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# ORLICZ SPACES, $\alpha$-DECREASING FUNCTIONS, AND THE $\Delta_2$ CONDITION

BY

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**Abstract.** We prove some quantitatively sharp estimates concerning the $\Delta_2$ and $\nabla_2$ conditions for functions which generalize known ones. The sharp forms arise in the connection between Orlicz space theory and the theory of elliptic partial differential equations.

In [4], the present author studied a class of uniformly elliptic equations which includes the $p$-Laplacian equation as a special case. The usual function $t^p$ is replaced by an arbitrary increasing function $G(t)$ satisfying the growth condition

$$\delta + 1 \le \frac{tG'(t)}{G(t)} \le \Gamma + 1 \tag{0.1}$$

for positive constants $\delta \le \Gamma$. Since [4] also looked at variational problems, such as minimizing the value of

$$\int_\Omega G(|Du|)\,dx$$

over all functions $u$ in a suitable function space, it is obvious that this paper actually looked at problems in Orlicz spaces of the form $W^{1,G}(\Omega)$, with $G$ satisfying a $\Delta_2$ condition and a $\nabla_2$ condition. However, the connection to Orlicz spaces was de-emphasized in that paper because many of the tools of Orlicz space theory were not useful for the techniques involved there.

Our goal here is to make this connection more explicit. We are interested in two specific connection points. First, Young's inequality for a Young function $Y$ states that, for any two positive numbers $s$ and $t$, we have

$$st \le Y(s) + \widetilde{Y}(t),$$

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where $\widetilde{Y}$ is the complementary function to $Y$, defined by

$$\widetilde{Y}(t) = \max_{s \geq 0}(st - Y(s)).$$

In [4], a substitute Young's inequality was used: If $g$ is an increasing function, and $s$ and $t$ are positive numbers, then

$$sg(t) \leq sg(s) + tg(t).$$

We shall show that, in a sense, these two versions of Young's inequality are equivalent in Section 1. Next, it is clear that condition (0.1) for a Young function $G$ implies that $G$ satisfies a $\Delta_2$ and a $\nabla_2$ condition, and [3, Theorem 4.3] states that any Young function $G$ which satisfies a $\Delta_2$ and a $\nabla_2$ condition also satisfies (0.1) (provided we take $G'$ to mean the derivative from the right for $G$). However, the relation between the constants $\delta$ and $\Gamma$ and the constants in the $\Delta_2$ and $\nabla_2$ condition is not very satisfactory in [3]. In Section 2, we explain this relation more carefully and then we show that, for a given Young function $Y$, we can find another Young function $G$, equivalent to $Y$, satisfying (0.1) with optimal constants $\delta$ and $\Gamma$.

**1. The complementary Young function.** We begin by recalling some basic definitions. A *Young function* is a convex, strictly increasing function $Y\colon [0,\infty) \to [0,\infty)$. We also say that two positive functions $Y$ and $Z$ are *equivalent* if there are constants $k_1$ and $k_2$ such that $Y(s) \leq Z(k_1 s)$ and $Z(s) \leq Y(k_2 s)$ for all $s \geq 0$.

To continue, we note that $\widetilde{Y}$ is also a Young function. The definition of the complementary function immediately implies Young's inequality, and hence, for a Young function $Y$, we have $\widetilde{Y} \leq Y \circ y^{-1}$ (with $y$ defined by $y(s) = Y(s)/s$). Furthermore, for $t = y^{-1}(s)$, we have $2st - Y(t) = Y(t)$, so $\widetilde{Y}(2s) \geq Y \circ y^{-1}(s)$, and hence $\widetilde{Y}$ is equivalent to $Y \circ y^{-1}$.

**2. The $\Delta_2$ and $\nabla_2$ conditions.** We say that $Y$ satisfies a $\Delta_2$ *condition*, or that $Y \in \Delta_2$, if there is a constant $K \geq 2$ such that

$$Y(2t) \leq KY(t) \tag{2.1}$$

for all $t \geq 0$. Similarly $Y$ satisfies a $\nabla_2$ *condition*, or $Y \in \nabla_2$, if there is a constant $L > 1$ such that

$$2LY(s) \leq Y(Ls) \tag{2.2}$$

for all $s \geq 0$. According to [7, Theorem 2.3.3], $Y \in \Delta_2$ if and only if $\widetilde{Y} \in \nabla_2$.

