdots\varphi(Tx)\varphi(x), & n \geq 1, \\ e, & n = 0, \\ \varphi(T^n x)^{-1}\varphi(T^{n+1}x)^{-1}\cdots\varphi(T^{-1}x)^{-1}, & n \leq -1. \end{cases}$$

Then the cocycle condition $\varphi^{(n+k)}(x) = \varphi^{(n)}(T^k x)\varphi^{(k)}(x)$ is satisfied. Thus a continuous map $\varphi$ defines a $\mathbb{Z}$-cocycle $\varphi^{(n)}$. Conversely, each $\mathbb{Z}$-cocycle $\Psi : \mathbb{Z}\times X \to G$ is of the form $\Psi(n,x) = \varphi^{(n)}(x)$, where $\varphi(x) = \Psi(1,x)$. Therefore we will call a continuous function $\varphi : X \to G$ a *$\mathbb{Z}$-cocycle*; moreover, $\varphi$ is a *coboundary* if $\varphi(x) = (f(Tx))^{-1}f(x)$ for some continuous function $f$. Suppose now that $\varphi : X \to G$ is a cocycle. Define a homeomorphism

$$T_\varphi : X \times G \to X \times G, \quad T_\varphi(x,g) = (Tx, \varphi(x)g). \tag{1}$$

The flow $(X \times G, T_\varphi)$ defined by (1) is said to be a *cocycle group extension*, or, indicating the group, a *cocycle $G$-extension* of $(X,T)$.

For a compact minimal flow $(X,T)$, a locally compact metric group $G$ and a cocycle $\varphi$ we define the notion of essential value of $\varphi$ in the following way. Denote by $G_\infty$ the Aleksandrov compactification of $G$: $G_\infty = G \cup \{\infty\}$. We say that $g \in G_\infty$ is an *essential value* of $\varphi$ if for each non-empty open set $U \subset X$ and each neighbourhood $V$ of $g$ there exists $N \in \mathbb{Z}$ such that

$$U \cap T^{-N}U \cap \{x \in X : \varphi^{(N)}(x) \in V\} \neq \emptyset. \tag{2}$$

The set of all essential values of $\varphi$ will be denoted by $E_\infty(\varphi)$. It is not empty as it always contains the neutral element of the group $G$ (take $N = 0$ in the definition of $E_\infty(\varphi)$). Put also $E(\varphi) = E_\infty(\varphi) \cap G$. The set $E(\varphi)$ is always a closed subgroup of $G$ ([11, Proposition 3.1]). By [11, Proposition 3.2], $T_\varphi$ is topologically transitive iff $E(\varphi) = G$. On the other hand, the $\mathbb{Z}$-cocycle $\varphi^{(\cdot)} : \mathbb{Z}\times X \to G$ is a bounded function iff $\varphi$ is a coboundary iff $E_\infty(\varphi) = \{0\}$ (see e.g. [11, Proposition 3.4]). In that case the cocycle $G$-extension $(X \times G, T_\varphi)$ is a union of compact minimal subsets of the form $\{(x, f(x)g) : x \in X\}$, where $\varphi(x) = (f(Tx))^{-1}f(x)$ and $g \in G$.

If $G$ is Abelian, then a cocycle $\varphi$ is called *regular* if there exists a continuous function $\psi : X \to E(\varphi)$ such that $\varphi$ and $\psi$ differ by a coboundary: $\varphi = \psi + f \circ T - f$ for some continuous $f : X \to G$ (for Abelian groups we will often use the additive notation). Note that for Abelian groups, $E_\infty(\varphi) = E_\infty(\psi)$ whenever the cocycles $\varphi$ and $\psi$ differ by a coboundary ([11, Proposition 3.2]).

In this paper we will concentrate on the following situation. The flows $(X,T)$ will be minimal rotations on compact metric monothetic groups, $G = \mathbb{R}$ or $G = \mathbb{R}^m$ for some positive integer $m$. In such cases the group $E(\varphi)$ of essential values of $\varphi$ is a linear subspace of $\mathbb{R}^m$ ([13, Theorem 3.5]), and, whenever $\varphi$ has zero mean, $\varphi$ is regular ([13, Theorem 4.9]). In particular, for $\varphi : X \to \mathbb{R}$ there is a trichotomy:

(a) either $\int_X \varphi\,d\mu \neq 0$ (then $T_\varphi$ is transient: all orbits are discrete); or
(b) $T_\varphi$ is topologically transitive; or
(c) $\varphi$ is a coboundary.

Let us now recall the Denjoy–Koksma inequality. For the definition of discrepancy and the proof of Theorem 1.1 below we refer to [8] (or [9]).

THEOREM 1.1 ([9, Chapter 2, Theorem 5.1], see also [8]). *Let $\varphi$ be a function of bounded variation on $[0,1]$ and $x_1,\dots,x_N \in [0,1)$. Then*

$$\left|\frac{1}{N}\sum_{n=1}^{N}\varphi(x_n) - \int_0^1 \varphi(t)\,dt\right| \le \operatorname{Var}(\varphi) D_N^*,$$

*where $D_N^*$ denotes the discrepancy of the sequence $\{x_1,\dots,x_N\}$.* ■

We use this theorem in the particular case when $\int_0^1 \varphi(t)\,dt = 0$, $x_n = x + n\alpha \bmod 1$ ($\alpha$ is irrational), $N = q_k$, where $(q_k)_{k\ge1}$ is the sequence of the denominators of the continued fraction expansion of $\alpha$. Since, in that case, $D^*_{q_k} \le 1/q_k + 1/q_{k+1}$ (see e.g. [9, Chapter 2, (3.17)]), we obtain the following estimate:

$$\|\varphi^{(q_k)}\| \le 2\operatorname{Var}(\varphi) \tag{3}$$

(here and in what follows, $\|\cdot\|$ denotes the supremum norm in the space of continuous functions); we will use it in the proof of Theorem 2.4.

