COLLOQUIUM MATHEMATICUM

VOL. 101 2004 NO. 1

STRONG CONTINUITY OF INVARIANT PROBABILITY CHARGES

BY

HARALD LUSCHGY (Trier) and SŁAWOMIR SOLECKI (Urbana, IL)

Abstract. Consider a semigroup action on a set. We derive conditions, in terms of the induced action of the semigroup on $\{0,1\}$-valued probability charges, which ensure that all invariant probability charges are strongly continuous.

1. Introduction. Let $\mathcal{A}$ be an algebra of subsets of some set X and let G be a semigroup acting from the left on X. Assume that $\mathcal{A}$ is G-invariant, that is, $g^{-1}A = \{x \in X : gx \in A\} \in \mathcal{A}$ for every $g \in G$, $A \in \mathcal{A}$. By a *probability charge* (p-charge) on $\mathcal{A}$ we mean a finitely additive probability measure on $\mathcal{A}$. In this setting a p-charge μ on $\mathcal{A}$ is said to be *G-invariant* of $g\mu = \mu$ for every $g \in G$, where $g\mu(A) = \mu(g^{-1}A)$. The p-charge μ is called *strongly continuous* if for every $\varepsilon > 0$ there exists a finite partition $\{A_1, \ldots, A_n\}$ of X in $\mathcal{A}$ with $\sup_{1\le i\le n} \mu(A_i) \le \varepsilon$.

In this paper, we are interested in conditions on $\mathcal{A}$, G, and the action of G on X which ensure that invariance of a p-charge implies its strong continuity. Note that a p-charge μ defined on a σ-algebra is strongly continuous if and only if μ is *strongly nonatomic*, i.e. $\{\mu(B) : B \in \mathcal{A}, B \subset A\} = [0, \mu(A)]$ for every $A \in \mathcal{A}$ (cf. [3, 5.1.6, 11.4.5]). A further interesting feature of strongly continuous p-charges can be found in [10, Theorem 3]. For existence of invariant p-charges the reader is referred to Wagon [16] and Paterson [12].

2. Main results. For p-charges λ and μ on $\mathcal{A}$ we write $\lambda \ll_w \mu$ if λ is weakly absolutely continuous with respect to μ, i.e. $\lambda(A) = 0$ whenever $A \in \mathcal{A}$ and $\mu(A) = 0$. The following theorem is fundamental to the other results in the paper.

THEOREM 1. *Let μ be a G-invariant p-charge on $\mathcal{A}$. If the orbit $G\lambda = \{g\lambda : g \in G\}$ is infinite for every $\{0,1\}$-valued p-charge λ on $\mathcal{A}$ with $\lambda \ll_w \mu$, then μ is strongly continuous.*

2000 *Mathematics Subject Classification*: 28A12, 43A07.

Key words and phrases: semigroup actions, invariant charges, strongly continuous charges, strongly nonatomic charges.

Research of S. Solecki supported by NSF grant DMS-0400931.

We present two proofs of the above result: the first one based on representing a p-charge as the barycenter of a regular probability measure on the space of all $\{0,1\}$-valued p-charges, and the second one of an elementary nature.

The first proof requires two lemmas. Let K denote the set of all p-charges on $\mathcal{A}$ equipped with the weak* topology and H the set of all $\{0,1\}$-valued p-charges on $\mathcal{A}$. Since K is a (compact) Bauer simplex and H is the extreme boundary of K, the map

$$K \to M^1(H), \qquad \mu \mapsto \varrho_\mu,$$

with

$$\mu(A) = \int_H \lambda(A)\, d\varrho_\mu(\lambda), \qquad A \in \mathcal{A},$$

is an affine homeomorphism, where $M^1(H)$ denotes the set of all regular probability measures on the Borel σ-algebra $\mathcal{B}(H)$ of H equipped with the vague topology (cf. [2, II. 4.2]). The semigroup G acts from the left on H by $\lambda \mapsto g\lambda$. This action is (weak*-)continuous and thus $\mathcal{B}(H)$ is G-invariant. The induced left action of G on $M^1(H)$ is given by

$$\varrho \mapsto g\varrho, \qquad g\varrho(B) = \varrho(g^{-1}B), \qquad B \in \mathcal{B}(H),$$

where regularity of $g\varrho$ can be seen as follows. For every $B \in \mathcal{B}(H)$, we have

$$\begin{aligned} g\varrho(B) &= \sup\{\varrho(F) : F \text{ compact}, F \subset g^{-1}B\} \\ &\le \sup\{\varrho(F) : F \text{ compact}, gF \subset B\} \\ &\le \sup\{\varrho(g^{-1}gF) : F \text{ compact}, gF \subset B\} \\ &\le \sup\{g\varrho(D) : D \text{ compact}, D \subset B\} \le g\varrho(B), \end{aligned}$$

which implies

$$g\varrho(B) = \inf\{g\varrho(O) : O \text{ open}, O \supset B\}.$$

LEMMA 1. *$K \to M^1(H)$, $\mu \mapsto \varrho_\mu$ is G-equivariant, that is, $\varrho_{g\mu} = g\varrho_\mu$ for every $g \in G, \mu \in K$.*

