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ON ITERATES OF STRONG FELLER OPERATORS ON ORDERED PHASE SPACES

BY

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Abstract. Let (X, d) be a metric space where all closed balls are compact, with a fixed σ-finite Borel measure μ. Assume further that X is endowed with a linear order $\preceq$. Given a Markov (regular) operator $P : L^1(\mu) \to L^1(\mu)$ we discuss the asymptotic behaviour of the iterates P^n. The paper deals with operators P which are Feller and such that the μ-absolutely continuous parts of the transition probabilities $\{P(x,\cdot)\}_{x\in X}$ are continuous with respect to x. Under some concentration assumptions on the asymptotic transition probabilities $P^m(y,\cdot)$, which also satisfy $\inf(\operatorname{supp} Pf_1) \preceq \inf(\operatorname{supp} Pf_2)$ whenever $\inf(\operatorname{supp} f_1) \preceq \inf(\operatorname{supp} f_2)$, we prove that the iterates P^n converge in the weak* operator topology.

Introduction. Let (X, d) be a metric space with the property that each closed ball $\overline{K}(x_0, r) = \{x \in X : \mathrm{d}(x_0, x) \le r\}$ is compact. Given a σ-finite measure μ on the Borel σ-algebra $\mathcal{B}$ of subsets in (X, d) we denote by $(L^1(\mu), \|\cdot\|)$ the Banach lattice of μ-integrable functions on X. Functions from $L^1(\mu)$ which are equal μ-almost everywhere are identified. Instead of $\mathcal{B}$ we will rather think of its μ-completion $\widetilde{\mathcal{B}}$. If not stated otherwise, also all inequalities are in the μ-a.e. sense. We denote by $\mathcal{D}$ the convex set $\{f \in L^1(\mu) : f \ge 0, \int_X f\,d\mu = 1\}$ of all *densities* in $L^1(\mu)$. A linear operator $P : L^1(\mu) \to L^1(\mu)$ which preserves $\mathcal{D}$ (i.e. $P(\mathcal{D}) \subseteq \mathcal{D}$) is called *Markov* (or *stochastic*). In this paper we will deal only with *regular* Markov operators, given by a family of transition probabilities $\{P(y, dx)\}_{y\in X}$ satisfying

- $(X, \mathcal{B}) \ni y \mapsto P(y, A)$ *is measurable for each fixed* $A \in \mathcal{B}$,
- $0 \le P(y, A) \le 1$, $P(y, \emptyset) = 0$ *and* $P(y, X) = 1$,
- $P(y, \bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(y, A_n)$ *whenever* $A_n \in \mathcal{B}$ *are pairwise disjoint*,
- $P(y, A) = 0$ *for* μ*-almost all* y *if* $\mu(A) = 0$.

Then $\int_A Pf(x)d\mu(x) = \int_X f(y)P(y, A)\,d\mu(y)$ for all $A \in \mathcal{B}$ and $f \in L^1(\mu)$.

By setting $P\nu(A) = \int_X P(y, A)\,d\nu(y)$ we extend P to the Banach lattice $(\mathcal{M}(X), \|\cdot\|)$ of all bounded signed measures ν on $(X, \mathcal{B})$. A posi-

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tive linear contraction $Q : L^1(\mu) \to L^1(\mu)$ is called a *kernel operator* if there exists a Borel measurable function $q : X \times X \to \mathbb{R}_+$ such that $Qf(x) = \int_X q(x,y)f(y)\,d\mu(y)$ and $\int q(x,y)\,d\mu(x) \le 1$ for every $y \in X$. In particular, a kernel Markov operator P satisfies $P(y,\cdot) \preccurlyeq \mu$ and obviously $\int q(x,y)\,d\mu(x) = 1$ for every $y \in X$. Moreover, it follows from

$$P\nu(A) = \int_X \int_X q(x,y)\mathbf{1}_A(x)\,d\mu(x)\,d\nu(y)$$

that $P\nu \in L^1(\mu)$ for any $\nu \in \mathcal{M}(X)$. We say that a Markov operator P is *Feller* if $y_n \to y_0$ implies that

$$\int f(x)\,P(y_n,dx) \to \int f(x)\,P(y_0,dx)$$

for each continuous and bounded function f. By $C_0(X)$ we denote the Banach sublattice of functions vanishing at infinity (i.e. for each positive ε there exists a compact set $K \subseteq X$ such that $|f(x)| \le \varepsilon$ for $x \notin K$) equipped with the sup-norm $\|\cdot\|_{\sup}$. In this paper we will additionally assume that $P^*C_0(X) \subseteq C_0(X)$. Clearly this property is inherited by powers P^{*k}. By $L_0^\infty(\mu)$ we denote the Banach sublattice of $L^\infty(\mu)$, endowed with the sup–norm $\|\cdot\|_{\sup}$, consisting of all functions h vanishing at infinity. It is clear that if $P^*(C_0(X)) \subseteq C_0(X)$ then $P^*(L_0^\infty(\mu)) \subseteq L_0^\infty(\mu)$. Clearly, for any E contained in a ball $K(x_0,r)$ and $\varepsilon > 0$ there exists t such that $P(y,E) < \varepsilon$ whenever $\mathrm{d}(y,x_0) > t$.