**2. The problem of minimality for cylinder extensions of minimal rotations on a circle.** In this section we will use the following standard notations. Let $\mathbb{T}$ be the unit circle on the complex plane with its natural topological group structure; we will often identify $\mathbb{T}$ with the interval $[0,1)$ mod 1. Then denote by $\mathrm{CBV}(\mathbb{T})$ the Banach space of continuous real functions on $\mathbb{T}$ with bounded variation: $\psi \in \mathrm{CBV}(\mathbb{T})$ iff $\psi$ is continuous and $\operatorname{Var}(\psi) < \infty$. Let $\mathrm{CBV}_0(\mathbb{T})$ be the subspace of $\mathrm{CBV}(\mathbb{T})$ consisting of all functions of zero mean with respect to the Lebesgue measure on $\mathbb{T}$. Note that if $\|\cdot\|$ denotes the supremum norm on the space $C(\mathbb{T})$, then the norm Var on $\mathrm{CBV}_0(\mathbb{T})$ satisfies $\|\psi\| \le \operatorname{Var}(\psi)$, in particular the usual norm $\|\cdot\| + \operatorname{Var}(\cdot)$ on $\mathrm{CBV}(\mathbb{T})$ is equivalent on $\mathrm{CBV}_0(\mathbb{T})$ to $\operatorname{Var}(\cdot)$. Let $\mathrm{AC}_0(\mathbb{T})$ be the subspace of $\mathrm{CBV}_0(\mathbb{T})$ of all absolutely continuous functions of zero mean.

The following theorem was essentially proved by Besicovitch in [2]. Although in [2] the case of $X = \mathbb{T}$ is considered, it is an immediate observation that Besicovitch used only compactness of $\mathbb{T}$. We repeat the proof of Besicovitch in our general situation for the paper to be more self-contained.

THEOREM 2.1 ([2]). *Let $(X,T)$ be a compact metric flow, and $\varphi : X \to \mathbb{R}$ a continuous map. Then $T_\varphi : X\times\mathbb{R} \to X\times\mathbb{R}$ is not minimal.*

*Proof.* We may assume that $(X,T)$ is minimal and $T_\varphi$ is topologically transitive. Let $(x_0,0)\in X\times\mathbb{R}$ be a topologically transitive point for $T_\varphi$, i.e. $\overline{\mathrm{Orb}}(x_0,0)=X\times\mathbb{R}$. By [5, Theorem 9.23] we may assume that $(x_0,0)$ is extensively transitive, i.e. both positive (this case will be used in our reasoning) and negative semi-orbits of $(x_0,0)$ are dense in $X\times\mathbb{R}$:

$$\overline{\{T_\varphi^n(x_0,0):n\ge 0\}}=X\times\mathbb{R},\qquad \overline{\{T_\varphi^n(x_0,0):n\le 0\}}=X\times\mathbb{R}.$$

Therefore we can find three sequences of integers $(m_j)_{j\ge1}$, $(n_j)_{n\ge1}$, $(s_j)_{j\ge1}$ such that $m_j<s_j<n_j$, $j\ge1$, and

$$\varphi^{(m_j)}(x_0)<-j,\quad \varphi^{(n_j)}(x_0)<-j,\quad \varphi^{(s_j)}(x_0)>j,\quad j\ge1.$$

As $\varphi$ is continuous and $X$ is compact,

$$m_j-s_j\to-\infty,\qquad n_j-s_j\to\infty. \tag{4}$$

We may assume that

$$\varphi^{(s_j)}(x_0)=\max\{\varphi^{(n)}(x_0):m_j\le n\le n_j\},\qquad j\ge1. \tag{5}$$

Consider the points

$$\begin{aligned}(x_j^n,r_j^n)&=T_\varphi^{s_j+n}(x_0,-\varphi^{(s_j)}(x_0))=(T^{s_j+n}x_0,\varphi^{(s_j+n)}(x_0)-\varphi^{(s_j)}(x_0))\\&=(T^{s_j+n}x_0,\varphi^{(n)}(T^{s_j}x_0)),\qquad n\in\mathbb{Z},j\ge1.\end{aligned}$$

Then, for $m_j-s_j\le n\le n_j-s_j$, $j\ge1$, we have

$$r_j^0=0,\qquad T_\varphi(x_j^n,r_j^n)=(x_j^{n+1},r_j^{n+1}). \tag{6}$$

Take a convergent subsequence $x_{j_k}^0\to\widetilde{x}$. Then, by (6), for each $n\in\mathbb{Z}$ the subsequence

$$(x_{j_k}^n,r_{j_k}^n)=(x_{j_k}^n,\varphi^{(n)}(T^{s_{j_k}}x_0))=T_\varphi^n(x_{j_k}^0,0),\qquad k\ge1,$$

is also convergent, $(x_{j_k}^n,r_{j_k}^n)\to(T^n\widetilde{x},\varphi^{(n)}(\widetilde{x}))$. By (4), for each given integer $n$ the inequalities $m_{j_k}-s_{j_k}<n<n_{j_k}-s_{j_k}$ hold for $k$ large enough, hence, by (6) and by (5), $\varphi^{(n)}(\widetilde{x})\le0$. In particular for each $(x,r)\in\overline{\mathrm{Orb}}(\widetilde{x},0)$ we have $r\le0$, therefore $\mathrm{Orb}(\widetilde{x},0)$ is not dense in $X\times\mathbb{R}$. ∎

COROLLARY 2.2. *Let $(X,T)$ be a compact metric flow, and $\varphi:X\to\mathbb{R}^m$ a continuous map. Then $T_\varphi:X\times\mathbb{R}^m\to X\times\mathbb{R}^m$ is not minimal.*

