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ONE-DIRECTED INDECOMPOSABLE PURE INJECTIVE MODULES OVER STRING ALGEBRAS

BY

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Abstract. We classify one-directed indecomposable pure injective modules over finite-dimensional string algebras.

1. Introduction. Let A be a finite-dimensional string algebra over a field $\Bbbk$ (as an example one may consider the Gelfand–Ponomarev algebra $G_{2,3}$ given by generators α, β and relations $\alpha\beta = \beta\alpha = \alpha^2 = \beta^3 = 0$). A classification of indecomposable finite-dimensional A-modules has been known since Butler and Ringel [2]: they are exactly the so-called string and band modules.

Although the classification of arbitrary infinite-dimensional modules over a string algebra A is hardly possible, some particular classes of such modules are of special interest. For instance, Ringel [13] announced a program to classify indecomposable pure injective modules over string algebras. It is known that over a finite-dimensional algebra pure injective modules may be characterized as direct summands of direct products of finite-dimensional modules (see [4, Ex. 7.10]).

Every indecomposable finite-dimensional module is pure injective, but there are less obvious examples. For every band (see Section 2) C over a string algebra A there are finitely many one-parameter families of "Prüfer" modules and finitely many one-parameter families of "adic" modules. Also there is one "generic" module corresponding to C. We will refer to these modules as *infinite-dimensional band modules*.

Moreover, if v is a one-sided almost periodic string or a two-sided biperiodic string over A, then Ringel [12] associated to v a module $M(v)$ which is, in his terminology, a direct sum, direct product or "mixed" module and which is pure injective and indecomposable.

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Roughly a finite-dimensional algebra is (*tame*) *domestic* if there are finitely many one-parameter families of finite-dimensional A-modules that cover all but finitely many finite-dimensional A-modules in each dimension (see [17, S. 14.4] for a precise definition).

CONJECTURE 1.1 (Ringel's conjecture—see [13, p. 48, p. 51]). *Let A be a finite-dimensional domestic string algebra. Then every infinite-dimensional indecomposable pure injective A-module is either a band module or is of the form $M(v)$, where v is either a one-sided almost periodic string or a two-sided biperiodic string.*

There is a natural construction which assigns to every element m of a pure injective module M over a string algebra A an (infinite) word $w(m)$. We will say that M is *one-directed* if for some $m \in M$, $w(m)$ is a one-sided word. Otherwise M is *two-directed*. For instance, every finite-dimensional string module is one-directed.

In this paper we classify one-directed indecomposable pure injective modules over a string algebra A. We prove in Theorem 5.4 that, if M is an indecomposable pure injective A-module, and $0 \neq m \in M$ is such that $w(m)$ is a one-sided word, then the isomorphism type of M is determined by $w(m)$. Moreover, for every one-sided word w there is an indecomposable pure injective A-module M and $m \in M$ such that $w(m) = w$, and we show that this correspondence is bijective for infinite words (up to inversion of words).

Thus one-directed indecomposable pure injective modules over a string algebra A are classified by one-sided words over A. Using this we show in Corollary 6.1 that over a non-domestic string algebra A there are precisely 2^{ω} non-isomorphic one-directed indecomposable pure injective modules.

However the methods used in the proofs do not give much information about the structure of such modules. For domestic string algebras, using Ringel's results, we are able to give a completely satisfactory description of one-directed indecomposable pure injective modules. Precisely, every such module has the form $M(v)$ from Ringel's list, and $M(v) \cong M(w)$ iff $v = w$ or $v = w^{-1}$.

Given a domestic string algebra A, we calculate the Cantor–Bendixson rank of the open set in the Ziegler spectrum formed by the one-directed indecomposable pure injective modules. We prove that this rank is equal to $n+1$, where n is the length of a maximal path in the bridge quiver of A. Note that conjecturally the Cantor–Bendixson rank of the Ziegler spectrum of a domestic string algebra A is equal to $n+2$ (Schröer's conjecture—see [14, p. 84]); we prove that the rank is at least $n+2$.

The paper consists of two parts. In the first part we show how to analyze the open subset of the Ziegler spectrum given by a chain in the lattice of pp-formulae. This part works for modules over an arbitrary ring. In the second part we apply these results to the family of uniserial functors constructed by Prest and Schröer [8] and combine them with Ringel's results.

Note that the problem of classifying indecomposable pure injective modules over a non-domestic string algebra appears to be extremely difficult. Some examples were collected in Baratella and Prest [1], and we use them in this paper to illustrate results. Recently Puninski [10] proved that every non-domestic string algebra over a countable field has pure injective modules without indecomposable direct summands.

So, it may be instructive to see that a general model-theoretic approach combined with relatively unsophisticated algebraic methods (what Ringel [13, p. 48] refers to as "bare hands") clarifies the situation without exhaustive calculations.

2. String algebras. Almost everywhere in this paper modules will be left modules over a finite-dimensional algebra A. Upper case letters such as C, D and E will always denote finite strings.

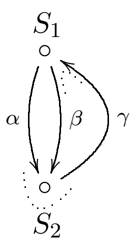
Let $Q = (Q_0, Q_1)$ be a finite quiver (that is, an oriented graph), where Q_0 is a set of vertices, and Q_1 is a set of arrows. Let $A = \Bbbk Q$ be a (possibly infinite-dimensional) $\Bbbk$-algebra with a $\Bbbk$-basis given by the paths in Q and with multiplication given by the composition of paths. For instance, for every vertex $S \in Q_0$ there is the path of length 0 which is an indecomposable idempotent $e_S \in A$. Given an arrow $\alpha \in Q_1$, its starting point will be denoted by $s(\alpha)$ and its end point will be denoted by $e(\alpha)$. Thus $\alpha\beta$ (β then α) is a path in Q if $s(\alpha) = e(\beta)$ (this fits with our convention that we consider left modules).

We impose some monomial relations (i.e., relations of the form $\alpha_1 \ldots \alpha_n = 0$, where α_i are arrows in Q forming a path) on A to make A finite-dimensional. Then A is a *string algebra* (see [2, S. 3]) if the following holds:

1) every vertex is the starting point of at most two arrows and the end point of at most two arrows;
2) if α, β, γ are arrows such that $e(\alpha) = s(\beta) = s(\gamma)$ (i.e., $\beta\alpha$ and $\gamma\alpha$ are paths in Q), then either $\beta\alpha = 0$ or $\gamma\alpha = 0$ is a relation on A;
3) if α, β, γ are arrows such that $s(\alpha) = e(\beta) = e(\gamma)$ (i.e., $\alpha\beta$ and $\alpha\gamma$ are paths in Q), then either $\alpha\beta = 0$ or $\alpha\gamma = 0$ is a relation on A.

For instance,

R_1



with relations $\gamma\alpha = 0$ and $\beta\gamma = 0$ is a string algebra (the relations are indicated by dotted curves).

For every arrow α we introduce a formal inverse α^{-1} with $s(\alpha^{-1}) = e(\alpha)$ and $e(\alpha^{-1}) = s(\alpha)$. A *string* (of length k) over A is a sequence of letters (that is, arrows or inverses of arrows) $C = c_1 \dots c_k$ such that

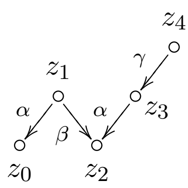
1) $s(c_i) = e(c_{i+1})$ for all $1 \le i \le k-1$;
2) $c_i c_{i+1}$ is neither of the form $\alpha\alpha^{-1}$ nor of the form $\beta^{-1}\beta$ for any arrows α or β, for $1 \le i \le k-1$;
3) $c_{i+1} \dots c_{i+t}$, $1 \le i+1 < i+t \le k$, is neither of the form $\alpha_1 \dots \alpha_t$ nor of the form $\alpha_t^{-1} \dots \alpha_1^{-1}$, where $\alpha_1 \dots \alpha_t = 0$ is any relation on A.

Roughly, a string represents a reduced, non-zero "walk" in Q where arrows may be traversed in either direction.

For instance, $\alpha\beta^{-1}\alpha\gamma$ is a string over R_1 (interpreted as "go along γ then along α, then lift through β and go along α again"). If $C = c_1 \dots c_k$ is a string, then set $e(C) = e(c_1)$ and $s(C) = s(c_k)$, that is, C ends before c_1 and starts after c_k.