We recall that a kernel Markov operator on $L^1(\mu)$ is *strong Feller in the strict sense* if the mapping

$$\text{(SFS)} \qquad (X,\mathrm{d}) \ni y \mapsto q(\,\cdot\,,y) \in (\mathcal{D},\|\cdot\|)$$

is continuous. It follows from (SFS) that P^*h is bounded and continuous whenever $h \in L^\infty(\mu)$, where P^* denotes the adjoint operator. This follows easily from $P^*h(y) = \int_X q(x,y)h(x)\,d\mu(x)$. The asymptotic properties of iterates of (SFS) operators and their role in modelling a cell cycle have been studied by the author in [B1], [B2] (following a series of papers [GL], [KM], [KT], [LM1]–[LM3], [LMT], [ŁR], [M], [T]). The current paper is a two-fold extension of [B1]. We replace assumption (SFS) by a weaker condition (PSFS) (see below), and proceeding to applications we discuss ordered phase spaces instead of $\mathbb{R}$. Moreover, our new proofs seem to be simplified.

We say that a variation norm bounded sequence of measures ν_n is *vaguely convergent* to ν if $\lim_{n\to\infty} \int_X h\,d\nu_n = \int_X h\,d\nu$ for all $h \in C_0(X)$. Since $\mathcal{M}(X)$ may be identified with the adjoint space $C_0(X)^*$, this is just the weak* convergence. Let us recall that the set of all subprobability positive measures on X is compact for the vague topology.

Definition 1.1. We say that a Markov operator $P : L^1(\mu) \to L^1(\mu)$ is *partially kernel* if there exists a nonnegative measurable kernel $q(x,y)$ such that $0 \le Q \le P$ (in particular $P_{\mathrm{ac}}(y,\cdot) \ge q(\cdot,y)d\mu$) and moreover

$$\inf_{y\in K} \eta(y) = \eta(K) > 0 \tag{1}$$

for each compact $K \subseteq X$, where $\eta(y) = \|q(\cdot,y)\|_1$.

Definition 1.2. A partially kernel Feller operator P is called *partially strong Feller in the strict sense* (PSFS) if $P^*(C_0(X)) \subseteq C_0(X)$ and the substochastic kernel operator Q has the property that

$$(X,d) \ni y \mapsto q(\cdot,y) \in (L^1_+(\mu), \|\cdot\|_1) \tag{PSFS}$$

is continuous.

Clearly a (PSFS) operator P does not necessarily map $L^\infty(\mu)$ into continuous functions as in the case of (SFS). This is because the singular part $P_{\sin}(y,\cdot)$ of the transition probabilities may be nontrivial and contribute towards discontinuity. The iterates of a (PSFS) Markov operator P also satisfy (PSFS). For this we notice (see [F1] for details) that the kernel operators form a two-sided ideal in the Banach lattice of bounded operators on $L^1(\mu)$. If $P = Q + R$ ($R = P - Q$) then $P^m = \sum_{j=1}^m P^{m-j}QR^{j-1} + R^m$. Denote by $q_m(x,y)$ the kernel corresponding to the substochastic operator $Q_m = \sum_{j=1}^m P^{m-j}QR^{j-1}$. In order to verify that $X \ni y \mapsto q_m(\cdot,y) \in L^1_+(\mu)$ is continuous we first note that the R^j are Feller. Therefore if $y_n \to y$ then $R^{*j}\delta_{y_n}$ converges to $R^{*j}\delta_y$ in the vague topology. The rest follows from the observation that the kernel substochastic operator Q maps weak* (vaguely) convergent sequences to norm convergent ones.

Markov kernels considered in this paper enjoy a kind of asymptotic concentration and have stochastically controlled jumps ((SCJ) for abbreviation). Long jumps come "mostly" from singular or discontinuous parts. In order to formally introduce this concept, for fixed $y_0 \in X$ and $m \ge 1$ we define

$$t_m(\varepsilon,r_0) = \inf\{t > 0 : P^m(y, K(y_0,r_0)) < \varepsilon \text{ for all } y \in X \text{ with } \mathrm{d}(y,y_0) > t\},$$
$$\eta_m(\varepsilon,r_0) = \inf\Big\{\int_X q_m(x,y)\,d\mu(x) : \mathrm{d}(y,y_0) \le t_m(\varepsilon,r_0)\Big\}.$$

Definition 1.3. We say that a (PSFS) Markov operator P has *property* (SCJ) if

$$\lim_{k\to\infty} \eta_k(\varepsilon,r_0) = 1 \quad \text{for all } \varepsilon > 0 \text{ and } r_0 > 0. \tag{SCJ}$$

Remark 1. We notice that if there exists $0 < a < 1$ such that $\eta(y) \ge a$ for every $y \in X$ then for each natural m the following estimate holds:

$$\eta_m(\varepsilon,r_0) \ge 1 - (1-a)^m$$

for any ε, r_0, y_0. In particular condition (SCJ) holds. Clearly all (SFS) Markov operators enjoy (SCJ).

For the convenience of the reader, we repeat some parts of [B1], mainly concerning notation and definitions. Most of them are standard and are borrowed from [F1].

The motivation for this paper comes from studying a class of Markov operators on $L^1([0,\infty))$, the Banach lattice of Lebesgue integrable functions on $[0,\infty)$, with kernels (see [LMT] for origins)

$$q(x,y)=\begin{cases} -\dfrac{\partial}{\partial x}H(Q(\lambda(x))-Q(y)) & \text{if } 0\le y\le \lambda(x),\\ 0 & \text{otherwise,}\end{cases} \tag{3}$$

where the functions $H,Q,\lambda:[0,\infty)\to[0,\infty)$ are assumed to be absolutely continuous. Moreover, they satisfy:

(H) $H(0)=1$, $\lim_{x\to\infty}H(x)=0$, H is nonincreasing,

(Qλ) $Q(0)=\lambda(0)=0$, $\lim_{x\to\infty}Q(x)=\lim_{x\to\infty}\lambda(x)=\infty$, and Q, λ are nondecreasing.