*Proof.* Let $\varphi=(\varphi_1,\dots,\varphi_m):X\to\mathbb{R}^m$ be a continuous map. By the above there exists a point $\widetilde{x}\in X$ such that the orbit of $(\widetilde{x},0)$ via $T_{\varphi_1}$ is not dense in $X\times\mathbb{R}$, hence the orbit of $(\widetilde{x},0,\dots,0)$ via $T_\varphi$ is not dense in $X\times\mathbb{R}^m$. In particular $T_\varphi$ is not minimal. ∎

Our next aim is to show that no continuous bounded variation cocycle on $\mathbb{T}$ admits minimal subsets (Theorem 2.4). The method of the proof of this theorem is similar to the proof of [7, Proposition 2] of Krygin's paper on the Poincaré sets for smooth cocycles, i.e. the vertical sections of limit sets in

$\mathbb{T}\times\mathbb{R}$. Together with Lemma 2.3 below some ideas of Krygin's proof of [7, Proposition 2], after modifications, will give the proof of our result.

LEMMA 2.3. *Let $(X,T)$ be a minimal rotation on a compact monothetic group, $\varphi : X \to \mathbb{R}^m$ a continuous map. Suppose $M \subset X\times\mathbb{R}^m$ is a $T_\varphi$-minimal set. Define $M_x = (\{x\}\times\mathbb{R}^m)\cap M$. Then* $\operatorname{card} M_x \le 1$ *for every $x\in X$.*

*Proof.* First consider the case $m=1$. If $T_\varphi$ is not topologically transitive, then either it is a coboundary or $T_\varphi$ is transient. In the first case $M$ is the graph of some continuous function $f : X\to\mathbb{R}$ (see [11, Proposition 5.1]), in the second $M$ is equal to the orbit via $T_\varphi$ of some point (see [11, Remark 4]). In both cases $\operatorname{card} M_x \le 1$. Thus we may assume that $T_\varphi$ is topologically transitive.

Observe that as $T$ is minimal, the set $D=\{x\in X : M_x\ne\emptyset\}$ is dense in $X$. Put $H=\{r\in\mathbb{R} : M+r=M\}$ (here $M+r=\{(x,s+r) : (x,s)\in M\}$). It is easy to see that $H$ is a closed subgroup. Similarly to [3, Lemma 3.1] or [16, Lemma 2.6.1] we see that if $M_x\ne\emptyset$ then $M_x=r+H$ for every $r\in\mathbb{R}$ such that $(x,r)\in M$.

First assume that $H=\mathbb{R}$. Then for $x\in D$ we have $M_x=\mathbb{R}$, which implies $M=X\times\mathbb{R}$, a contradiction with Theorem 2.1.

Now let $H=a\mathbb{Z}$. Take a $T_\varphi$-transitive point $(x,0)\in X\times\mathbb{R}$. Find a sequence $x_i\in D$, $i\ge 1$, that converges to $x$, and numbers $r_i\in\mathbb{R}$ such that $(x_i,r_i)\in M$. Since $H=a\mathbb{Z}$, the numbers $r_i$ may be chosen from $[0,a)$, thus by passing to a subsequence if necessary, we may assume that the points $(x_i,r_i)$ converge to $(x,r)\in M$. But $(x,r)$ is a transitive point, a contradiction.

It follows that $H=\{0\}$, which gives the result.

Suppose now $m$ is arbitrary. For any linear functional $L:\mathbb{R}^m\to\mathbb{R}$ define a factor map $\widetilde{L} : X\times\mathbb{R}^m\to X\times\mathbb{R}$ by setting $\widetilde{L}(x,r)=(x,L(r))$. Then the set $N^L=\widetilde{L}(M)$ is minimal and $\operatorname{card} N^L_x\le 1$ for each $x\in X$. Suppose $r,s\in M_x$, $r=(r_1,\ldots,r_m)$, $s=(s_1,\ldots,s_m)$. Fix $i$, $1\le i\le m$, and take $L=p_i$, the projection onto the $i$th coordinate. Then $\widetilde{p_i}(x,r)=(x,r_i)\in N_x^{p_i}$ and $\widetilde{p_i}(x,s)=(x,s_i)\in N_x^{p_i}$, hence $r_i=s_i$. We have shown $r=s$ and the result follows. ■

THEOREM 2.4. *Let $T$ be a minimal rotation on $\mathbb{T}$. If $\varphi\in\mathrm{CBV}_0(\mathbb{T})$ and $\varphi$ is not a coboundary then $T_\varphi$ has no minimal subsets.*

*Proof.* Identify $\mathbb{T}$ with $[0,1)$ mod 1 and let $Tx=x+\alpha \mod 1$, where $\alpha$ is an irrational number. Let $(q_n)_{n\ge1}$ be the sequence of denominators in the continued fraction expansion of $\alpha$.

Assume that $M\subset\mathbb{T}\times\mathbb{R}$ is a $T_\varphi$-minimal set and $(x,0)\in M$. Choose $\varepsilon>0$. By Lemma 2.3 we find $\delta>0$ such that the positive semi-orbit

$\{(T_\varphi)^n(x,0) : n > 0\}$ of $(x,0)$ intersects neither $B^- = (x-\delta, x+\delta) \times (-\varepsilon - 2\operatorname{Var}(\varphi), -\varepsilon)$ nor $B^+ = (x-\delta, x+\delta) \times (\varepsilon, \varepsilon + 2\operatorname{Var}(\varphi))$.