Every string $C = c_1 \dots c_k$ over A defines a *string module* $M(C)$ as follows. $M(C)$ is a $(k+1)$-dimensional vector space with basis $z_0, z_1, \dots, z_k$. Informally c_i will be between z_{i-1} and z_i in $M(C)$ and the action of c_i will be to map z_i to z_{i-1} or vice versa. If c_i is a *direct arrow* (that is, an arrow), say $c_i = \alpha$, then put $z_{i-1} = \alpha z_i$. If c_i is an inverse arrow, say $c_i = \beta^{-1}$, then set $\beta z_{i-1} = z_i$. For each such relation $\alpha z_i = z_j$, say $s(\alpha) = k$ and $e(\alpha) = l$, set $e_k z_i = z_i$ and $e_l z_j = z_j$. All the remaining actions of generators of $\Bbbk Q$ on these basis elements z_i are defined to be zero. It is easy to check that $M(C)$ is a left A-module.

In what follows we will draw direct arrows from the upper right to the lower left and inverse arrows from the upper left to the lower right. Thus the string module $M(\alpha\beta^{-1}\alpha\gamma)$ over R_1 has the following diagram:



It is known (see [2]) that any string module is indecomposable, and $M(C) \cong M(D)$ iff either $C = D$ or $C^{-1} = D$.

An infinite sequence of letters $v = c_1 c_2 \dots$ is called a *one-sided string* if $c_1 \dots c_k$ is a string for every k. Similarly we can define a one-sided string $v = \dots c_{-2} c_{-1}$ directed to the left. For instance, ${}^{\infty}(\beta^{-1}\alpha)$, meaning $\dots \beta^{-1}\alpha\beta^{-1}\alpha$, is a one-sided string over R_1, and analogously for $(\beta^{-1}\alpha)^{\infty}$, meaning $\beta^{-1}\alpha\beta^{-1}\alpha \dots$.

For every one-sided string $v = c_1c_2\dots$ we define a *direct sum module* with basis $z_0, z_1, \dots$ such that c_i acts between z_i and z_{i-1} as in a finite-dimensional string module. If we admit arbitrary (not necessarily finite) tuples and use the same action "pointwise", we obtain a *direct product module.*

A one-sided string v is called *almost periodic* if $v = CD^\infty$ or $v = {}^\infty DC$ for finite strings C and D. According to Ringel [12, Prop. 1] every almost periodic string is either "expanding" or "contracting" (depending on whether the last letter of D is direct or inverse); we will not need the definitions of these terms here.

FACT 2.1 ([12, p. 424], [13, p. 50]). *Let v be a one-sided almost periodic string over a string algebra A. If v is expanding then the direct product module*, *which we denote by $M(v)$*, *is pure injective and indecomposable. If v is contracting*, *then the direct sum module*, *denoted $M(v)$*, *is pure injective and indecomposable.*

Note that we use $M(v)$ to denote either the direct sum or direct product module depending on whether v is contracting or expanding (in the above references, $\overline{M}(v)$ is used for the direct product module).

A *band* over A is a string $C = c_1 \dots c_k$ such that:

1) every power C^m is defined;
2) C is not a power of a proper substring;
3) c_1 is a direct arrow and c_k is an inverse arrow.

Thus every band C over A is of the form $\alpha \dots \beta^{-1}$, and clearly $\alpha \neq \beta$. Note that then $C^{-1} = \beta \dots \alpha^{-1}$ is also a band. For instance, over R_1 we have the following bands: $C = \alpha\beta^{-1}$ and $C^{-1} = \beta\alpha^{-1}$.

For the following it will be convenient to use the ε-σ-formalism as in [2, p. 158]. Precisely, we introduce two functions $\varepsilon, \sigma : Q_1 \to \{-1, 1\}$ with the following properties:

1) if $\alpha \neq \beta$ are arrows with $s(\alpha) = s(\beta)$, then $\sigma(\alpha) = -\sigma(\beta)$.
2) If $\alpha \neq \beta$ are arrows with $e(\alpha) = e(\beta)$, then $\varepsilon(\alpha) = -\varepsilon(\beta)$.
3) If α, β are arrows such that $s(\alpha) = e(\beta)$ and $\alpha\beta$ is not a relation in A, then $\sigma(\alpha) = -\varepsilon(\beta)$.

Now extend these functions to finite strings in the following way. Set $\varepsilon(\alpha^{-1}) = \sigma(\alpha)$ and $\sigma(\alpha^{-1}) = \varepsilon(\alpha)$. If $C = c_1 \dots c_k$ is a string, then put $\varepsilon(C) = \varepsilon(c_1)$ and $\sigma(C) = \sigma(c_k)$. It particular, if CD is a string, then $\sigma(C) = -\varepsilon(D)$.

For every vertex S of Q_0 we introduce two strings $1_{S,1}$ and $1_{S,-1}$, where $e(1_{S,t}) = s(1_{S,t}) = S$ and $1_{S,t}^{-1} = 1_{S,-t}$. Set $\sigma(1_{S,t}) = -t$ and $\varepsilon(1_{S,t}) = t$. If C is a string of length ≥ 1, then $C1_{S,t} = C$ if $\sigma(C) = -\varepsilon(1_{S,t}) = -t$, and define

$C1_{S,t}$ to be zero otherwise. Similarly, let $1_{S,t}C = C$ if $\varepsilon(C) = -\sigma(1_{S,t}) = t$, and $1_{S,t}C = 0$ otherwise.

From now on we consider any string algebra to be provided with a fixed pair of such functions ε and σ. For instance, for R_1 we can take $\sigma(\alpha) = -1$, $\sigma(\beta) = \sigma(\gamma) = 1$; and $\varepsilon(\alpha) = \varepsilon(\gamma) = 1$, $\varepsilon(\beta) = -1$.

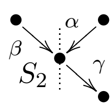
As soon as ε and σ are fixed, we can separate strings going in and out of a particular vertex into two classes. Precisely, if S is a vertex, we define $H_{-1}(S)$ to be the set of strings C such that $s(C) = S$ and $C1_{S,1} = C$, that is, $\sigma(C) = -1$. Similarly, let $H_1(S)$ consist of strings D such that $e(D) = S$ and $1_{S,1}D = D$, that is, $\varepsilon(D) = 1$.

For instance, if $A = R_1$ and $S = S_2$, then $H_{-1}(S)$ consists of $1_{S,1}$ and all strings $C = c_1 \dots c_k$, where $c_k = \beta^{-1}$. Also $H_1(S)$ consists of $1_{S,1}$ and all strings $D = d_1 \dots d_l$, where $d_1 = \alpha$ or $d_1 = \gamma^{-1}$.

Indeed, α^{-1}, β^{-1} and γ is a complete list of (direct and inverse) arrows starting in S_2. We have $\sigma(\alpha^{-1}) = \varepsilon(\alpha) = 1$, hence $\alpha^{-1} \notin H_{-1}(S)$. Also $\sigma(\beta^{-1}) = \varepsilon(\beta) = -1$, hence $\beta^{-1} \in H_{-1}(S)$. Since $\sigma(\gamma) = 1$, it follows that $\gamma \notin H_{-1}(S)$.

Similarly, α, β and γ^{-1} is a complete list of arrows ending in S_2, and $\beta \notin H_1(S)$, but $\alpha, \gamma^{-1} \in H_1(S)$.

In the following picture the strings in $H_{-1}(S_2)$ are drawn to the left of the dotted line, and strings in $H_1(S_2)$ are drawn to the right.



Thus we may consider the strings in $H_{-1}(S)$ as "left handed" and the strings in $H_1(S)$ as "right handed". Note that the sets $H_{-1}(S)$ and $H_1(S)$ are the main ingredients of the hammock order in Schröer [16, S. 3].

Suppose that CD is a string such that $s(C) = e(D) = S$. Then either $C \in H_{-1}(S)$ and $D \in H_1(S)$, or $D^{-1} \in H_{-1}(S)$ and $C^{-1} \in H_1(S)$. Thus we may always assume that CD is oriented such that $C \in H_{-1}(S)$ and $D \in H_1(S)$.

Now we define a (linear) order on the set of right handed strings connected with a vertex S. Let C, D be finite strings in $H_1(S)$. Define $C < D$ if one of the following holds:

1) $D = C\beta D'$, or
2) $C = D\gamma^{-1}C'$, or
3) $C = E\gamma^{-1}C'$ and $D = E\delta D'$ for some strings C', D', E and arrows β, γ, δ.