The class of Markov operators with kernels (3) and H, Q, λ satisfying conditions (H), (Qλ) is denoted by LMT (after Lasota, Mackey and Tyrcha whose contribution to mathematical modelling of cell cycles is crucial). Clearly LMT operators satisfy (SFS) and preserve $C_0(X)$, and for each m and $\varepsilon>0$ we have $\eta_m(\varepsilon,r_0)=1$. It has been proved in [B1] that LMT operators have convergent iterates in the weak* operator topology. We get the same for more general (PSFS) Markov operators on an abstract ordered phase. The following easy example shows that our generalization is meaningful.

Example 1. Let P_0 be an (SFS) Markov operator preserving $C_0([0,\infty))$ with a kernel $q(x,y)$, let $\varphi:[0,\infty)\to[0,\infty)$ be a continuous mapping satisfying $\lim_{y\to\infty}\varphi(y)=\infty$ and let $\eta:[0,\infty)\to(0,1)$ be continuous. Define

$$P(y,\cdot)=(1-\eta(y))\delta_{\varphi(y)}+\eta(y)q(\cdot,y)d\lambda.$$

Obviously P does not belong to (SFS) but satisfies (PSFS).

Remark 2. Applying the above remark we easily get an important class of (PSFS) operators. Consider any LMT operator P_0. Every convex mixture of P_0 with a deterministic perturbation produces a (PSFS) operator which is not (SFS). Another important class of (PSFS) operators are convolution operators P_μ (i.e. operators of the form $P_\mu f(x)=\int_{\mathbb{R}}f(x-y)\,d\mu(y)$), where the probability measure μ has a nontrivial absolutely continuous part.

Generalizing [B1] we study weak convergence of iterates P^nf of a (PSFS) operator P. We will prove that under some mild extra conditions they converge to Sf, where S is a substochastic projection onto $L^1_*(\mu)$ (i.e.

$S^2 = S \geq 0$ and $\int_X Sf\,d\mu \leq 1$ for every $f \in \mathcal{D}$). Moreover, if $F \in \mathcal{B}$ denotes the minimal (modulo sets of measure zero) set which carries supports of all P-invariant densities, then $S_F = S\restriction_{L^1(F,\mu)}$ is a Markov projection onto P-invariant functions, and $\|P^n f - S_F f\| \to 0$ for all $f \in L^1(F,\mu)$. We note that $L^1(F,\mu)$ is P-invariant and that F is well defined, because $L^1(\mu)$ is separable. If $F_j = \operatorname{supp} f_{*j}$, where f_{*j} is an ergodic invariant density, then $P_j = P\restriction_{L^1(F_j,\mu)}$ is clearly asymptotically stable on its domain. We note that for $f \in \mathcal{D}$ we have $\int_{F_j} Sf\,d\mu = \lim_{n\to\infty} \int_{F_j} P^n f\,d\mu = \Lambda_j(f)$, where the last sequence converges as it is bounded and nondecreasing (F_j is invariant). This implies $Sf = \Lambda_j(f) f_{*j}$ on F_j. If there are no invariant densities at all, then obviously $S \equiv 0$, and $\lim_{n\to\infty} \int_0^\infty P^n f h\,d\mu = 0$ for every $h \in C_0[0,\infty)$. In particular, P is sweeping with respect to the family of compact sets.

The rest of our notation follows [F1]. We shall also use some of its results, which we now briefly recall. Given a Markov operator P on $L^1(\mu)$, the space X may be divided into two disjoint parts C and D. The *conservative part* C is characterized by $C = \{x \in X : f \geq 0 \Rightarrow \sum_{n=0}^\infty P^n f(x) \text{ is either } 0 \text{ or } \infty\}$. Because $P^*\mathbf{1}_C \geq \mathbf{1}_C$, the Markov operator $P_C f = P(1_C f)$ is well defined. D is called the *dissipative part*. Obviously $P^*\mathbf{1}_D \leq \mathbf{1}_D$ and $\sum_{n=0}^\infty P^n f(x) < \infty$ for all $f \in L^1(\mu)$ and $x \in D$. In particular, $\lim_{n\to\infty} P^n f(x) = 0$ for $x \in D$. Even more can be proved: there exists a sequence D_m of measurable subsets of D such that $\lim_{n\to\infty} P^{*n}\mathbf{1}_{D_m}(x) = 0$ and $\bigcup_{m=1}^\infty D_m = D$. If $C = X$ then we say that the operator P is *conservative*. We denote by $\Sigma_{\mathrm{i}}(P)$ the σ-algebra of all invariant sets A, i.e. such that $P^*\mathbf{1}_A = \mathbf{1}_A$. If $\Sigma_{\mathrm{i}}(P) = \{\emptyset, X\}$ then P is called *ergodic*. The *deterministic σ-algebra* $\Sigma_{\mathrm{d}}(P)$ is defined as $\{B \in \mathcal{B} : P^{*n}\mathbf{1}_B = \mathbf{1}_{B_n} \text{ for every natural } n\}$. We say that $B_0, B_1, \ldots, B_{d-1}$ (from $\Sigma_{\mathrm{d}}(P)$) form a *cycle* if $P^{*d}\mathbf{1}_{B_0} = \mathbf{1}_{B_0}$ and $P^*\mathbf{1}_{B_j} = \mathbf{1}_{B_{j+1}}$ for all $0 \leq j \leq d-2$. Clearly $\Sigma_{\mathrm{i}}(P) \subseteq \Sigma_{\mathrm{d}}(P)$, but in general these two σ-algebras may differ. If $\Sigma_{\mathrm{i}}(P) = \Sigma_{\mathrm{d}}(P)$ then we say that the Markov operator P *does not allow cycles.*