Choose a positive odd integer $n$ so large that $\|q_n\alpha\| < \delta$ (here $\|\beta\|$ denotes the distance of the number $\beta$ from the set of integers). Then $I = [x - \|q_{n+1}\alpha\|, x + \|q_n\alpha\|) \subset (x-\delta, x+\delta)$. If $t \in I$ and $m_I(t)$ denotes the first return time of $t$ to $I$, then

$$m_I(t) = \begin{cases} q_{n+1}, & t \in [x, x + \|q_n\alpha\|), \\ q_n, & t \in [x - \|q_n\alpha\|, x). \end{cases}$$

It follows that every point of the orbit of $x$ under $T$ has the first return time to $I$ equal either to $q_n$ or to $q_{n+1}$. Now, by (3), we have $|\varphi^{(l)}(x)| \le \varepsilon$ whenever $T^l x \in I$, $l > 0$, since the positive semi-orbit of $(x,0)$ does not intersect $B^- \cup B^+$. Moreover, the set $\{l > 0 : T_\varphi^l(x,0) \in I \times [-\varepsilon, \varepsilon]\} \subset \mathbb{N}$ has bounded gaps, thus the positive semi-orbit of $(x,0)$ is bounded. Therefore, by [5, Theorem 14.11], $\varphi$ is a coboundary. We have reached a contradiction. ∎

REMARK 2.5. In [17] Sidorov constructs for each irrational rotation on $\mathbb{T}$ a topologically transitive cocycle without discrete orbits (recall that a discrete orbit is always a minimal set). Below, using rather standard methods, we generalize this by showing that over every irrational rotation there exists a cocycle without minimal sets.

On the other hand, Besicovitch ([2]) constructs a particular irrational rotation and a topologically transitive cocycle that admits a discrete orbit. It remains an open problem whether there exist topologically transitive cylinder cocycles with minimal sets other than discrete orbits.

Now we will show the following:

*For every minimal rotation $T$ on $\mathbb{T}$ there exists $\varphi \in \mathrm{AC}_0(\mathbb{T})$ that is topologically transitive.*

*Proof.* Assume that $T$ is a minimal rotation on $\mathbb{T}$, $Tx = x + \alpha$, such that all $\varphi \in \mathrm{AC}_0(\mathbb{T})$ are coboundaries, i.e. for every $\varphi \in \mathrm{AC}_0(\mathbb{T})$ there exists $g_\varphi \in C(\mathbb{T})$ such that $\varphi = g_\varphi - g_\varphi \circ T$. By minimality of $T$ we may assume that $g_\varphi$ is a zero mean function. We have obtained a well defined linear map $\mathrm{CBV}_0(\mathbb{T}) \supset \mathrm{AC}_0(\mathbb{T}) \ni \varphi \mapsto g_\varphi \in C_0(\mathbb{T})$. For the purpose of this proof we consider the space $\mathrm{AC}_0(\mathbb{T})$ with the variation norm Var. With this norm the map $\varphi \mapsto g_\varphi$ is continuous by the Closed Graph Theorem. Thus there is a constant $M \ge 0$ such that

$$\|g_\varphi\| \le M \cdot \operatorname{Var}(\varphi) \tag{7}$$

for every $\varphi \in \mathrm{AC}_0(\mathbb{T})$. However, we will see that there exists a sequence $(P_N)_{N\ge0}$ of real polynomials on $\mathbb{T}$ such that

$$\lim \frac{\|P_N\|}{\mathrm{Var}(P_N - P_N \circ T)} = \infty \tag{8}$$

(which will give a contradiction to (7)).

To see this consider $P_N(x) = \sum_{-N}^{N} a_n e^{2\pi i n x}$, $N \geq 0$, such that $a_n = a_{-n}$ for $n \geq 0$, $a_n = 0$ for $n \neq q_k$, $a_{q_k} > 0$ (here $(q_k)_{k\geq 1}$ is the sequence of denominators of the continued fraction expansion of $\alpha$), and

$$\lim \frac{\sum a_{q_k}}{(\sum a_{q_k}^2)^{1/2}} = \infty. \tag{9}$$

We have

$$P_N' - P_N' \circ T = 2\pi i \sum q_k a_{q_k} (1 - e^{2\pi i q_k \alpha}) e^{2\pi i q_k x}.$$

Since $|1 - e^{2\pi i q_k \alpha}| \leq \|q_k \alpha\|$ and $q_k \|q_k \alpha\| \leq 1$ (recall that $\|\beta\|$ denotes the distance of $\beta$ from the set of integers), we have

$$\begin{aligned} \mathrm{Var}(P_N - P_N \circ T) &= \|P_N' - P_N' \circ T\|_{L^1} \\ &\leq \|P_N' - P_N' \circ T\|_{L^2} \leq \mathrm{const} \cdot \Big(\sum a_{q_k}^2\Big)^{1/2}. \end{aligned} \tag{10}$$

Now, the equality $\|P_N\| = 2\sum a_{q_k}$ (recall that $\|\cdot\|$ denotes the supremum norm), (10), and the assumption (9) imply (8). We have obtained a contradiction with (7) and we are done. ∎

For more general results on the existence of topologically transitive cocycles over irrational rotations see [12, Theorems 3 and 6].

REMARK 2.6. Consider a linear map $\Phi : C_0(\mathbb{T}) \ni f \mapsto f - f \circ T \in C_0(\mathbb{T})$. Assume for a moment that $\Phi(C_0(\mathbb{T})) \subset \mathrm{CBV}_0(\mathbb{T})$. Then, by the Closed Graph Theorem, the map $\Phi : C_0(\mathbb{T}) \to \mathrm{CBV}_0(\mathbb{T})$ is continuous. Thus there is a constant $M > 0$ such that $\mathrm{Var}(f - f \circ T) \leq M\|f\|$ for all $f \in C_0(\mathbb{T})$. On the other hand, it is not difficult to find $f \in C_0(\mathbb{T})$ with $\|f\| = 1$ and arbitrarily large variation of $f - f \circ T$, which is a contradiction. This rather standard consideration shows the following:

*For every minimal rotation $T$ on $\mathbb{T}$ there exists $f \in C_0(\mathbb{T})$ such that $f - f \circ T$ is not of bounded variation.*

Theorem 2.4 together with Remarks 2.5 and 2.6 shows that there are also unbounded variation cocycles without minimal sets.