Clearly $<$ is a linear order. Also every string $C \in H_1(S)$ (except the maximal one) has an immediate successor C^+ with respect to this order. For instance, if $C\beta$ is a string for some arrow β, then $C^+ = C\beta\gamma^{-1}\dots$ (as many inverse arrows as possible).

For example, if $A = R_1$ and $S = S_2$, then $\alpha\beta^{-1}, \alpha \in H_1(S)$ and $\alpha\beta^{-1} < \alpha$. Also, if $D = \alpha\beta^{-1}$, then $D^+ = \alpha\beta^{-1}\alpha\beta^{-1}\gamma^{-1}\alpha^{-1}$.

This order obviously can be extended to infinite strings v such that $e(v) = S$ and $1_{S,1}v = v$; we will consider these strings as "right handed". Then v defines a *cut* on the set of finite strings in $H_1(S)$: the "lower part" of this cut is $\{C \mid C < v\}$ and the "upper part" is $\{D \mid D > v\}$.

Similarly we may define the "left order" $<'$ on the set of strings from $H_{-1}(S)$. Precisely, if $C, D \in H_{-1}(S)$, then C^{-1} and D^{-1} are "right handed" strings such that $C^{-1} = 1_{S,-1}C^{-1}$ and $D^{-1} = 1_{S,-1}D^{-1}$. We may compare C^{-1} and D^{-1} in a right order $<$ as above. Then set $C <' D$ if $C^{-1} < D^{-1}$. The immediate successor of C with respect to this order will be denoted by ${}^+C$.

Note that $<$ and $<'$ are defined separately for each particular vertex S of the quiver of A.

3. Some model theory. We recall some basic notions from the model theory of modules. For more on this the reader is referred to [5]. A more algebraic approach to the theory of pure injective (= algebraically compact) modules can be found in [4, Ch. 7].

A *pp-formula* $\varphi(x)$ (in one free variable x) is a formula of the form $\exists \bar{y}$ $(B\bar{y}^t = \bar{b}^t x)$, where $\bar{y}^t$ is the column transpose of the row $\bar{y} = (y_1, \dots, y_n)$, B is a $k \times n$ matrix over A and $\bar{b}^t$ is a column over A with k rows. This pp-formula is interpreted as "B divides $\bar{b}^t x$". For instance, a *divisibility formula* is a pp-formula of the form $\exists y\ (ry = x)$ where $r \in A$; we write $r \mid x$ for short.

Let φ be a pp-formula as above and let m be an element of a module M. We say that φ is *satisfied* by m in M, written $M \models \varphi(m)$, if there is a tuple $\overline{m} = (m_1, \dots, m_n) \in M^n$ such that $B\overline{m}^t = \bar{b}^t m$. Then $\varphi(M) = \{m \in M \mid M \models \varphi(m)\}$ is a *pp-definable subgroup* of M. Note that $\varphi(M)$ is a (right) S-submodule of M, where $S = \mathrm{End}(M)$. For instance, for a divisibility formula we have $(r \mid x)(M) = rM$.

Let φ and ψ be pp-formulae. We write $\psi \to \varphi$ (ψ implies φ) if $\psi(M) \subseteq \varphi(M)$ for every module M. The implication relation is reflexive and transitive, therefore defines a (quasi-) order on the set of all pp-formulae. Thus we will often write $\psi \leq \varphi$ instead of $\psi \to \varphi$. We say that pp-formulae φ and ψ are *equivalent* if $\psi \leq \varphi \leq \psi$, that is, $\psi(M) = \varphi(M)$ for every module M.

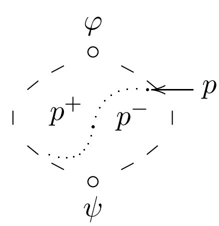
Factorizing the set of all pp-formulae by the equivalence relation, we obtain a partial order $L(A)$. In fact $L(A)$ is a modular lattice, where the meet operation $\wedge$ ("and") is conjunction of pp-formulae and the join operation

$+$ is given by the rule $(\varphi + \psi)(x) = \exists y\ (\varphi(y) \wedge \psi(x - y))$. If $\psi < \varphi$ are pp-formulae then (φ/ψ) will, in this paper, denote the interval $[\psi; \varphi]$ in $L(A)$.

A *pp-type* $p(x)$ is a collection of pp-formulae which is closed with respect to implication and (finite) conjunction. For instance, if m is an element of a module M, then the set of all pp-formulae satisfied by m in M is a pp-type, denoted $\mathrm{pp}_M(m)$. By [5, Ch. 4] for every pp-type p there is a "minimal" pure injective module $M = N(p)$ containing an element $m \in M$ such that $p = \mathrm{pp}_M(m)$. This module is unique (up to isomorphism over m) and will be called a *pure injective envelope* of p.

We say that a pp-type p is *indecomposable* if $N(p)$ is an indecomposable module. The *positive part*, p^+, of a pp-type p consists of all pp-formulae $\varphi \in p$ (i.e., $p^+ = p$) and its *negative part* p^- consists of those pp-formulae ψ with $\psi \notin p$.

We say that an interval (φ/ψ) is *open in a pp-type* p, written $p \in (\varphi/\psi)$, if $\varphi \in p^+$ and $\psi \in p^-$. In this case p defines a cut on (φ/ψ), whose "upper" part consists of pp-formulae in p^+ (and below φ) and whose "lower" part consists of pp-formulae in p^- (and above ψ):



The following result says that the pure injective envelope of an indecomposable pp-type p is uniquely determined by any (local) cut of p.

FACT 3.1. *Let $\psi < \varphi$ be pp-formulae and let $p, q \in (\varphi/\psi)$ be indecomposable pp-types which define the same cut on the interval (φ/ψ). Then $N(p) \cong N(q)$.*

Proof. We have $\varphi \in p, q$ and $\psi \in p^-, q^-$. If $N(p)$ and $N(q)$ were non-isomorphic then, by a result of Ziegler ([18], see [5, Lemma 9.2]), there would exist a pp-formula θ such that $\psi < \theta < \varphi$ and either $\theta \in p \setminus q$ or $\theta \in q \setminus p$. Thus p and q would define different cuts on the interval (φ/ψ), a contradiction. ∎

In general not every cut on an interval (φ/ψ) leads to an indecomposable pp-type. But this is the case if (φ/ψ) is a chain.

LEMMA 3.2. *Given any cut on a chain (φ/ψ) there is an indecomposable pp-type q which defines this cut on (φ/ψ). Moreover the (indecomposable* *pure injective*) *module* $N(q)$ *is uniquely* (*up to isomorphism*) *determined by the original cut.*

Proof. Since (φ/ψ) is a chain, the upper part of the cut, denote it p^+, is closed with respect to conjunctions, and the lower part of the cut, p^-, is closed with respect to sums. Also the set $p^+ \cup \neg p^-$ of formulas is consistent.

Let us extend $p^+ \cup \neg p^-$ to a maximal pp-type q (that is, such that $q^+ \supseteq p^+$ and is maximal with respect to $q^+ \cap p^- = \emptyset$). From [5, Thm. 4.33] it follows that q is indecomposable. By the construction, q defines the original cut on the chain (φ/ψ). Suppose that q' is another indecomposable pp-type that defines the same cut on (φ/ψ). Then $N(q) \cong N(q')$ by Fact 3.1. ■

We say that an indecomposable pure injective module M *opens an interval* (φ/ψ), written $M \in (\varphi/\psi)$, if there is $m \in M$ such that $m \in \varphi(M) \setminus \psi(M)$, that is, $p \in (\varphi/\psi)$, where $p = \mathrm{pp}_M(m)$.

Thus we obtain the following "rough" classification of indecomposable pure injective modules living on the chain.

THEOREM 3.3. *Let* (φ/ψ) *be a chain in the lattice of all pp-formulae over* A. *Then there is a natural surjection from the set of cuts on* (φ/ψ) *to the set of* (*isomorphism types of*) *indecomposable pure injective* A*-modules opening this interval.*

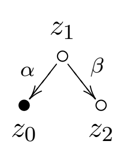
Proof. This follows from Lemma 3.2. ■

In general this map is not monic: different cuts may lead to isomorphic indecomposable pure injective modules.