To discuss the connection to (0.1), we first introduce the following terminology: A function $G$ is said to be $\alpha$*-increasing* for some $\alpha \in \mathbb{R}$ if the function $G_\alpha$, defined by $G_\alpha(s) = s^{-\alpha}G(s)$, is increasing; similarly, $G$ is $\alpha$*-decreasing* if $G_\alpha$ is decreasing. Note that condition (0.1) is equivalent, for a $C^1$ function $G$, to the joint statements that $G$ is $(\delta+1)$-increasing and $(\Gamma+1)$-decreasing. Next, [3, Theorem 4.1] states that a Young function $Y \in \Delta_2$ if and only if there is a constant $\alpha > 1$ such that $Y$ is $\alpha$-decreasing. The issue that we address here is the relation between $\alpha$ and $K$ in (2.1). From the proof of [3, Theorem 4.1], we see that, if $Y$ satisfies (2.1), then it is $\alpha$-increasing with $\alpha = K$ but, if $Y$ is $\alpha$-increasing, then it satisfies (2.1) with $K = 2^\alpha$. By using ideas from [4, Lemma 1.6] (see also [5, pp. 301–302]), we shall show that, if $Y$ satisfies (2.1), then there is an equivalent function $G$ which is $\alpha$-increasing with $K = 2^\alpha$. (Our philosophy is to note that almost increasing functions are equivalent to increasing functions. We refer the interested reader to Section 2.1 of [1], especially equations (2.4) and (2.4′) of that work for more explanation. In addition, [6] looks at some similar issues, and some of our ideas are also present in the discussion of the Matuszewska index in [2, Section 2.2].)

Our first step is a general lemma, which allows a comparison to other results.

LEMMA 2.1. *Let $k > 1$, $K > 1$, and $M \geq 1$ be constants. Suppose that $Y$ be a positive function such that*

$$Y(ks) \leq KY(s) \quad \text{for all } s \geq 0, \tag{2.3a}$$

$$Y(s) \leq MY(\sigma) \quad \text{for all } \sigma \geq s \geq 0. \tag{2.3b}$$

*Set $\alpha = \log_k K$. Then there is an increasing, $\alpha$-decreasing function $H$ such that*

$$\frac{H(s)}{M} \leq Y(s) \leq MH(ks) \tag{2.4}$$

*for all $s \geq 0$.*

*Proof.* Set

$$H(s) = \sup_{0\leq t\leq s} t^\alpha \inf_{0\leq\tau\leq t} \tau^{-\alpha} Y(\tau).$$

Then clearly $H$ is increasing, and

$$H(s) \leq \sup_{0\leq t\leq s} t^\alpha t^{-\alpha} Y(t) \leq MY(s).$$

Next, we have

$$\inf_{t/k\leq\tau\leq t} \tau^{-\alpha} Y(\tau) = \inf_{t/k^2\leq\tau\leq t/k} (k\tau)^{-\alpha} Y(k\tau) \leq \inf_{t/k^2\leq\tau\leq t/k} \tau^{-\alpha} Y(\tau),$$

and, by induction,

$$\inf_{t/k\leq\tau\leq t} \tau^{-\alpha} Y(\tau) \leq \inf_{t/k^{m+1}\leq\tau\leq t/k^m} \tau^{-\alpha} Y(\tau)$$

for any positive integer $m$. Hence

$$\inf_{0\leq\tau\leq t} \tau^{-\alpha} Y(\tau) = \inf_{t/k\leq\tau\leq t} \tau^{-\alpha} Y(\tau) \geq \frac{1}{M} t^{-\alpha} Y(t/k),$$

so

$$MH(s) \geq \sup_{0\leq t\leq s} Y(t/k) \geq Y(s/k),$$

which implies that $MH(ks) \geq Y(s)$.

Finally, if $s \leq \sigma$, then

$$\sigma^{-\alpha} \sup_{0\leq t\leq s} t^{\alpha} \inf_{0\leq \tau\leq t} \tau^{-\alpha} Y(\tau) = \sigma^{-\alpha} H(s) \leq s^{-\alpha} H(s)$$

and

$$\begin{aligned}\sigma^{-\alpha} \sup_{s\leq t\leq \sigma} t^{\alpha} \inf_{0\leq \tau\leq t} \tau^{-\alpha} Y(\tau) &= \sigma^{-\alpha} \sup_{s^2/\sigma\leq t\leq s} (t\sigma/s)^{\alpha} \inf_{0\leq \tau\leq t\sigma/s} \tau^{-\alpha} Y(\tau) \\ &= s^{-\alpha} \sup_{s^2/\sigma\leq t\leq s} t^{\alpha} \inf_{0\leq \tau\leq t\sigma/s} \tau^{-\alpha} Y(\tau) \\ &\leq s^{-\alpha} \sup_{0\leq t\leq s} t^{\alpha} \inf_{0\leq \tau\leq t\sigma/s} \tau^{-\alpha} Y(\tau) \\ &\leq s^{-\alpha} \sup_{0\leq t\leq s} t^{\alpha} \inf_{0\leq \tau\leq t} \tau^{-\alpha} Y(\tau) = s^{-\alpha} H(s).\end{aligned}$$