2. Main result. The structure of the present paper resembles [B1]. In this section we describe the asymptotic properties of iterates of (PSFS) Markov operators acting on abstract $L^1(\mu)$ spaces. We start with

DEFINITION 2.1. Let B be a μ-measurable subset of X. The *essential closure* of B is the smallest closed subset D of X such that $\mu(B \cap D^{\mathrm{c}}) = 0$. It is denoted by $\overline{B}^{\mathrm{ess}}$.

Clearly, $\overline{B}^{\mathrm{ess}}$ is well defined. It is enough to take any density f with support B (it exists as μ is σ-finite) and set $\overline{B}^{\mathrm{ess}} = \operatorname{supp}(f d\mu)$.

LEMMA 2.2. *Let P be a* (PSFS) *Markov operator on $L^1(X,\mathcal{B},\mu)$. If $B_0, B_1 \subseteq C$ are two invariant measurable sets* (*i.e. $P(x,B_j) \geq \mathbf{1}_{B_j}(x)$ for*

μ-almost all x, where $j = 0, 1$) which are disjoint μ-almost everywhere (*i.e.* $\mu(B_0 \triangle B_1) = 0$) *then* $\overline{B}_0^{\mathrm{ess}} \cap \overline{B}_1^{\mathrm{ess}} = \emptyset$.

Proof. Let $Q(x, A) = \int_A q(z, x)\, d\mu(z)$. It follows from the (PSFS) assumption that the sets

$$\{x : Q(x, B_0) = 0\} \quad \text{and} \quad \{x : Q(x, B_1) = 0\}$$

are closed. Since $\mu(B_j \setminus \{x : Q(x, B_{1-j}) = 0\}) = 0$ for $j = 0, 1$, it follows that $\overline{B}_j^{\mathrm{ess}} \subseteq \{x : Q(x, B_{1-j}) = 0\}$. Suppose that $x_0 \in \overline{B}_1^{\mathrm{ess}} \cap \overline{B}_2^{\mathrm{ess}}$. Then by weak continuity of $x \mapsto P(x, \cdot)$ we get $P(x_0, \overline{B}_j^{\mathrm{ess}}) = 1$ for $j = 0, 1$. On the other hand, we can choose sequences $B_0 \ni y_n \to x_0$ and $B_1 \ni z_n \to x_0$ such that $P(y_n, B_1) = 0$, $P(y_n, B_0) = 1$, $P(z_n, B_0) = 0$, and $P(z_n, B_1) = 1$ for each n. By the norm continuity of $x \mapsto Q(x, \cdot)$ and the property (1) which guarantees that

$$\eta(\overline{K}(x_0, 1)) > \delta$$

for some strictly positive δ, we obtain

$$0 < \delta \le Q(x_0, \overline{B}_0^{\mathrm{ess}}) = \lim_{n\to\infty} Q(z_n, B_0) \le P(z_n, B_0) \equiv 0,$$

a contradiction.

LEMMA 2.3. *Let P be a* (PSFS) *Markov operator on $L^1(X, \mathcal{B}, \mu)$ such that P_C does not allow cycles. Then*

$$\lim_{n\to\infty} \int_{K\cap(C\setminus F)} P^n f\, d\mu = 0 \tag{4}$$

for every compact set $K \subseteq X$ and arbitrary $f \in L^1(X, \mathcal{B}, \mu)$.

Proof. The proof is essentially the same as that of Lemma 1 in [B1]. Here we only simplify some arguments. Repeating the first part of the above mentioned proof, for all atoms $B \in \Sigma_{\mathrm{d}}(P_C)$ we get

$$\lim_{n\to\infty} \sup_{f\in\mathcal{D}} \|P_B^n f - P_B^{n+1} f\| = 0 \tag{5}$$

(i.e. the zero alternative in the so-called “0-2 law”, see [F2] for details).

Then we notice that if $\nu = \lim_{j\to\infty} P^{n_j} f$ for some $n_j \nearrow \infty$, where $f \in \mathcal{D}_\mu$ and the convergence is in the vague topology, then ν is absolutely continuous (if nonzero). In fact, by (5) we get $P\nu = \nu$. Since P is partially kernel we have $(P\nu_{\mathrm{sin}})_{\mathrm{ac}} \ne 0$ whenever the singular part ν_{sin} of ν is nonzero (otherwise ν is already absolutely continuous and there is nothing to prove). It follows from

$$\begin{aligned}\int \nu_{\mathrm{ac}}\, d\mu &= \int (P\nu)_{\mathrm{ac}} = \int [P(\nu_{\mathrm{ac}} + \nu_{\mathrm{sin}})]\, d\mu \\ &= \int P\nu_{\mathrm{ac}}\, d\mu + \int (P\nu_{\mathrm{sin}})_{\mathrm{ac}}\, d\mu = \int \nu_{\mathrm{ac}}\, d\mu + \int (P\nu_{\mathrm{sin}})_{\mathrm{ac}}\, d\mu\end{aligned}$$

that $\nu_{\sin} = 0$. We conclude that each P-invariant probability measure is absolutely continuous.