4. Preliminary results. Let S be a vertex of a quiver of a string algebra A, and let $D = d_1 \dots d_l$ be a finite string over A from $H_1(S)$, in particular $e(D) = S$. Thus $D = 1_{S,1}$, or $d_1 = \alpha$, where α is a direct arrow such that $e(\alpha) = S$ and $\varepsilon(\alpha) = 1$; or $d_1 = \beta^{-1}$, where β^{-1} is an inverse arrow such that $s(\beta) = S$ (hence $e(\beta^{-1}) = S$) and $\sigma(\beta) = 1$ (hence $\varepsilon(\beta^{-1}) = \sigma(\beta) = 1$).

Now we define "right handed" pp-formulae $(.D)$. Given $D \in H_1(S)$, let CD be a maximal string such that $C \in H_{-1}(S)$ consists of direct arrows. Let $M = M(CD)$ be the corresponding string module with a canonical basis $z_0, \dots, z_k$ such that z_i goes before D. Define $(.D)$ to be a pp-formula equivalent to the pp-type of z_i in M.

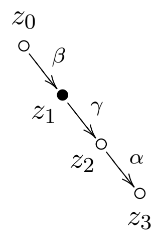
For instance, let $A = R_1$, $S = S_2$ and $D = \alpha\beta^{-1}$. Then we cannot add a direct arrow before D, hence $(.D)$ generates the pp-type of z_0 in $M(D)$:



Thus $(.D)$ is the pp-formula $\exists z_1, z_2\ (x = \alpha z_1 \wedge e_1 z_1 = z_1 \wedge \beta z_1 = z_2 \wedge \gamma z_2 = 0)$, where e_1 is a basic idempotent corresponding to the vertex S_1.

Now we define "left handed" pp-formulae $(C.)$. Given $C \in H_{-1}(S)$, let CD be a maximal string such that $D \in H_1(S)$ consists of inverse arrows. Let $M = M(CD)$ be the corresponding string module with a canonical basis $z_0, \dots, z_k$ such that z_i goes after C. Define $(C.)$ to be a pp-formula equivalent to the pp-type of z_i in M.

For instance, if $A = R_1$, $S = S_2$ and $C = \beta^{-1} \in H_{-1}(S)$, then $D = \gamma^{-1}\alpha^{-1}$, hence $M = M(CD)$ is the following module:



Thus $(C.)$ is the pp-formula

$$\exists z_0, z_2, z_3\ (e_1 z_0 = z_0 \wedge \alpha z_0 = 0 \wedge \beta z_0 = x \wedge \gamma x = z_2 \wedge \alpha z_2 = z_3),$$

which is equivalent to $\exists z_0\ (\alpha z_0 = 0 \wedge \beta z_0 = x)$.

If $C \in H_{-1}(S)$ and $D \in H_1(S)$, then $(C.D)$ will denote the conjunction of $(C.)$ and $(.D)$. In most cases where the previous definition applies, CD is a string. But, since formulae $(C.)$ and $(.D)$ are defined separately, this definition makes sense even if (CD) is not a string.

We say that a pair (M, m) is a *free realization* of a pp-formula $\varphi(x)$ if M is a finitely presented (= finite-dimensional in the context of modules over finite-dimensional algebras) module, $M \models \varphi(m)$, and $\varphi \to \psi$ for every pp-formula ψ such that $M \models \psi(m)$. In particular $\mathrm{pp}_M(m)$ is generated as a pp-type by φ. By [5, Ch. 8] every pp-formula has a free realization. For instance, the pair (A, r) is a free realization of the formula $r \mid x$.

The following example of a free realization will be of special importance. Let $M = M(CD)$ be a string module over a string algebra A (we allow $C = 1_{S,1}$ or $D = 1_{S,1}$) and let z be the element of a canonical basis of M lying between C and D (in the sense of the construction of string modules).

Remark 4.1. *Let $M = M(CD)$ be a string module and let z be an element of the canonical basis of M between C and D. Then (M, z) is a free realization of $(C.D)$.*

Proof. By definition $(C.D) \in p = \mathrm{pp}_M(z)$. Let $\psi \in \mathrm{pp}_M(z)$; we need to prove that $\varphi \to \psi$, that is, $\varphi(N) \subseteq \psi(N)$ for every module N. Let $N \models \varphi(n)$ for some $n \in N$. From the description of $(C.D)$ it is easy to construct a morphism $f : M \to N$ such that $f(z) = n$. From $M \models \psi(z)$ we obtain $N \models \psi(n)$, as desired. ∎

If M is a module and D is a string then $(.D)(M)$ is a pp-subgroup of M. It is quite straightforward to check that, if $E, F \in H_1(S)$ are finite strings such that $E \leq F$, then $(.F) \to (.E)$. Similarly, if $C, D \in H_{-1}(S)$ are finite strings such that $C \leq' D$, then $(D.) \to (C.)$.

The following lemma says that $(.D)$ defines a homogeneous subspace in every direct sum or direct product module (we allow the sum below to be infinite in a direct product module).

Lemma 4.2. *Let $M = M(v)$ be either the direct sum or direct product module corresponding to a one-sided (or two-sided) string v and let $D \in H_1(S)$ be a finite string. Then $(.D)(M)$ is a homogeneous subspace of M, that is, $\sum_i \lambda_i z_i \in (.D)(M)$ if and only if $z_i \in (.D)(M)$ for every i such that $\lambda_i \neq 0$ (where z_i are elements of a canonical basis of $M(v)$).*

Proof. Similar to [1, Lemma 3.4]. ∎

The following (almost obvious) lemma will be useful in what follows.

Lemma 4.3. *Let (M, m) be a free realization of a pp-formula φ. Suppose that $m = n + k$, that (M, n) is a free realization of φ_1, and that $M \models \varphi_2(k)$. If ψ is a pp-formula such that $\varphi_2 \to \psi$, then $\varphi + \psi$ is equivalent to $\varphi_1 + \psi$.*

Proof. Since (M, m) is a free realization of φ, and $M \models (\varphi_1 + \varphi_2)(m)$, we obtain $\varphi \to \varphi_1 + \varphi_2 \to \varphi_1 + \psi$. Therefore $\varphi + \psi \to \varphi_1 + \psi$.

So it remains to prove that $\varphi_1 + \psi \to \varphi + \psi$. Since $(M, n = m - k)$ is a free realization of φ_1, and $M \models (\varphi + \varphi_2)(n)$, we conclude that $\varphi_1 \to \varphi + \varphi_2$. Then $\varphi_1 + \psi \to \varphi + \varphi_2 + \psi = \varphi + \psi$. ∎

The next result is a key one in what follows.

Lemma 4.4. *Let CD be a string over a string algebra A. Then every formula in the interval $(C.D)/(^+C.D)$, apart from $(^+C.D)$, is equivalent to a formula $(^+C.D) + (C.D_i)$ for some $D_i \geq D$ such that CD_i is a string. In particular the interval $(C.D)/(^+C.D)$ is a chain.*

Proof. All this follows from [8, Thm. 3.2]. We just add some explanations.

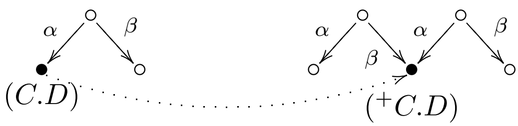
It is clear that the formulae $(C.D_i)$ with $D_i \geq D$ are linearly ordered, therefore the same is true for the formulae $(^+C.D) + (C.D_i)$. Thus it suffices to prove that every pp-formula strictly between $(C.D)$ and $(^+C.D)$ is of the required form. Note that such a formula can be obtained in the following way: take any pp-formula φ below $(C.D)$ and add $(^+C.D)$.

Let z be the element of a canonical basis of $M = M(CD)$ between C and D. Let (N, m) be a free realization of φ. Since $(C.D) \geq \varphi$, there is a morphism $f : M \to N$ taking z to m. Since any sum of pp-formulas of the form $({}^{+}C.D) + (C.D_i)$ is equivalent to a single one of them we may assume that N is indecomposable, therefore is either a string or a band module.

If N is a band module, then from the proof of Theorem 3.2 in [8] it follows that $\varphi \to ({}^{+}C.D)$, therefore φ is taken to $({}^{+}C.D)$ by summation.