Hence $H$ is $\alpha$-decreasing. ■

Note that, since $H$ is $\alpha$-decreasing, we also have $Y(s) \leq MKH(s)$. Moreover, if $Y$ satisfies conditions (2.3) with positive constants $k$, $K$, and $M$ with $k > 1$, then clearly we must have $M \geq 1$, and we also have

$$Y(k^m) \leq K^m Y(1) \leq MK^m Y(k^m)$$

for any positive integer $m$. Hence we must have $MK^m \geq 1$ for any positive integer $m$, and therefore $K \geq 1$.

Let us also note the special case $K = 1$ so $\alpha = 0$. In this case, $H$ is increasing and 0-decreasing, and therefore constant. Alternatively, we can use (2.3) to see that

$$\frac{Y(\sigma)}{M} \leq Y(s) \leq MY(\sigma)$$

for all $s$ and $\sigma$, so we can also use $H \equiv Y(\sigma)$ for any choice of $\sigma$.

Let us compare this result to the one on pp. 301–302 of [5]. There, a function $F$ is given satisfying the conditions

$$F(t) \geq c_1 F(4t), \quad c_2 F(t)t \geq F(s)s$$

for all $s \geq t > 0$, and an increasing, $\alpha$-decreasing function $g$ is constructed with $\alpha = 1 - \log_4 c_1$ such that

$$c_2 c_1 g(t) \leq 4tF(t), \quad c_1 tF(t) \leq 4g(t).$$

In the present notation, we set $Y(t) = tF(t)$. Then $Y$ satisfies (2.3) with $k = 4$, $K = 4/c_1$, and $M = 1/c_2$, so

$$Y/K \leq g \leq MKY.$$

From Lemma 2.1, we have an increasing, $\alpha$-decreasing function $H$ such that

$$Y/MK \leq H \leq MY,$$

which improves the result in [5] if $M \leq K$.

In fact, the function $g$ from [5] is also $C^1$. Our next result shows how this additional constraint affects the present arguments.

COROLLARY 2.2. *With* $Y$, $K$, $k$, $M$, *and* $\alpha$ *as in Lemma* 2.1, *there is a* $C^1(0,\infty)$, *increasing*, $\alpha$*-decreasing function* $G$ *such that*

$$\frac{G(s)}{M} \leq Y(s) \leq 2MG(2ks) \tag{2.5}$$

*for all* $s > 0$.

*Proof.* With $H$ as in Lemma 2.1, set

$$G(s) = \frac{1}{s}\int_0^s H(t)\,dt.$$

Then $G$ is clearly $C^1(0,\infty)$.

Since $H$ is increasing, we have

$$G(s) \leq \frac{1}{s}\int_0^s H(s)\,dt = H(s) \leq MY(s),$$

and

$$G(2s) \geq \frac{1}{2s}\int_s^{2s} H(t)\,dt \geq \frac{1}{2s}\int_s^{2s} H(s)\,dt = \frac{1}{2}\,H(s),$$

which proves (2.5).

Next, $G'(s) = (H(s) - G(s))/s \geq 0$, so $G$ is increasing.

Finally, let $\sigma \geq s > 0$. Then

$$\begin{aligned}\sigma^{-\alpha}G(\sigma) &= \sigma^{-1-\alpha}\int_0^\sigma H(t)\,dt = \sigma^{-1-\alpha}\int_0^s H\left(\frac{\sigma}{s}\,t\right)\frac{\sigma}{s}\,dt \\ &\leq \sigma^{-1-\alpha}\int_0^s \left(\frac{\sigma}{s}\right)^\alpha H(t)\,\frac{\sigma}{s}\,dt = s^{-\alpha}G(s),\end{aligned}$$

so $G$ is $\alpha$-decreasing. ∎

Since $G$ is $\alpha$-decreasing and $C^1$, we have

$$0 \leq \frac{tG'(t)}{G(t)} \leq \alpha$$

for all $t > 0$.

In addition, we can use the ideas in this proof to prove the following alternative version of [4, Lemma 1.1(b)]: If $g$ is an increasing, $\Gamma$-decreasing function, and if

$$G(s) = \int_0^s g(t)\,dt,$$

then

$$G(s) \ge \frac{1}{1+\Gamma}\,sg(s)$$

for all $s$. (In [4], $g$ was also assumed $C^1$.) We just write

$$G(s) \ge \int_0^s g(s)\,\frac{s^\Gamma}{t^\Gamma}\,dt = \frac{1}{1+\Gamma}\,sg(s).$$

From our lemma, we can also deduce a quantitative connection between the $\Delta_2$ and $\nabla_2$ condition and condition (0.1).