**3. The problem of minimality for cylinder extensions of adding machines.** Let $\overline{r} = (r_n)_{n\geq 1}$ be a sequence of integers such that $r_n \geq 2$, $n \geq 1$. Set $\lambda_0 = 1$, $\lambda_n = r_1 \cdots r_n$, $n \geq 1$. Let

$$\mathbb{Z}(\overline{r}) = \Big\{ \sum_{n=0}^{\infty} a_n \lambda_n : a_n \in \{0, 1, \ldots, r_{n+1} - 1\} \Big\} \tag{11}$$

be the compact group of $\overline{r}$-adic numbers with the product topology induced from $\prod_{n=0}^{\infty}\{0,1,\dots,r_n-1\}$. This topology may be defined by the metric $d(\sum a_n\lambda_n,\sum b_n\lambda_n)=1/\lambda_m$, where $m=\min\{n: a_n\neq b_n\}$.

For $m\geq 1$ and $0\leq k<\lambda_m$ define the sets $W_k^m=[a_0\,a_1\,\dots\,a_{m-1}]$ by

$$W_k^m=\Big\{x\in\mathbb{Z}(\overline{r}): x=\sum_{n=0}^{\infty}x_n\lambda_n,\ x_i=a_i,\ i=0,1,\dots,m-1\Big\}, \tag{12}$$

where $a_i\in\{0,1,\dots,r_{i+1}-1\}$ are such that $\sum_{i=1}^{m}a_i\lambda_i=k$. Let $\mathcal{W}^m=\{W_0^m,W_1^m,\dots,W_{\lambda_m-1}^m\}$. Clearly the sets $W_k^m$ are closed-open and $\bigcup\mathcal{W}^m=\mathbb{Z}(\overline{r})$. Let $\mu$ denote the normalized Haar measure on $\mathbb{Z}(\overline{r})$. Observe that $\operatorname{diam}(W_k^m)=\mu(W_k^m)=1/\lambda_m$. We define a homeomorphism $T:\mathbb{Z}(\overline{r})\to\mathbb{Z}(\overline{r})$ by $Tx=x+1$, thus obtaining a minimal rotation on the compact metric monothetic group $\mathbb{Z}(\overline{r})$. Then the metric $d$ defined above as well as the measure $\mu$ are $T$-invariant. Moreover $TW_k^m=W_{k+1}^m$, where $k+1$ is taken mod $\lambda_m$. The flow $(\mathbb{Z}(\overline{r}),T)$ is called an *adding machine*.

Denote by $C(\mathbb{Z}(\overline{r}))$ the space (algebra) of all continuous real functions on $\mathbb{Z}(\overline{r})$. Equip $C(\mathbb{Z}(\overline{r}))$ with the topology of uniform convergence. Observe that each real function that is constant on elements of some $\mathcal{W}^m$ is continuous.

DEFINITION 3.1. We say that $\varphi\in C(\mathbb{Z}(\overline{r}))$ has *bounded variation* if

$$\operatorname{Var}(\varphi):=\sup_{m\geq 0}\mathrm{V}_m(\varphi)<\infty,\quad\text{where}\quad \mathrm{V}_m(\varphi)=\sum_{k=0}^{\lambda_m-1}(\max_{W_k^m}\varphi-\min_{W_k^m}\varphi).$$

The family of such functions is denoted by $\mathrm{CBV}(\mathbb{Z}(\overline{r}))$; moreover, as usual, $\mathrm{CBV}_0(\mathbb{Z}(\overline{r}))$ stands for the subfamily of $\mathrm{CBV}(\mathbb{Z}(\overline{r}))$ consisting of all functions with zero mean with respect to $\mu$.

REMARK 3.2. Let $(X,d)$ be a compact metric space and $\varphi: X\to\mathbb{R}$ be a continuous function. Recall that a function $M_\varphi=M:\mathbb{R}_+\to\mathbb{R}$ defined by

$$M(h)=\sup_{d(x,y)\leq h}|\varphi(x)-\varphi(y)|$$

is called a *continuity modulus* of $\varphi$. Now take $X=\mathbb{Z}(\overline{r})$ and consider a family of functions $\varphi\in C(\mathbb{Z}(\overline{r}))$ such that $M_\varphi(1/\lambda_k)=O(1/\lambda_k)$ (i.e. the sequence $(\lambda_k M_\varphi(1/\lambda_k))_{k\geq 1}$ is bounded). Since obviously

$$\mathrm{V}_m(\varphi)\leq\lambda_m M(1/\lambda_m),$$

this family is contained in $\mathrm{CBV}(X)$; actually this inclusion may be strict in general.

We intend to prove that a topologically transitive bounded variation cocycle over an adding machine does not admit minimal subsets. Such a cocycle is constructed in Example 4.2. We start with a lemma that contains a kind of Denjoy–Koksma inequality for cocycles over adding machines.

LEMMA 3.3. *Let* $\varphi \in \mathrm{CBV}(\mathbb{Z}(\overline{r}))$. *Then*
$$\left|\frac{1}{\lambda_m}\varphi^{(\lambda_m)}(x) - \int\limits_{\mathbb{Z}(\overline{r})}\varphi(t)\,dt\right| \le \frac{1}{\lambda_m}\mathrm{Var}(\varphi)$$
*for every* $x \in \mathbb{Z}(\overline{r})$.