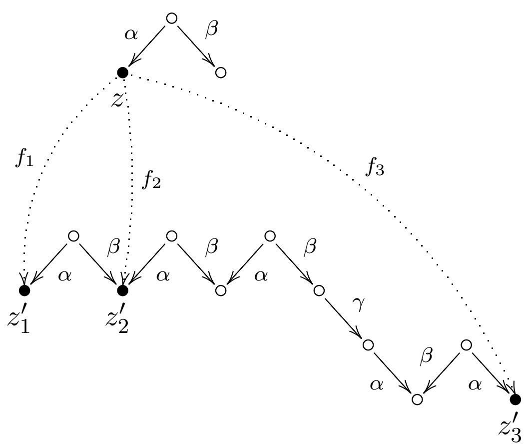
Otherwise N is a string module, therefore, by Crawley-Boevey [3], $f = \sum_i \lambda_i f_i$, $\lambda_i \in \Bbbk$, is a linear combination of graph maps $f_i : M \to N = M_i = M(C_i D_i)$, $i = 1, \ldots, n$, where $f_i(z) = z'_i$ with z'_i lying between C_i and D_i. We will assume that each z'_i is located in N to the left of z'_{i+1} (so z'_1 is utmost left, and z'_n is utmost right).

To understand the situation better let us look at the following example of pp-formulas over R_1, where $S = S_2$ and $(C.D)$ is the formula $\exists z_1\ (x = \alpha z_1 \wedge e_1 z_1 = z_1 \wedge \gamma\beta z_1 = 0)$ (that is stronger than $\alpha \mid x$):



Thus $C = 1_{S,1}$ in this case, and $({}^{+}C.)$ is $\beta \mid x$.

Let $f_i : M \to N$, $i = 1, 2, 3$, extend the map $z \mapsto z'_i$ as the following diagram shows:



Let $f = f_1 + f_2 + f_3 : M \to N$, in particular $m = f(z) = z'_1 + z'_2 + z'_3$.

Note that f_1, f_2 preserve the orientation of M, but f_3 flips it over. By Remark 4.1, $(C_i.D_i)$ generates the pp-type p_i of z'_i in N. We show how to "eliminate" z'_2 and z'_3 from consideration, that is, we prove that $({}^{+}C.D) + \varphi$ is equivalent to $({}^{+}C.D) + (C_1.D_1)$.

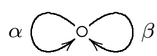
Clearly there is an endomorphism h of N which sends z'_1 to z'_3 such that h is in the Jacobson radical of $\mathrm{End}(N)$. Since $1 + h$ is an automorphism of N, the pp-type of $(1+h)(z_1') = z_1' + z_3' = n$ in N is $p_1 = pp(z_1')$, hence is generated by $\varphi_1 = (C_1.D_1)$.

Note that p_2 includes $\beta \mid x$, hence $\varphi_2 = (C_2.D_2) \to \psi = ({}^+C.D)$. If $k = z_2'$, then $m = n + k$. By Lemma 4.3, $\psi + \varphi$ (that is, $({}^+C.D) + \varphi$) is equivalent to $\psi + \varphi_1 = \psi + (C_1.D_1)$, that is, to $({}^+C.D) + (C_1.D_1)$.

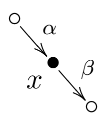
Now we consider the general case. We may assume that z_1' goes after C at the left end of N. Indeed, otherwise the pp-type of z_1' contains the formula $({}^+C.D)$, hence f_1 can be eliminated from the sum $\sum_i \lambda_i f_i$. Similarly we may assume that the right end of N after z_n' is C^{-1}. Then the pp-type p_1 of $\lambda_1 z_1'$ in N and the pp-type p_n of $\lambda_n z_n'$ in N are comparable. Since $E \neq E^{-1}$ for every finite string E, p_1 and p_n do not coincide. By symmetry we may assume that $p_1 \subset p_n$.

As above, using Lemma 4.3 (i.e., dropping the elements $z_2', \ldots, z_n'$), we conclude that $({}^+C.D) + \varphi$ is equivalent to $({}^+C.D) + (C_1.D_1)$, that is, to $({}^+C.D) + (C.D_1)$. ■

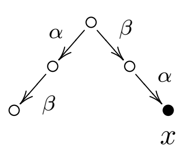
As we have already mentioned, the formula $({}^+C.D)$ makes sense even if ${}^+CD$ is not a string. For example, let A be the following string algebra:



with relations $\alpha^2 = \beta^2 = \alpha\beta\alpha = \beta\alpha\beta = 0$. Let $C = \alpha^{-1}$ and $D = \beta^{-1}$, hence $M = M(CD)$ is the following module:



Now ${}^+C = \beta\alpha\beta^{-1}\alpha^{-1}$, hence $({}^+C.D)$ is equivalent to the pp-type of x in the module $M(\beta\alpha\beta^{-1}\alpha^{-1})$:



Thus, adding $({}^+C.)$ we impose a new relation $\beta x = 0$ on x.

5. One-directed pure injective modules. Let M be an indecomposable pure injective A-module, and let $m \in e_S M$, where e_S is the basic idempotent corresponding to some vertex S. As we have mentioned before, we can separate strings going in and out of the vertex S into two classes, such that the notions of a "right hand" string and a "left hand" string make sense. If $D \in H_1(S)$ is a (finite) right hand string, then $m \in DM$ means $m \in (.D)(M)$. We will consider an infinite string $v = v_1 v_2 \dots$ to be right handed if every finite string $v \,|\, n = v_1 \dots v_n$ is in $H_1(S)$. The notion $m \in vM$ is defined as for finite strings (or see [11]).

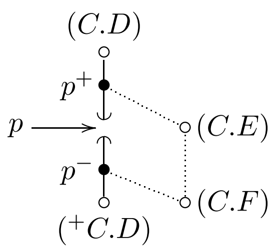
For every n let u_n be a maximal (with respect to $<$) string of length $\leq n$ such that $m \in u_n M$. Then (see [11, p. 29]) there is a (usually infinite) string u such that $u \,|\, n = u_n$ for every n. (Since M is pure injective we even may assume that $m \in uM$, but we do not use this fact in what follows.) Similarly m determines a left hand (infinite) string v. Then $w(m) = vu$ is a (two-sided) string constructed using m.

Definition 5.1. An indecomposable pure injective module M is said to be *one-directed* if M opens an interval $(C.D)/(^+C.D)$ for some string CD (then say that M ends with C on the left) or M opens an interval $(C.D)/(C.D^+)$ (then say that M ends with D on the right). Otherwise we say that M is *two-directed*.

Clearly this is the same as to say that there exists $m \in M$ such that the string $w(m)$ is one-sided.

For instance, every finite-dimensional string module $M(CD)$ opens both pairs $(C.D)/(^+C.D)$ and $(C.D)/(C.D^+)$, hence $M(CD)$ is one-directed. Also, if v is a one-directed almost periodic string, then (the direct sum or direct product) module $M(v)$ from Ringel's list is one-directed.

Let M be a one-directed indecomposable pure injective module and, with notations as in the definition, choose $m \in (C.D)(M) \setminus (^+C.D)(M)$ (and such that $m \in e_S M$ for a basic idempotent that corresponds to the vertex S between C and D). Then the pp-type $p = \mathrm{pp}_M(m)$ defines a cut, by intersection with p^+ and p^-, in the chain $(C.D)/(^+C.D)$:



Moreover, by Fact 3.1, M is determined up to isomorphism by this cut. However this cut (therefore this pp-type) may be "non-homogeneous". For instance, it is (at least conjecturally) possible to have $(C.E) \in p^-$ but $(^+C.D) + (C.E) \in p^+$. To avoid this possibility we will improve p slightly. We say that a pp-type $p \in (C.D)/(^+C.D)$ is *homogeneous* (with respect to this chain) if $(^+C.D) + (C.E) \in p^+$ implies $(C.E) \in p^+$ for every $E \geq D$ such that CE is a string.

LEMMA 5.2. *Let $p \in (C.D)\backslash({}^{+}C.D)$ be an indecomposable pp-type. Then there is a homogeneous pp-type q such that $N(p) \cong N(q)$.*

Proof. First include in q^+ all pp-formulae $(C.E)$ such that $({}^{+}C.D) + (C.E) \in p^+$. Since C is fixed, these formulae in q^+ form a chain. Now include in q^- all pp-formulae $({}^{+}C.D) + (C.F) \in p^-$. These also form a chain.