Theorem 2.3. *Suppose $Y$ is a positive increasing function on $(0,\infty)$ and that there are constants $K \ge 2$ and $L > 2$ such that (2.1) and (2.2) hold. Set $\alpha = \log_2 K$, $\beta = \log_2 L + 1$, and $\theta = \beta/(\beta-1)$, and suppose that $\theta \le \alpha$. Then there is a $C^1$, convex function $G$ satisfying (0.1) with $\delta = \alpha - 1$ and $\Gamma = \theta - 1$. Further,*

$$2^{1-\beta}G(s) \le Y(s) \le 2^{\beta-1-\theta}\alpha G(2s) \tag{2.6}$$

*for all $s \ge 0$.*

*Proof.* First, define $Z$ by $Z(s) = s^{-\theta}Y(s)$. Then

$$Z(2s) = (2s)^{-\theta}Y(2s) \le 2^{\alpha-\theta}Z(s).$$

Next, let $s \le \sigma$, choose $m$ to be the positive integer such that

$$2^{(1-\beta)m}\sigma < s \le 2^{(1-\beta)(m-1)}\sigma,$$

and set $t = 2^{(1-\beta)m}\sigma/s$, so $2^{1-\beta} \le t < 1$. Then

$$\begin{aligned} Z(\sigma) &= 2^{m(1-\beta)\theta}(st)^{-\theta}Y(2^{m(\beta-1)}st) \ge 2^{m(1-\beta)\theta}2^{m\beta}(st)^{-\theta}Y(st) \\ &= (st)^{-\theta}Y(st) \ge 2^{1-\beta}s^{-\theta}Y(s) = 2^{1-\beta}Z(s). \end{aligned}$$

It then follows from Lemma 1.1 (with $Z$ in place of $Y$, $\alpha-\theta$ in place of $\alpha$, $k=2$, $2^{\alpha-\theta}$ in place of $K$, and $M = 2^{\beta-1}$) that there is an increasing, $(\alpha-\theta)$-decreasing function $H$ such that $2^{1-\beta}H(s) \le Z(s) \le 2^{\beta-1}H(2s)$ for all $s$.

Now define $H_1$ by $H_1(s) = s^\theta H(s)$. It follows that $H_1$ is $\alpha$-decreasing and $\theta$-increasing with

$$2^{1-\beta}H_1(s) \le Y(s) \le 2^{\beta-1-\theta}H_1(2s).$$

Finally, we define $G$ by

$$G(s) = \int_0^s \frac{H_1(t)}{t}\,dt.$$

Since $\theta \geq 1$, it follows that $H_1$ is 1-increasing, which means that the integrand here is nonnegative, continuous and increasing on $(0, \infty)$. Hence $G$ is convex. In addition, since $H_1$ is 1-increasing, we have $G \leq H_1$. Since $H_1$ is $\alpha$-decreasing, we also have

$$G(s) \geq \int_0^s \frac{H_1(s)}{t} \frac{t^\alpha}{s^\alpha} \, ds = \frac{H_1(s)}{\alpha}.$$

Hence (2.6) holds.

The proof is completed by noting that the argument in Corollary 2.2 shows that $G$ is $\alpha$-decreasing and $\theta$-increasing. ■

Note that $Y$ is not assumed to be a Young function. Moreover, if $\theta = \alpha$, then we can show by other means that $G(s) = Cs^\alpha$ satisfies the conclusion of this theorem for a suitable positive constant $C$. Finally, we remark that the condition $\theta \leq \alpha$ is not just an artifact of our method. Of course such a condition is necessary for (0.1) to hold, but, more significantly, if $\beta$ is rational, then conditions (2.1) and (2.2) imply that $\alpha \geq \theta$. To prove this statement, we suppose that $\beta = p/q$ for some positive integers $p$ and $q$, and we note that $p > q$. Then

$$Y(2^{q(\beta-1)}s) = Y(2^{p-q}s) \leq K^{p-q} Y(s) = 2^{\alpha(p-q)} Y(s)$$

from (2.1) while

$$Y(s) \leq (2L)^{-q} Y(L^q s) = 2^{-\beta q} Y(2^{q(\beta-1)}s)$$

from (2.2). The combination of these two inequalities implies that

$$2^{\alpha(p-q)-\beta q} \leq 1,$$

so $\alpha(p-q) \geq \beta q$ or $\alpha q(\beta-1) \geq q\beta$, so $\alpha(\beta-1) \geq \beta$. This last inequality immediately gives $\alpha \geq \theta$.

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