Since the deterministic σ-field is atomic, the conservative part C is a countable (or finite) union of atoms. As in [B1], in order to prove

$$\lim_{n\to\infty} \int\limits_{K\cap(C\setminus F)} P^n f\, d\mu = 0,$$

it is enough to show that

$$\lim_{n\to\infty} \int\limits_{K\cap B} P^n f\, d\mu = 0$$

for every atom $B \subseteq C \setminus F$. It follows from Lemma 1 that $\overline{B}^{\mathrm{ess}} \cap \overline{F}^{\mathrm{ess}} = \emptyset$. Suppose that

$$\overline{\lim_{n\to\infty}} \int\limits_{K\cap B} P^n f\, d\mu > 0 \quad \text{for some } f \in \mathcal{D}.$$

Since B is invariant we may assume that f is (μ-a.e.) concentrated on $\overline{B}^{\mathrm{ess}}$. Choosing a subsequence, let $P^{n_j} f \to \nu$ in the vague topology, where ν is a nonzero positive measure. By (5) we have $\|P\nu - \nu\| = 0$ and by Lemma 1 we infer that ν is totally concentrated on $\overline{B}^{\mathrm{ess}}$, which is disjoint from $\overline{F}^{\mathrm{ess}}$. In particular, $d\nu/d\mu = f_\nu \in L^1(\mu)$ is invariant and concentrated outside $\overline{F}^{\mathrm{ess}}$, a contradiction. ∎

The next lemma is an easy consequence of a corollary from [BB]. Actually its proof is the same as that of Lemma 2 in [BB], as the latter is not affected by passing to (PSFS).

LEMMA 2.4. *Let P be a partially kernel Markov operator on $L^1(\mu)$. If P_F does not allow cycles then for every $f \in L^1(F, \mu)$,*

$$\lim_{n\to\infty} \|P_F^n(f) - \mathcal{E}(f|\Sigma_{\mathrm{i}}(P_F))\|_1 = 0.$$

The last lemma of this section is a modification of Lemma 3 in [B1].

LEMMA 2.5. *Let P be a* (PSFS) *Markov operator satisfying the* (SCJ) *condition. Then for every compact set $K \subseteq X$ and $f \in \mathcal{D}$ we have*

$$\lim_{n\to\infty} \int\limits_{K\cap D} P^n f\, d\mu = 0.$$

Proof. Given a compact set $K \subseteq X$ suppose that $\lim_{j\to\infty} \int_{K\cap D} P^{n_j} f\, d\mu > 4\varepsilon$ for some sequence $n_j \nearrow \infty$. Since P is defined by transition probabilities, we have

$$\begin{aligned}P^{*k}\mathbf{1}_{D_m\cap K}(y) &= \int\limits_X \mathbf{1}_{D_m\cap K}(x)\,P^k(y,dx)\\ &\nearrow \int\limits_X \mathbf{1}_{D\cap K}(x)\,P^k(y,dx) = P^{*k}\mathbf{1}_{D\cap K}(y)\end{aligned}$$

as $m\nearrow\infty$, for all $y\in X$ and any $k\ge 1$.

Now fix $k\ge 1$ large enough so that $\eta_k(\varepsilon,r_0)>1-\varepsilon$, where r_0 is such that $K\subseteq K(y_0,r_0)$. As above,

$$\lim_{m\to\infty} Q_k^*\mathbf{1}_{D_m\cap K}(z) = \lim_{m\to\infty}\int q_k(x,z)\mathbf{1}_{D_m\cap K}(x)\,d\mu(x) = Q_k^*\mathbf{1}_{D\cap K}(z)$$

for each $z\in X$, and moreover the sequence is nondecreasing. Since both $Q_k^*\mathbf{1}_{D_m\cap K}$ and the limit function $Q_k^*\mathbf{1}_{D\cap K}$ belong to $C_0(X)$, by a Dini argument the convergence is uniform on X. We find $m_\diamondsuit$ such that for all $m\ge m_\diamondsuit$ we have

$$\sup_{z\in X}|Q_k^*\mathbf{1}_{D_m\cap K}(z)-Q_k^*\mathbf{1}_{D\cap K}(z)|<\varepsilon.$$

If $z\notin\overline{K}(y_0,t_k(\varepsilon,r_0))$ then by the definition of $t_m(\varepsilon,r_0)$ we have

$$|P^{*k}\mathbf{1}_{D\cap K}(z)-P^{*k}\mathbf{1}_{D_m\cap K}(z)|\le P^{*k}\mathbf{1}_{D\cap K}(z)+P^{*k}\mathbf{1}_{D_m\cap K}(z)\le 2\varepsilon.$$

If $z\in\overline{K}(y_0,t_k(\varepsilon,r_0))$ then

$$\begin{aligned}|P^{*k}\mathbf{1}_{D\cap K}(z)-P^{*k}\mathbf{1}_{D_m\cap K}(z)| &\le |P^{*k}\mathbf{1}_{D\cap K}(z)-Q_k^*\mathbf{1}_{D\cap K}(z)|\\ &\quad+|Q_k^*\mathbf{1}_{D\cap K}(z)-Q_k^*\mathbf{1}_{D_m\cap K}(z)|\\ &\quad+|Q_k^*\mathbf{1}_{D_m\cap K}(z)-P^{*k}\mathbf{1}_{D_m\cap K}(z)|\\ &\le\varepsilon+\varepsilon+\varepsilon=3\varepsilon\qquad\text{whenever } m\ge m_\diamondsuit.\end{aligned}$$