We prove that $q^+ \cup \neg q^-$ is consistent. Indeed, otherwise we deduce that $(C.E) \to ({}^{+}C.D)+(C.F)$ for some E and F such that $({}^{+}C.D)+(C.E) \in p^+$ and $({}^{+}C.D) + (C.F) \in p^-$. Note that $(.E) > (.F)$. If $M = M(CE)$ and z is between C and E in the canonical basis of M, then, by Remark 4.1, (M, z) is a free realization of $(C.E)$.

Since $z \in (C.E)(M)$ and $(C.E) \to ({}^{+}C.D) + (C.F)$, it follows that $z \in ({}^{+}C.D)(M) + (C.F)(M)$. By Lemma 4.2 we deduce that either $z \in ({}^{+}C.D)(M)$ or $z \in (C.F)(M)$, therefore either $(C.E) \to ({}^{+}C.D)$ or $(C.E) \to (C.F)$.

If $(C.E) \to ({}^{+}C.D)$, then $({}^{+}C.D) = ({}^{+}C.D) + (C.E) \in p^+$, a contradiction. Also $(C.E) \to (C.F)$ implies $({}^{+}C.D) + (C.F) \in p^+$, a contradiction again.

Thus $q^+ \cup \neg q^-$ is consistent. Now we extend this type to a maximal pp-type containing q^+ and omitting q^-. The result (denote it also by q) will be indecomposable by [5, Thm. 4.33] and $N(p) \cong N(q)$ by Fact 3.1 and Lemma 4.4. ∎

Recall that the Ziegler spectrum Zg_A of A is a topological space whose points are isomorphism types of indecomposable pure injective A-modules (e.g. see [7]). The topology on Zg_A is given by basic open sets $(\varphi/\psi) = \{M \in \mathrm{Zg}_A \mid \psi(M) < \varphi(M)\}$, where $\psi < \varphi$ are pp-formulae. It is known that Zg_A is quasi-compact.

LEMMA 5.3. *Let q be a homogeneous pp-type as in Lemma* 5.2. *Then the pairs $(C.E)/(({}^{+}C.D) + (C.F))$, where $D \leq E < F$ are such that CF is a string, $(C.E) \in q^+$ and $({}^{+}C.D) + (C.F) \in q^-$, form a neighborhood basis of open sets for $N(q)$.*

Proof. Since p opens the interval $(C.D)/({}^{+}C.D)$, by Ziegler [18, Thm. 4.9], a neighborhood basis of $N(q)$ can be taken to be those pairs (φ/ψ) such that $({}^{+}C.D) \leq \psi < \varphi \leq (C.D)$. It remains to apply Lemma 4.4 and homogeneity of q. ∎

Now we are in a position to prove the main theorem of the paper.

THEOREM 5.4. *Let A be a finite-dimensional string algebra. Then there is a natural one-to-one correspondence between the set of pairs $\{v, v^{-1}\}$ of one-sided strings over A and the set of isomorphism types of one-directed indecomposable pure injective A-modules.*

Proof. Let M be a one-directed indecomposable pure injective A-module. First we assign to M a one-sided string $w = w(M)$.

Since M is one-directed, M opens a pair, say $(C.D)/(^+C.D)$, on a (nonzero) $m \in M$ (such that $m \in e_S M$ for some basic idempotent e_S).

Shifting along C we may further assume that $C = 1_{S,1}$. Thus M opens the interval $((.D)/(^+.D))$, and this interval is a chain by Lemma 4.4.

Moreover, by Lemmas 5.2 and 3.2, we may assume that m is such that $p = \mathrm{pp}_M(m)$ is a homogeneous pp-type, so $M = N(p)$. Then the isomorphism type of M is determined by the cut of p on the above interval (see Lemma 3.2). If E is a string, then $(.E) + (^+.D) \in p^+ \cap (.D)/(^+.D)$ iff E is an initial part of the one-sided string w determined by m (as before Definition 5.1). Thus the cut and the string determine each other and we assign this string to M.

Conversely, let w be a one-sided infinite (right handed) string. Take any finite string $D \in H_1(S)$ such that $D \leq w$. Then the interval $(.D)/(^+.D)$ is a chain and w defines a cut on it as above. By Lemmas 5.2 and 3.2, there is an indecomposable (homogeneous) pp-type p such that p defines on this interval the same cut as w.

Then we assign to w the (one-directed) indecomposable pure injective module $N(p)$. Since $N(p)$ is determined by w, we may use the notation $N(w)$.

It remains to prove that for different one-sided strings $v \neq w$, both infinite to the right, the corresponding modules $M = N(v)$ and $N = N(w)$ are not isomorphic. Assume first that v and w start at the same vertex (so we may compare v and w with respect to the ordering $<$ on strings).

Looking for a contradiction, we may assume that $v < w$ and $M \cong N$. By Lemma 5.3, a basis for N in Zg_A can be chosen to consist of pairs of the form $(.G)/((.H) + (^+.))$, where $G \leq w < H$ are finite strings. Choose G, H such that $G \,|\, n = H \,|\, n$ for some n large enough that the initial segments of v and w of length n are different. In particular, $M \cong N \in (.G)/((.H) + (^+.))$.

Similarly, a basis of open sets for M can be chosen to consist of pairs of the form $(.E)/((.F) + (^+.))$, where $E \leq v < F$ are finite strings. We have two neighborhood bases of M so we may choose E, F such that $(.E)/((.F)+ (^+.)) \subseteq (.G)/((.H) + (^+.))$. We prove that this leads to a contradiction.

Indeed, let v_k be an initial part of v of length k. If k is large enough, $E \leq v_k < F$, and also $v_k < G$ by the choice of G and H. Let $M_k = M(v_k)$ be the corresponding indecomposable string module with the basis $z_0, \dots, z_k$. Clearly $M_k \in (.E)/((.F) + (^+.))$, where z_0 realizes the corresponding pp-type. By choice of E and F, $M_k \in (.G)/((.H) + (^+.))$, therefore there is $z \in M_k$ which opens this pair.

By homogeneity (see Lemma 4.2) we may assume that z is one of the basis elements z_i. Since $(^+.)$ is in the pp-type of z_i for $1 \leq i \leq k-1$, we conclude that $z = z_0$ or $z = z_k$. From $v_k < G$ it follows that the pp-type of z_0 does not contain $(.G)$, hence we must have $z = z_k$.

Thus for every (large enough) k, the pp-type of z_k in M_k would open the pair $(.G)/(.H)$, in particular the n-initial part of the string defined from z_k in M_k would be equal to $w \mid n$, which is clearly not possible (as k varies).

Now assume that v and w start at different vertices, but $N(v)$ and $N(w)$ are isomorphic. We show that this leads to a contradiction.

As above, choose a basis for $N(v)$ in Zg_A consisting of pairs $(.E)/(.F) + ({}^{+}.)$, where $E \le v < F$ are strings ending in S. Similarly $N(w)$ has a basis in Zg_A consisting of pairs $(.G)/(.H) + ({}^{+}.)$, where $G \le w < H$ are strings ending at a vertex S', $S' \ne S$.

Since $N(v) \cong N(w)$, we obtain $(.E)/(.F) + ({}^{+}.) \subseteq (.G)/(.H) + ({}^{+}.)$ for some E, F, G and H as above. Let v_k be an initial part of v of length k, and let $M_k = M(v_k)$ have a canonical basis $z_0, \dots, z_k$. If k is large enough, then $M_k \in (.E)/(.F) + ({}^{+}.)$, hence $M_k \in (.G)/(.H) + ({}^{+}.)$.

By homogeneity, some element z_i of the canonical basis of M_k should satisfy $(.G)$ but not $({}^{+}.)$. Since z_i does not satisfy $({}^{+}.)$, we must have $i = 0$ or $i = k$, that is, z_i is either the first or the last element of the basis of M_k. Since z_i satisfies $(.G)$, we conclude that $e_{S'} z_i = z_i$, hence z_i must be the last element of the basis. Thus we have $e_{S'} z_k = z_k$ for every large enough k, a contradiction. ∎

COROLLARY 5.5. *Let M, N be one-directed indecomposable pure injective modules over a string algebra A. If M and N are topologically indistinguishable in Zg_A, then they are isomorphic.*

6. Applications. Given a string algebra A, $n_d(A)$ will denote the number of one-parameter families required to cover all but finitely many indecomposable A-modules of dimension d. We say that A is *domestic* if there is N such that for every d, $n_d(A) \le N$.