*Proof.* Fixing $m \ge 1$ and $x \in \mathbb{Z}(\overline{r})$ we may assume that $x \in W_0^m$. Then
$$\begin{aligned}\left|\varphi^{(\lambda_m)}(x) - \lambda_m\int\limits_{\mathbb{Z}(\overline{r})}\varphi(t)\,dt\right| &\le \sum_{k=0}^{\lambda_m-1}\left|\varphi(T^k x) - \lambda_m\int\limits_{W_k^m}\varphi(t)\,dt\right| \\ &\le \sum_{k=0}^{\lambda_m-1}\lambda_m\int\limits_{W_k^m}|\varphi(T^k x) - \varphi(t)|\,dt \\ &\le \sum_{k=0}^{\lambda_m-1}\lambda_m\int\limits_{W_k^m}(\max_{W_k^m}\varphi - \min_{W_k^m}\varphi)\,dt = \mathrm{V}_m(\varphi) \\ &\le \mathrm{Var}(\varphi). \ \blacksquare\end{aligned}$$

In the case of $\varphi \in \mathrm{CBV}_0(X)$ the inequality from Lemma 3.3 takes the form
$$\|\varphi^{(\lambda_m)}\| \le \mathrm{Var}(\varphi). \tag{13}$$

THEOREM 3.4. *Let* $(X,T)$ *be an adding machine. If* $\varphi \in \mathrm{CBV}_0(X)$ *and* $\varphi$ *is not a coboundary then* $T_\varphi$ *admits no minimal subsets.*

*Proof.* The proof is similar to that of Theorem 2.4. We consider the rigidity time $(\lambda_m)_{m\ge1}$ instead of $(q_k)_{k\ge1}$ and use (13) instead of (3). The interval $I$ is replaced by one of the levels of $\mathcal{W}^n$ for appropriate $n$. ■

**4. Existence of topologically transitive cocycles over compact monothetic groups.** Our next aim is to show that all minimal rotations on compact infinite metrizable monothetic groups admit topologically transitive real cocycles. Gottschalk and Hedlund ([5]) have developed a theory of real cocycles over minimal rotations on connected and locally connected monothetic groups. However also rotations on disconnected monothetic groups may admit transitive cocycles.

EXAMPLE 4.1. Let $X = \mathbb{Z}_2 \times [0,1)$ and $T(i,x) = (i+1, x+\alpha)$, where the addition on the first coordinate is taken mod 2 while the addition on the second coordinate is taken mod 1. Take a continuous function $\varphi : [0,1) \to \mathbb{R}$ and define $\widetilde{\varphi} : X \to \mathbb{R}$ by $\widetilde{\varphi}(i,x) = \varphi(x)$. Assume $\widetilde{\varphi}$ is a coboundary, i.e. there exists a continuous $\widetilde{g} : X \to \mathbb{R}$ with $\widetilde{\varphi}(i,x) = \widetilde{g}(i+1,x+\alpha) - \widetilde{g}(i,x)$. Putting $g : [0,1) \to \mathbb{R}$, $g(x) = \frac{1}{2}(\widetilde{g}(0,x) + \widetilde{g}(1,x))$ we have
$$g(x+\alpha) - g(x) = \frac{1}{2}\left(\widetilde{\varphi}(1,x) + \widetilde{\varphi}(0,x)\right) = \varphi(x).$$

Now, taking a transitive cocycle $\varphi$ on $([0,1),\alpha)$ we get, by the construction above, a transitive cocycle $\widetilde{\varphi}$ over a disconnected, locally connected monothetic group.

Notice that in the example above, $g$ is an integral of $\widetilde{g}$ with respect to the normalized Haar measure on the kernel of the projection onto the second coordinate. This simple observation gives rise to Lemma 4.3.

Now we show that also adding machines that are not locally connected admit topologically transitive cocycles.

EXAMPLE 4.2. Let $X=\mathbb{Z}(\overline{r})$ and $T$ be a minimal rotation on $X$. Put

$$\chi_s\Big(\sum_{n=0}^{\infty} a_n\lambda_n\Big)=\exp\Big(2\pi i\,\frac{\sum_{n=0}^{s-1}a_n\lambda_n}{\lambda_s}\Big).$$

Observe that (the character group) $\widehat{X}$ is $\{\chi_s^l : s\geq 1,\ 0\leq l\leq \lambda_s-1\}$. Define $\varphi : X\to\mathbb{R}$ by

$$\varphi(x)=\sum_{s=1}^{\infty}\varphi_s=\sum_{s=1}^{\infty}\frac{\chi_s+\chi_s^{-1}}{\lambda_s}.$$

Clearly $\varphi\in C_0(X)$. Assume that $\varphi$ is a coboundary, $\varphi=g\circ T-g$ for some $g\in C_0(X)$. Represent $g=\sum_{s\geq 1}\sum_{0<t<\lambda_s} a_{s,t}\chi_s^t$. For $s\geq 1$ we have

$$a_{s,t}=\begin{cases}\dfrac{1}{\lambda_s(\exp(2\pi i/\lambda_s)-1)} & \text{if } t=1,\\ 0 & \text{if } 1<t<\lambda_s-1,\\ \dfrac{1}{\lambda_s(\exp(-2\pi i/\lambda_s)-1)} & \text{if } t=\lambda_s-1.\end{cases}$$

Simple calculations show that

$$|a_{s,1}|=|a_{s,\lambda_s-1}|=\frac{1}{2\lambda_s\sin(\pi/\lambda_s)}\geq\frac{1}{2\pi},$$

which is impossible. Therefore $\varphi$ is a topologically transitive cocycle over the adding machine. It turns out that $\varphi\in \mathrm{CBV}_0(X)$. To see this observe first that $\chi_s$, hence also $\varphi_s$, is constant on the levels of $\mathcal{W}^t$ for $t\geq s$. Moreover $\varphi_s$ takes on levels of $\mathcal{W}^s$ the values $2\cos(2\pi l/\lambda_s)/\lambda_s$, $0\leq l<\lambda_s$, as $\chi_s$ takes the values $\exp(2\pi il/\lambda_s)$, $0\leq l<\lambda_s$. Therefore

$$\max_{W_k^m}\varphi-\min_{W_k^m}\varphi\leq\sum_{s>m}(\max_{W_k^m}\varphi_s-\min_{W_k^m}\varphi_s)\leq\sum_{s>m}\frac{4}{\lambda_s}$$

for $m\geq 1$. Thus

$$\mathrm{V}_m(\varphi)=\sum_{j=0}^{\lambda_m-1}(\max_{W_k^m}\varphi-\min_{W_k^m}\varphi)\leq 4\sum_{s>m}\frac{\lambda_m}{\lambda_s}<\frac{8}{r_{m+1}}\leq 4.$$

Consequently,

$$\mathrm{Var}(\varphi) = \sup_{m\ge 1} \mathrm{V}_m(\varphi) \le 4$$

and $\varphi$ has bounded variation.