Therefore

$$\|P^{*k}\mathbf{1}_{D\cap K}-P^{*k}\mathbf{1}_{D_m\cap K}\|_\infty\le 3\varepsilon$$

if $m\ge m_\diamondsuit$. Now

$$\begin{aligned}\int\limits_{D_m\cap K} P^{n_j}f\,d\mu &= \Big(\int\limits_{D_m\cap K}-\int\limits_{D\cap K}+\int\limits_{D\cap K}\Big)P^{n_j}f\,d\mu\\ &\ge 4\varepsilon-\Big|\int P^{n_j-k}f(P^{*k}\mathbf{1}_{D_m\cap K}-P^{*k}\mathbf{1}_{D\cap K})\,d\mu\Big|\\ &\ge 4\varepsilon-\int P^{n_j-k}f\|P^{*k}\mathbf{1}_{D_m\cap K}-P^{*k}\mathbf{1}_{D\cap K}\|_{\sup}\,d\mu>\varepsilon\end{aligned}$$

for all $j=1,2,\ldots$. On the other hand, $f\cdot P^{*n_j}\mathbf{1}_{D_m\cap K}\to 0$ as $j\to\infty$ for μ-almost all x. Finally by the Lebesgue dominated convergence theorem,

$$\lim_{j\to\infty}\int\limits_{D_m\cap K}P^{n_j}f\,d\mu=\lim_{j\to\infty}\int\limits_X f\cdot P^{*n_j}\mathbf{1}_{D_m\cap K}\,d\mu=0,$$

and we arrive at a contradiction. ∎

Combining the above lemmas we easily obtain the main result of this section. Its proof is omitted as it is straightforward and the same as in [B1].

THEOREM 2.6. *Let* (X, d) *be a metric space such that all closed balls are compact. If a* (PSFS) *Markov operator* P *on* $L^1(X, \mathcal{B}, \mu)$ *has property* (SCJ) *and* $\Sigma_{\mathrm{i}}(P_C) = \Sigma_{\mathrm{d}}(P_C)$, *then for every compact set* $K \subseteq X$ *and every* $f \in L^1(X, \mathcal{B}, \mu)$ *we have*

$$\lim_{n\to\infty} \int_K P^n f \, d\mu = \int_K Sf \, d\mu,$$

where $S : L^1(\mu) \to L^1_*(\mu)$ *is a substochastic projection onto the sublattice of* P*-invariant functions. Moreover*, *on* $L^1(F, \mathcal{B}_F, \mu|_F)$ *the above convergence is in the* L^1 *norm.*

REMARK 2.7. Note that if P is (PSFS) then the singular parts of $P^n\nu$ tend to 0 (in the variation norm). Thus in the above theorem f may be replaced by any bounded signed measure ν on $(X, \mathcal{B})$. Clearly in that case the substochastic projection S maps $M(X)$ to $L^1_*(\mu)$.

3. (PSFS) operators on ordered phase spaces. In this section we apply the abstract theorem of the previous section to a special class of (PSFS) operators. Even though they are more general than LMT we still get their weak asymptotic stability. The phase space of LMT operators is the nonnegative half-line $[0, \infty)$. Here the phase spaces (X, d) have both a metric structure and are partially ordered by a relation $\preceq$ (if $x \preceq y$ and $x \neq y$ then we write $x \prec y$). This extension seems justified because of possible applications in cell models. Here we may think of multiparameter models with parameters (well) ordered according to their influence on the growth of the cell. Therefore we will assume that

(i) $\preceq$ is linear,
(ii) each closed subset $F \subseteq X$ has a least element (denoted by $\bigwedge(F)$).

Of course $\mathbb{R}_+$ with ordinary order and topology satisfies the above conditions. Another natural case is

EXAMPLE 3.1. Let $X = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq 0 \text{ and } \sum_{j=2}^n x_j^2 \leq r(x_1)\}$, where r is a continuous function defined on $[0, \infty)$ satisfying $r(0) = 0$ and $r(x) > 0$ for $x > 0$. The order $\preceq$ on X can be taken lexicographical.

Returning to general considerations, for each $\mu \in \mathcal{M}(X)$ we set $\bigwedge(\mu) = \bigwedge(\operatorname{supp} \mu)$, where as before $\operatorname{supp} \mu$ stands for the topological support of the measure μ (for a density $f \in \mathcal{D}$ we set $\bigwedge(f) = \bigwedge(\operatorname{supp} f \, d\mu)$).

DEFINITION 3.2. We say that the transition probabilities of a Markov operator P acting on $L^1(\mu)$ have *nondecreasing lower bounds* (abbreviated to (NLB)) if

$$x \preceq y \ \Rightarrow\ \bigwedge(P(x,\cdot)) \preceq \bigwedge(P(y,\cdot)). \tag{6}$$

EXAMPLE 3.3. LMT operators are (NLB). This follows easily from the monotonicity of the functions Q, λ and H. In fact, $\bigwedge(P(y,\cdot))$ is the smallest $x \in \mathbb{R}_+$ such that $\lambda(x) \le y$ and $H(Q(\lambda(x+h))-Q(y)) < H(Q(\lambda(x))-Q(y))$ for any $h > 0$.