COROLLARY 6.1. *Let A be a non-domestic string algebra. Then there are 2^{ω} one-directed indecomposable pure injective modules over A.*

Proof. Since A is non-domestic, by Ringel [12, Prop. 2], A has 2^{ω} one-sided (non-periodic) strings. So we can apply Theorem 5.4. ∎

If the field $\Bbbk$ is countable, the existence of 2^{ω} points in the Ziegler spectrum of A was already known and can be proved as follows. By [16, Prop. 2] there is a dense chain of pp-formulae over A. Since A is countable, we can apply [6, p. 450].

We have defined a one-directed pure injective module M to be a module with an element m such that the string $w(m)$ is one-sided. A negative answer to the following question would allow us to separate one-directed and two-directed pure injective modules completely.

QUESTION 6.2. *Let M be a one-directed indecomposable pure injective module. Is it possible to have $m \in M$ such that the string $w(m)$ is two-sided?*

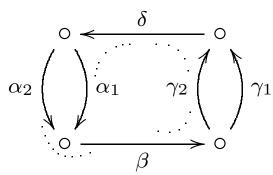
It will follow from what we show below that the answer to this question is negative for domestic string algebras.

The following characterization of domestic string algebras is contained in [12, Prop. 8.2].

FACT 6.3. *A string algebra A is domestic if and only if every one-sided string over A is almost periodic.*

For instance, the following string algebra

R_2 :



(all zero-relations have length 2 and are shown by dotted curves) is domestic. Indeed up to inversion every (one-sided or two-sided) string v over R_2 is a substring of either

$$^{\infty}(\alpha_1^{-1}\alpha_2)\delta\gamma_1(\gamma_2^{-1}\gamma_1)^n\beta(\alpha_1\alpha_2^{-1})^{\infty}, \qquad n \in \mathbb{Z},$$

or

$$^{\infty}(\alpha_1^{-1}\alpha_2)\delta\gamma_1(\gamma_2^{-1}\gamma_1)^{\infty}, \quad \text{or} \quad ^{\infty}(\gamma_2^{-1}\gamma_1)\beta(\alpha_1\alpha_2^{-1})^{\infty},$$

hence v is almost periodic.

THEOREM 6.4. *Let M be a one-directed indecomposable pure injective module over a domestic string algebra A. Then M is isomorphic to a module $M(v)$ from Ringel's list, where v is a one-sided string. Moreover, $M(v) \cong M(w)$ if and only if $v = w$ or $v = w^{-1}$.*

Proof. As in the proof of Theorem 5.4, we may assume that $0 \neq m \in M$ such that $u(m)$ (the left handed string determined by m) is $1_{S,1}$, hence $v = w(m)$ is a right handed string. Moreover, we may further assume that $M = N(v) = N(p)$, where p is a pp-type homogeneous in the interval $(.D)/(^+.D)$ for some finite string $D \leq v$.

Since A is domestic, Fact 6.3 implies that v is almost periodic. Let $M = M(v)$ be the direct sum or direct product module determined by v. Let z_0 be the first element of a standard basis of M, and let $q = \mathrm{pp}_M(z_0)$. By Lemma 4.2, q is homogeneous with respect to $(.D)/(^+.D)$.

Calculating in $M(v)$ we see that q and p coincide on formulas $(.E)$, $E \geq D$ (since realizations of q and p define the same string v). Since p and q

are homogeneous, they define the same cut on the chain $(.D)/(^{+}.D)$. Then $N(v) \cong M(v)$ by Fact 3.1.

Also, by Theorem 5.4, $N(v) \cong N(v')$ iff $v = v'$ or $v^{-1} = v'$. ■

QUESTION 6.5. *What is the algebraic structure of one-directed indecomposable pure injective modules over a non-domestic string algebra?*

To highlight that some new effects may occur in a non-domestic case, let us consider some examples.

EXAMPLE 6.6 ([1]). *Let $A = G_{2,2}$ be the Gelfand–Ponomarev algebra, that is, a $\Bbbk$-algebra with generators α, β and relations $\alpha^2 = \beta^2 = \alpha\beta = \beta\alpha = 0$. Suppose that the characteristic of $\Bbbk$ is not equal to 2, and let M be the following direct sum module:*

$$\begin{tikzcd}
 & z_1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & & z_3 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & & z_5 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & & \\
z_0 & & z_2 & & z_4 & & z_6 & \cdots
\end{tikzcd}$$

Let $p = \mathrm{pp}_M(z_0)$. If $D = \alpha\beta^{-1}$, then p is homogeneous in $(.D)/(^{+}.D)$ but decomposable. Moreover the embedding of M into the corresponding direct product module is not pure.

Proof. The pp-type p is homogeneous by Lemma 4.2. Also we have $z_0 = \frac{1}{2}\cdot(z_0+z_2)+\frac{1}{2}\cdot(z_0-z_2)$. But clearly $(\alpha\pm\beta) \mid x \in \mathrm{pp}_M(z_0\pm z_2)$. Calculating strings on $z_0 \pm z_2$ as in [1, p. 26] we obtain $p \subset \mathrm{pp}_M(z_0 \pm z_2)$. That p is decomposable then follows from [5, Cor. 4.30].

Since $(\alpha-\beta)(z_1+z_3+\cdots) = z_0$, we see that $\alpha-\beta$ divides z_0 in the corresponding direct product module $\overline{M}$. Also clearly $\alpha-\beta$ does not divide z_0 in M. Thus M is not pure in $\overline{M}$. ■

Note that in this example the defining string $v = (\alpha\beta^{-1})^{\infty}$ for M is expanding. Therefore in the direct product module $M(v)$ the pp-type of z_0 is indecomposable.

But in general there are indecomposable pp-types of a completely different shape. If $v = v_1v_2\ldots$ is a one-sided string, then $v(i)$ will denote the string $v_iv_{i+1}\ldots$, and recall that $v\,|\,i$ denotes the string $v_1\ldots v_i$.

LEMMA 6.7. *Let v be a one-sided string over a string algebra A and let $M = M(v)$ be a direct sum module with the standard basis $z_0, z_1, \ldots$. Suppose that there is i such that for some n, $v(i)\,|\,n \neq v(j)\,|\,n$ for every $j \neq i$. Then the pp-type $\mathrm{pp}_M(z_i)$ is indecomposable.*

Proof. Similar to [1, Prop. 6.2]. ■

EXAMPLE 6.8. *Let* $A = G_{2,3}$, $a = \alpha\beta^{-1}$, $b = \alpha\beta^{-2}$ *and* $v = aba^2ba^3b\ldots$. *Let* $M = M(v)$ *be the corresponding direct sum module, and let* z_i *be an element of the canonical basis of* M. *Then the pp-type* $p_i = \mathrm{pp}_M(z_i)$ *is indecomposable.*

Proof. Using Lemma 6.7 it is easy to check that p_i is indecomposable. For instance, for z_0 we may take $n = 5$, i.e., ab, as the required part of v. ∎

Nevertheless, even in this case we do not know if the pure injective envelope of M is indecomposable.

7. The Cantor–Bendixson rank. The Cantor–Bendixson analysis on Zg_A runs as follows. At the first step we delete from Zg_A the isolated points, that is, by [5, Cor. 13.4] exactly the indecomposable finite-dimensional A-modules. What remains is a closed subset, Zg'_A, the first derivative of Zg_A. Removing isolated points from this space we obtain the second derivative Zg'_A and so on. At limit stages we put $\mathrm{Zg}_A^{(\lambda)} = \bigcap_{\mu<\lambda} \mathrm{Zg}_A^{(\mu)}$.

If this process reaches the empty set at stage $\lambda + 1$, then the CB-rank of Zg_A is defined to be λ. In this case for every point $M \in \mathrm{Zg}_A$ we may define the CB-rank of M to be the least μ such that $M \in \mathrm{Zg}_A^{(\mu)} \setminus \mathrm{Zg}_A^{(\mu+1)}$.

Note that if V is an open subset in Zg_A, then the CB-rank of every point in V can be calculated inside V. We define $\mathrm{CB}(V)$ as the supremum of CB-ranks of points in V.