LEMMA 4.3. *Let $\pi : X \to Y$ be a continuous group epimorphism of compact metric monothetic groups. Let $T : X \to X$, $Tx = x + \alpha$, and $S : Y \to Y$, $Sy = y + \beta$, where $\beta = \pi(\alpha)$, be minimal rotations. Then for any topologically transitive cocycle $\varphi : Y \to \mathbb{R}$ the cocycle $\widetilde{\varphi} : X \to \mathbb{R}$ defined by $\widetilde{\varphi}(x) = \varphi(\pi(x))$ is also topologically transitive.*

*Proof.* Assume that $\varphi : Y \to \mathbb{R}$ is a topologically transitive cocycle. Then $\int \varphi\, d\mu_Y = 0$, hence $\int \widetilde{\varphi}\, d\mu_X = 0$, where $\mu_X$ and $\mu_Y$ denote the normalized Haar measures on $X$ and $Y$ respectively. By [11, Theorem 1], either $T_{\widetilde{\varphi}}$ is topologically transitive or $\widetilde{\varphi}$ is a coboundary. Assume $\widetilde{\varphi} : X \to \mathbb{R}$, $\widetilde{\varphi} = \varphi \circ \pi$, is a coboundary over the rotation by $\alpha \in X$, i.e. there exists a continuous $\widetilde{g} : X \to \mathbb{R}$ with $\widetilde{\varphi}(x) = \widetilde{g}(x + \alpha) - \widetilde{g}(x)$. Set $K = \ker \pi$ and identify $Y$ with $X/K$; then $\beta = \alpha + K$. Define a continuous function $g : Y \to \mathbb{R}$ by $g(x + K) = \int_K \widetilde{g}(x + k)\, dk$. We have

$$\begin{aligned} g(x+\alpha+K) - g(x+K) &= \int_K (\widetilde{g}(x+\alpha+k) - \widetilde{g}(x+k))\, dk \\ &= \int_K \widetilde{\varphi}(x+k)\, dk = \int_K \varphi(x+K)\, dk = \varphi(x+K). \end{aligned}$$

Thus we have shown $\varphi$ to be a coboundary over the rotation by $\beta$, which is a contradiction. ■

In the following theorem we generalize Lemma 4.3.

THEOREM 4.4. *Let $\pi : X \to Y$ be a continuous group epimorphism of compact metric monothetic groups, and let $T : X \to X$ and $S : Y \to Y$, where $\pi \circ T = S \circ \pi$, be minimal rotations. Then $E(\varphi) = E(\varphi \circ \pi)$ for each continuous cocycle $\varphi : Y \to \mathbb{R}^m$.*

*Proof.* If $\varphi$ is transient, then clearly $\varphi \circ \pi$ is also transient. Suppose $S_\varphi$ is conservative, hence regular (see [13, Theorem 4.9]). Let $L : \mathbb{R}^m \to \mathbb{R}$ be linear. By [13, Theorem 3.5], both $E(\varphi)$ and $E(\varphi \circ \pi)$ are linear subspaces of $\mathbb{R}^m$. Then, by [14, Proposition 3.1],

$$E(L \circ \varphi) = L(E(\varphi)), \qquad E(L \circ \varphi \circ \pi) = L(E(\varphi \circ \pi)).$$

By Lemma 4.3, $E(L \circ \varphi) = E(L \circ \varphi \circ \pi)$ so $L(E(\varphi)) = L(E(\varphi \circ \pi))$. As $L$ is arbitrary, $E(\varphi) = E(\varphi \circ \pi)$. ■

REMARK 4.5. Define $\varphi : [0, 1) \to \mathbb{R}$ by setting

$$\varphi(x) = \sum_{k=1}^{\infty} \frac{1}{q_k} \cos 2\pi q_k x = \frac{1}{2} \sum_{k\ge 1} \frac{1}{q_k} \left(e^{2\pi i q_k x} + e^{-2\pi i q_k x}\right)$$

(by Euler's formula). Since $\sum 1/q_k$ converges, $\varphi$ is a well defined continuous (and zero mean) function. We will show that $\varphi$ is topologically transitive (cf. [5, 14.14]). Suppose to the contrary that $\varphi$ is a coboundary, i.e. $\varphi = g - g\circ T$ for some continuous function $g$. Let $a_n = \int_{\mathbb{T}} g(x)e^{-2\pi nix}\,dx$, $n \in \mathbb{Z}$. By the Lebesgue–Riemann Lemma, $\lim a_n = 0$. Simple calculations show that

$$\frac{1}{2}\cdot\frac{1}{q_k} = a_{\pm q_k}(1-e^{\pm 2\pi i q_k\alpha}),\quad k\ge 1,\qquad a_n = 0,\quad n\ne \pm q_k,\ k\ge 1.$$

However $|e^{2\pi i q_k\alpha}-1| < 8/q_{k+1}$ and it follows that

$$16|a_{q_k}| = \frac{8}{q_k}\cdot\frac{1}{|e^{2\pi i q_k\alpha}-1|} > \frac{q_{k+1}}{q_k}\ge 1,$$

which gives a contradiction. Thus the cocycle $\varphi$ is not a coboundary, hence $\varphi$ is topologically transitive.