For general $(X, \preceq)$ the condition (NLB) imposed on transition probabilities is not very useful in our proofs. Therefore we introduce

DEFINITION 3.4. We say that a Markov operator P acting on $L^1(\mu)$ *preserves the order of lower bounds of densities* (abbreviated to (NLBD)) if for any pair of densities $f_1, f_2 \in \mathcal{D}$ we have

$$\bigwedge(f_1) \preceq \bigwedge(f_2) \ \Rightarrow\ \bigwedge(Pf_1) \preceq \bigwedge(Pf_2). \tag{7}$$

THEOREM 3.5. *Let P be a Markov operator on $L^1(X,\mathcal{B},\mu)$, where $(X, \mathrm{d}, \preceq)$ is a metric ordered space and μ is a fixed σ-finite measure. Further, assume that closed balls in X are compact and $\preceq$ satisfies conditions* (i) *and* (ii) *above. If P is* (PSFS) *and conditions* (NLBD) *and* (SCJ) *hold then for every compact set $K \subseteq X$ and every $f \in L^1(X,\mathcal{B},\mu)$ we have*

$$\lim_{n\to\infty} \int_K P^n f\, d\mu = \int_K Sf\, d\mu,$$

where $S : L^1(\mu) \to L^1_*(\mu)$ is a substochastic projection onto the sublattice of P-invariant functions. Moreover, on $L^1(F, \mathcal{B}_F, \mu|_F)$ the above convergence is in the L^1 norm.

Proof. It is sufficient to show that $\Sigma_{\mathrm{i}}(P_C) = \Sigma_{\mathrm{d}}(P_C)$. Without loss of generality we may assume that P is conservative ($C = X$). Suppose that there are atoms B_j ($j = 0, 1, \ldots, d-1$) such that $P^{*n}\mathbf{1}_{B_j} = \mathbf{1}_{B_{j+n}}$, where $j+n$ is understood mod d. It follows from the (PSFS) condition that

$$P^*\mathbf{1}_{B_j}(x) = P(x, B_j) = Q(x, B_j) + R(x, B_j) > 0$$

for x from some open neighbourhood of $\overline{B}^{\mathrm{ess}}_{j+1}$. Hence the $\overline{B}^{\mathrm{ess}}_j$ are pairwise disjoint as $P^*\mathbf{1}_{B_j}(x) = 0$ for μ-almost all $x \in \bigcup_{k\ne j+1} B_k$. Define $c_j = \bigwedge(\overline{B}^{\mathrm{ess}}_j) = \bigwedge(f_j)$, where $f_j \in \mathcal{D}$ is a density with $\operatorname{supp} f = \overline{B}^{\mathrm{ess}}_j$. It follows that all c_j must be distinct. We note that $\operatorname{supp}(Pf_j) = \operatorname{supp} f_{j-1}$. Indeed,

$$\int_{B_{j-1}} Pf_j\, d\mu = \int f_j P^*\mathbf{1}_{B_{j-1}}\, d\mu = \int f_j \mathbf{1}_{B_j}\, d\mu = 1$$

and for every $A \subseteq B_{j-1}$ of positive measure, $\int_A Pf_j\, d\mu = \int f_j P^*\mathbf{1}_A\, d\mu > 0$ (see [KL]). Suppose that $d > 1$. Without loss of generality we may assume that $c_0 \prec \min\{c_j : j = 1, \ldots, d-1\}$. It follows that $\bigwedge(Pf_0) \succeq \min\{c_j : j = 1, \ldots, d-1\}$. Applying this argument several times we obtain $\bigwedge(P^n f_0) \succeq c_0$ for all n, contrary to $\operatorname{supp}(P^d f_0) = \overline{B}^{\mathrm{ess}}_0$. ∎

By showing that LMT operators do not allow cycles ($\Sigma_{\mathrm{i}}(P) = \Sigma_{\mathrm{d}}(P)$) it has been proved in [B1] that a general LMT operator P has convergent iterates (in the weak* operator topology). The proof relied on algebraic properties of LMT kernels (actually the (NLB) property was exploited). In this section we obtain the weak asymptotic stability of (PSFS)+(NLB) operators, generalizing the results of [B1] (and its predecessors [BL], [B2], [GL], [KM], [LMT], [M], and [T]).

LEMMA 3.6. *Let $(X, \mathrm{d}, \mu, \preceq)$ be a metric space such that all closed balls are compact and $\preceq$ satisfies* (i) *and* (ii). *If a regular Markov operator $P : L^1(\mu) \to L^1(\mu)$ is Feller and has the* (NLB) *property then for each density $f \in \mathcal{D}$ we have*

$$\bigwedge(Pf) \preceq \bigwedge(P(\bigwedge(f), \cdot)).$$

Proof. For each $z < \bigwedge(Pf)$ there exists $r_z > 0$ such that

$$0 = \int\limits_{\overline{K}(z,r_z)} Pf\, d\mu(x) = \int P(x, \overline{K}(z, r_z)) f(x)\, d\mu.$$

It follows that $P(x, \overline{K}(z, r_z)) = 0$ for μ-almost all $x \in \operatorname{supp} f$. The Feller property implies that $P(x, \overline{K}(z, r_z)) = 0$ for all $x \in \operatorname{supp} f$. In particular,

$$P(\bigwedge(f), \overline{K}(z, r_z)) = 0.$$

The order interval $I = \{z \in X : z \prec \bigwedge(f)\}$ (if nonempty) may be covered by $\bigcup_{z\in I} K(z, r_z) = \mathcal{U}_I$. Clearly $\mathcal{U}_I$, as an open subset of a separable metric space, is a countable union of base sets (say some balls $K(z, r_z)$). In particular, $P(\bigwedge(f), \mathcal{U}_I) = 0$. We conclude that $\bigwedge(P(\bigwedge(f), \cdot)) \succeq \bigwedge(Pf)$. ■