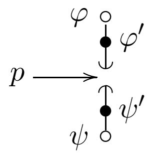
A similar analysis is possible for any theory T of A-modules such that the class of models of T is closed with respect to direct products. For instance, Zg_T is a closed subset of Zg_A whose points are the indecomposable pure injective modules that are direct summands of models of T. If φ and ψ are pp-formulae, $\varphi \leq_T \psi$ means that $\varphi(M) \subseteq \psi(M)$ for every $M \in \mathrm{Zg}_T$. Using the corresponding equivalence relation $\sim_T$ on the lattice of all pp-formulae L, we get the lattice L_T which is a factor of L. For more on this see [5].

Let $\psi < \varphi$ be pp-formulae. We say that (φ/ψ) is a *minimal pair* in T if the interval $[\psi, \varphi]$ in the lattice L_T is simple.

If (φ/ψ) is a minimal pair in T, then, by [5, Cor. 9.3], there exists a unique indecomposable pure injective module in Zg_T opening this pair. The question whether every isolated point in Zg_T is isolated by a minimal pair is still open. We say that a theory T has an *isolation property* (see [7, p. 382]) if for any theory T' such that $\mathrm{Zg}_{T'} \subseteq \mathrm{Zg}_T$, every isolated point in $\mathrm{Zg}_{T'}$ is isolated by a minimal pair.

The notion of m-dimension of a lattice L, $\mathrm{mdim}(L)$, can be found in [5, Ch. 10]. For instance, the m-dimension of a finite lattice is zero and $\mathrm{mdim}(\omega + 1) = 1$.

Let (φ/ψ) be a chain in the lattice of pp-formulae over A and let $p \in (\varphi/\psi)$ be an indecomposable pp-type. We define the m-dimension of p, mdim(p), as the infimum of m-dimensions of intervals (φ'/ψ') such that $\psi \leq \psi' < \varphi' \leq \varphi$ and $p \in (\varphi'/\psi')$.



PROPOSITION 7.1. *Let A be an arbitrary ring. Let (φ/ψ) be a chain in the lattice of all pp-formulae over A and let $p \in (\varphi/\psi)$ be an indecomposable pp-type. Then* $\mathrm{CB}(N(p)) = \mathrm{mdim}(p)$. *Also* $\mathrm{mdim}(\varphi/\psi)$ *is the supremum of m-dimensions of indecomposable pp-types $p \in (\varphi/\psi)$.*

Proof. The proof of [9, Thm. 3.1] can be applied in this situation to show that the isolation property holds true for the open set (φ/ψ): for every theory T of A-modules every isolated point in $T \cap (\varphi/\psi)$ is isolated by a minimal pair. Now the result is easily proved by induction, similarly to [5, Prop. 10.19]. ■

It follows from [15, Prop. 2] that for every non-domestic string algebra A, the CB-rank of Zg_A is undefined. For a domestic string algebra Schröer conjectured (see [14, p. 84]) that $\mathrm{CB}(\mathrm{Zg}_A)$ is finite and can be calculated from the bridge quiver of A.

The precise definition of the bridge quiver of a domestic string algebra A can be found in [16, S. 4]. We hope that from the following example it will be clear how to calculate the bridge quiver for a particular string algebra.

Let A be the domestic string algebra R_2 (see after Fact 6.3). The bands over A are the following: $C = \alpha_1\alpha_2^{-1}$, $C^{-1} = \alpha_2\alpha_1^{-1}$, and $D = \gamma_1\gamma_2^{-1}$, $D^{-1} = \gamma_2\gamma_1^{-1}$.

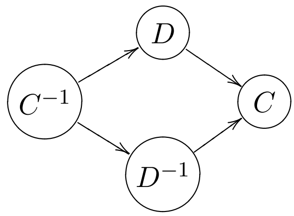
From the description of the two-sided strings over A (see above) we read off the following paths in the bridge quiver of A:

$$\alpha_2\alpha_1^{-1} \xrightarrow{\alpha_2\delta} \gamma_1\gamma_2^{-1} \xrightarrow{\gamma_1\beta} \alpha_1\alpha_2^{-1}$$

Inverting this we obtain:

$$\alpha_2\alpha_1^{-1} \xrightarrow{\beta^{-1}\gamma_1^{-1}} \gamma_2\gamma_1^{-1} \xrightarrow{\delta^{-1}\alpha_2^{-1}} \alpha_1\alpha_2^{-1}$$

Gluing these together we get the bridge quiver of A:



FACT 7.2 ([16, Lemma 4.2]). *Let A be a domestic string algebra. Then the bridge quiver of A is a finite oriented graph without oriented cycles.*

Note that, directly from the definition, for a string algebra A the one-directed indecomposable pure injective A-modules form an open subset in Zg_A. In the following theorem we calculate the CB-rank of this set.

THEOREM 7.3. *Let A be a domestic string algebra and let n be the maximal length of a path in the bridge quiver of A. Let U be the open set in Zg_A formed by the one-directed indecomposable pure injective A-modules. Then $\mathrm{CB}(U) = n+1$.*

Proof. Given a string CD, U_{CD} will denote the open set $(C.D)/(^+C.D)$ in Zg_A. We prove that $\mathrm{CB}(U_{CD}) \le n+1$. Since U is a union of such sets it will then follow that $\mathrm{CB}(U) \le n+1$.

Proposition 7.1 yields $\mathrm{CB}(U_{CD}) = \mathrm{mdim}(\varphi/\psi)$, where $\varphi = (C.D)$ and $\psi = (^+C.D)$. From Lemma 4.4 it follows that $\mathrm{mdim}(\varphi/\psi)$ is equal to the m-dimension of the chain $\{(C.D_i) \mid D_i \ge D,\ CD_i \text{ is a string}\}$. Then the result is easily derived from [16, Thm. 4.3].

For the converse let $C_0, \dots, C_n$ be bands such that $C_0 \dots C_i \dots C_n$ is a path of maximal length in the bridge quiver of A. Along this path we obtain one-sided strings $v_i = C_0^{k_0} \dots C_1^{k_1} \dots C_i^{\infty}$, where $k_0, k_1, \dots$ are arbitrary natural numbers, and dots replace bridges between bands. By induction on $i = n, \dots, 0$ we prove that (the direct product) module $M(v_i)$ has CB-rank $\ge n-i+1$.

For $i = n$, the module $M(v_0) = M(C_0^{k_0} \dots C_n^{\infty})$ is infinite-dimensional, therefore its CB-rank is not less than 1.

For $i < n$ note that $M(v_i)$ is in the Ziegler closure of the modules $M_k = M(w_k)$, where $w_k = C_0^{k_0} \dots C_i^{k} \dots C_{i+1}^{\infty}$, $k = 1, \dots$. Indeed take any finite string $D \le v_i$. By Lemma 4.4 and [18, Thm. 4.9] a basis of open neighborhoods of $M(v_i)$ can be chosen as $\{(.E)/((^+.D)+(.F)) \mid D \le E \le v_i < F\}$. Clearly for every such pair there exists k such that $E \le w_k < F$. Since a leftmost standard basis element z_0 opens this pair in M_k, we infer that $M_k \in (.E)/((^+.D)+(.F))$.

By the induction assumption, $\mathrm{CB}(M_k) \ge n-(i+1)+1 = n-i$ for every k. By the definition of CB-rank we deduce that $\mathrm{CB}(M(v_i)) \ge n-i+1$.

Finally for $i = 0$ we have $\mathrm{CB}(M(C_0^\infty)) \geq n + 1$, therefore $\mathrm{CB}(U) = n + 1$. ■

COROLLARY 7.4. *Let A be a domestic string algebra and let n be the maximal length of a path in the bridge quiver of A. Then* $\mathrm{CB}(\mathrm{Zg}_A) \geq n+2$.

Proof. From the proof of Theorem 7.3 we have $\mathrm{CB}(M) = n + 1$ where $M = M(C_0^\infty)$. Clearly it suffices to prove that the theory T of M contains a non-isolated point.

Indeed, otherwise M is the only isolated point of T. Let $C_0 = \alpha \ldots \beta^{-1}$ and let φ be the pp-formula $\alpha \mid x \wedge \beta \mid x$. Then $\varphi(M)$ is a uniserial right S-module, where $S = \mathrm{End}(M)$. As in [9, Thm. 3.1] it follows that M is isolated in T by a minimal pair. By [5, Prop. 10.17] the interval $(\varphi/x = 0)$ in T has finite length. Therefore $\varphi(M)$ has finite length as an S-module, a contradiction. ■

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