Using the topologically transitive cocycle $\varphi$ we have defined above one may construct for any $m$ a topologically transitive cocycle $\widetilde{\varphi} : \mathbb{T}\to\mathbb{R}^m$. To see this consider $m$ pairwise disjoint subsequences $(c_{k,j})_{k\ge 1}$, $j = 1,\dots,m$, of the sequence $(q_k)_{k\ge1}$ such that none of the sequences $(1/c_{k,j}|e^{2\pi i c_{k,j}\alpha}-1|)_{k\ge 1}$, $j=1,\dots,m$, is convergent. Setting

$$\varphi_j(x) = \sum_{k\ge 1}\frac{1}{c_{k,j}}\cos 2\pi c_{k,j}x,\qquad j=1,\dots,m,$$

we see that no non-zero combination $b_1\varphi_1+\ldots+b_m\varphi_m$ is a coboundary. By Atkinson's theorem ([1, Theorem 1] or [13, Proposition 4.8]), the cocycle $\widetilde{\varphi} = (\varphi_1,\dots,\varphi_m)$ is topologically transitive.

Using Example 4.2, Lemma 4.3 and Remark 4.5 we get the following.

THEOREM 4.6. *Assume that $X$ is an infinite compact metric monothetic group. Let $T : X\to X$ be a minimal rotation on $X$. Then $(X,T)$ admits a topologically transitive real cocycle.* ∎

Remark 4.5 allows us to give a slight generalization of Theorem 4.6.

COROLLARY 4.7. *Assume that $X$ is an infinite compact metric monothetic group. Let $T : X\to X$ be a minimal rotation. Then for each integer $m\ge 1$ and for each linear subspace $V\subset\mathbb{R}^m$ there exists a continuous cocycle $\varphi : X\to\mathbb{R}^m$ such that $E(\varphi)=V$.*

*Proof.* Suppose $V\subset\mathbb{R}^m$ is a linear subspace. Let $\psi = (\varphi_1,\dots,\varphi_m) : X\to\mathbb{R}^m$ be a topologically transitive cocycle. If $\dim V = 0$, then any coboundary is suitable. Suppose $\dim V = k > 0$. Let $\overline{e}_1,\dots,\overline{e}_m$ be the standard basis of $\mathbb{R}^m$. Without loss of generality we may assume that $V$ is generated by $\overline{e}_1,\dots,\overline{e}_k$. Indeed, by [13, Theorem 4.9], all zero mean cocycles with values in $\mathbb{R}^m$ are regular, and application of [14, Proposition 3.1] finishes the argument. Let $\phi = (\varphi_1,\dots,\varphi_k,0,\dots,0)$. Again by [14, Proposition 3.1], we have $E(\varphi)=V$. ∎

*REFERENCES*

[1] G. Atkinson, *A class of transitive cylinder transformations*, J. London Math. Soc. (2) 17 (1978), 263–270.

[2] A. S. Besicovitch, *A problem on topological transformations of the plane. II*, Proc. Cambridge Philos. Soc. 47 (1951), 38–45.

[3] E. Glasner, M. K. Mentzen and A. Siemaszko, *A natural family of factors for minimal flows*, in: Contemp. Math. 215, Amer. Math. Soc., Providence, RI, 1998, 19–42.

[4] S. Glasner and D. Maon, *Rigidity in topological dynamics*, Ergodic Theory Dynam. Systems 9 (1989), 309–320.

[5] W. H. Gottschalk and G. A. Hedlund, *Topological Dynamics*, Amer. Math. Soc. Colloq. Publ. 36, Amer. Math. Soc., Providence, RI, 1955.

[6] A. B. Krygin, *The $\omega$-limit sets of a cylindrical cascade*, Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), 879-898 (in Russian).

[7] —, *$\omega$-limit sets of smooth cylindrical cascades*, Mat. Zametki 23 (1978), 873–884 (in Russian).

[8] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Pure and Appl. Math., Wiley-Interscience, New York, 1974.

[9] —, —, *Uniform Distribution of Sequences*, Nauka, Moscow, 1985 (in Russian).

[10] P. Le Calvez et J.-Ch. Yoccoz, *Un théorème d'indice pour les homéomorphismes du plan au voisinage d'un point fixe*, Ann. of Math. (2) 146 (1997), 241–293.

[11] M. Lemańczyk and M. K. Mentzen, *Topological ergodicity of real cocycles over minimal rotations*, Monatsh. Math. 134 (2002), 227–246.

[12] P. Liardet and D. Volný, *Sums of continuous and differentiable functions in dynamical systems*, Israel J. Math. 98 (1997), 29–60.

[13] M. K. Mentzen, *On groups of essential values of topological cylinder cocycles over minimal rotations*, Colloq. Math. 95 (2003), 241–253.

[14] —, *Some applications of groups of essential values of cocycles in topological dynamics*, Topol. Methods Nonlinear Anal. 23 (2004), 357–375.

[15] H. Poincaré, *On Curves Defined by Differential Equations*, Ogiz, Moscow 1947 (in Russian).

[16] A. Siemaszko, *Ergodic and topological properties of distal dynamical systems*, thesis, Toruń 1996.

[17] E. A. Sidorov, *A certain class of minimal sets*, Uspekhi Mat. Nauk 28 (1973), no. 4, 225–226 (in Russian).

[18] P. Walters, *An Introduction to Ergodic Theory*, Grad. Texts in Math. 79, Springer, New York, 1982.

Faculty of Mathematics and
Computer Science
Nicolaus Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: mentzen@mat.uni.torun.pl

Faculty of Mathematics and
Information Technology
University of Warmia and Mazury in Olsztyn
Żołnierska 14A
10-561 Olsztyn, Poland
E-mail: artur@uwm.edu.pl

*Received 27 January 2004;*
*revised 2 June 2004* (4420)