In general, conditions (6) and (7) are not even comparable. However, if $X = \mathbb{R}_+$ then we have

LEMMA 3.7. *Let $P : L^1(\mathbb{R}_+) \to L^1(\mathbb{R}_+)$, where $\mathbb{R}_+$ is equipped with the ordinary metric $|\cdot|$ and order $\le$, be a regular Markov operator. If P is Feller and* (NLB) *holds then for each density $f \in \mathcal{D}$ we have*

$$\bigwedge(Pf) = \bigwedge(P(\bigwedge(f), \cdot)). \tag{7}$$

In particular (NLBD) *holds.*

Proof. We only have to show that $\bigwedge(Pf) \ge \bigwedge(P(\bigwedge(f), \cdot))$. Suppose $a = \bigwedge(Pf) < \bigwedge(P(\bigwedge(f), \cdot)) = b$. For some $r > 0$ we have $P(\bigwedge(f), [0, a+r]) = 0$. It follows from the (NLB) assumption that $\bigwedge P(x, \cdot) \ge a+r$ for all $x \ge \bigwedge(f)$. In particular, $P(x, [0, a + r/2]) = 0$ for all $x \in \operatorname{supp} f$. We obtain

$$\int\limits_{[a-r/2,a+r/2]} Pf(x)\, dx = \int f(x) P(x, [a - r/2, a + r/2])\, dx = 0,$$

contradicting $a \in \operatorname{supp}(Pf)$. ■

Combining Theorem 3.5 with the above lemma we obtain

THEOREM 3.8. *Let P be an LMT Markov operator on $L^1(\mathbb{R}_+)$, where $\mathbb{R}_+$ is equipped with the ordinary metric, order $\leq$, and Lebesgue measure λ. Then for every compact set $K \subseteq \mathbb{R}_+$ and every $f \in L^1(\mathbb{R}_+)$ we have*

$$\lim_{n\to\infty} \int_K P^n f\, d\lambda = \int_K Sf\, d\lambda,$$

where $S : L^1(\mathbb{R}_+) \to L^1_*(\mathbb{R}_+)$ is a substochastic projection onto the sublattice of P-invariant functions. Moreover, on $L^1(F)$ the above convergence is in the L^1 norm.

Final remarks. As mentioned before, conditions (NLB) and (NLBD) are not comparable. To see this let $X = [0,1]$ be equipped with the ordinary metric and order structure, and let λ denote the Lebesgue measure restricted to X. We define kernel transition probabilities by

$$q(x,y) = \begin{cases} y\mathbf{1}_{[0,1]}(x) + (1-y)\mathbf{1}_{[1,2]}(x) & \text{if } 0 \leq y \leq 1, \\ \mathbf{1}_{[0,1]}(x) & \text{if } y \in [1,2]. \end{cases} \tag{8}$$

It is clear that the Markov operator P defined by the above kernel satisfies condition (SFS) and $\bigwedge(Pf d\lambda) = 0$ for each density f. In particular, condition (NLBD) is satisfied. On the other hand,

$$\bigwedge(P(0,\cdot)) = 1 > \bigwedge(P(x,\cdot)) = 0$$

for each $0 < x \leq 2$. Hence (NLB) does not hold.

In order to show that (NLB) does not imply (NLBD) consider $X = \{(x,y) \in \mathbb{R}^2 : x \geq 0,\ 0 \leq y \leq x\}$ equipped with the ordinary 2-dimensional Euclidean metric, lexicographical order $\preceq$ and the Lebesgue measure λ_2 restricted to X. Let g be any density concentrated on the region

$$D = \{(x,y) \in X : x \geq 1,\ y \geq 2 - x,\ 0 \leq y \leq x\}$$

with unbounded support (for instance take $g(u,v) = \frac{\text{const}}{1+(u^2+v^2)^2}\mathbf{1}_D((u,v))$). For $x > 1$ we define the triangle ACD_x, where $A = (1,1)$, $C = (2,0)$, and $D_x = (1 + 1/x, 0)$. Let

$$g_x(\cdot) = \frac{2}{1 - 1/x}\,\mathbf{1}_{\triangle ACD_x}(\cdot)$$

denote a uniform density on this triangle. We define a Markov operator P on $L^1(X, \lambda_2)$ by the kernel

$$q(\cdot,(x,y)) = \begin{cases} g(\cdot) & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq x, \\ \dfrac{1}{x}g(\cdot) + \left(1 - \dfrac{1}{x}\right)g_x(\cdot) & \text{if } x > 1 \text{ and } 0 \leq y \leq x. \end{cases} \tag{9}$$

We notice that $\bigwedge(P((x,y),\cdot)) = (1,1)$ for any $(x,y) \in X$, hence (NLB) holds. On the other hand, if f_1 is a uniformly distributed density on the triangle $\{(x,y) : 0 \le x \le 1, 0 \le y \le x\}$ and f_2 is any density with unbounded support then $\bigwedge(f_1) = (0,0) \preceq \bigwedge(f_2)$. It follows from (9) that $\bigwedge(Pf_1) = (1,1) \succ (1,0) = \bigwedge(Pf_2)$, hence (NLBD) is not satisfied.

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