Proof. Let $\mathcal{C}$ denote the algebra of subsets of H which are both compact and open. It is well known that $\varphi : \mathcal{A} \to \mathcal{C}$, $\varphi(A) = \{\lambda \in H : \lambda(A) = 1\}$, is an isomorphism of algebras. (Note that $\varphi(A)$ coincides with the weak* closure of $\{\varepsilon_x : x \in A\}$.) We have $\mu(\varphi^{-1}C) = \varrho_\mu(C)$ and $\varphi^{-1}(g^{-1}C) = g^{-1}(\varphi^{-1}C)$, which gives

$$g\varrho_\mu(C) = \varrho_\mu(g^{-1}C) = \mu(g^{-1}(\varphi^{-1}C)) = g\mu(\varphi^{-1}C) = \varrho_{g\mu}(C)$$

for every $C \in \mathcal{C},\ g \in G$. Since $\mathcal{C}$ is a basis of the weak* topology on H and $g\varrho_\mu$ and $\varrho_{g\mu}$ are regular, this yields $g\varrho_\mu = \varrho_{g\mu}$. ∎

For $\varrho \in M^1(H)$, let $\mathrm{supp}(\varrho)$ denote the topological support of ϱ. Part (c) of the following lemma is well known.

LEMMA 2. *Let $\mu \in K$.*

(a) $\operatorname{supp}(\varrho_\mu) = \{\lambda \in H : \lambda \ll_w \mu\}$.

(b) $\{\lambda \in H : \varrho_\mu(\{\lambda\}) > 0\} = \{\lambda \in H : c_\lambda \lambda \le \mu$ *for some* $c_\lambda > 0\}$.

(c) *μ is strongly continuous if and only if ϱ_μ is continuous.*

Proof. Let $\mathcal{C}$ and φ be defined as in the proof of Lemma 1.

(a) Let $\lambda \in H$. Then $\lambda \in \operatorname{supp}(\varrho_\mu)$ if and only if $\varrho_\mu(C) > 0$ for every $C \in \mathcal{C}$ with $\lambda \in C$. Since $\varrho_\mu(\varphi(A)) = \mu(A)$, $A \in \mathcal{A}$, the latter condition means $\mu(A) > 0$ for every $A \in \mathcal{A}$ with $\lambda(A) = 1$ and this is equivalent to $\lambda \ll_w \mu$.

(b) For $\lambda \in H$, let $\mathcal{A}_\lambda = \{A \in \mathcal{A} : \lambda(A) = 1\}$. Then $\bigcap_{A \in \mathcal{A}_\lambda} \varphi(A) = \{\lambda\}$ and regularity of ϱ_μ gives

$$\varrho_\mu(\{\lambda\}) = \inf_{A \in \mathcal{A}_\lambda} \varrho_\mu(\varphi(A)) = \inf_{A \in \mathcal{A}_\lambda} \mu(A).$$

If $\varrho(\{\lambda\}) > 0$, choose $c_\lambda = \inf_{A \in \mathcal{A}_\lambda} \mu(A)$. Then $c_\lambda \lambda \le \mu$. If $c_\lambda \lambda \le \mu$ for some $c_\lambda > 0$, then $\mu(A) \ge c_\lambda$ for every $A \in \mathcal{A}_\lambda$ and hence $\varrho_\mu(\{\lambda\}) > 0$.

(c) follows e.g. from (b) and the fact that μ is strongly continuous if and only if the right hand side of (b) is empty (cf. e.g. [11, Example (2)]). ∎

First proof of Theorem 1. By Lemma 1, ϱ_μ is G-invariant. Therefore

$$\varrho_\mu(\{g\lambda\}) = \varrho_\mu(g^{-1}\{g\lambda\}) \ge \varrho_\mu(\{\lambda\})$$

for every $\lambda \in H$, $g \in G$. The assumption yields $\varrho_\mu(\{\lambda\}) = 0$ for every $\lambda \in H$ with $\lambda \ll_w \mu$. Hence, according to Lemma 2(a), ϱ_μ is continuous. The assertion follows from Lemma 2(c). ∎

Second proof of Theorem 1. Fix a p-charge μ on an algebra $\mathcal{A}$ of subsets of X. For a p-charge λ on $\mathcal{A}$, we write $\lambda \ll \mu$ if for any $\varepsilon > 0$ there is $\delta > 0$ such that for $A \in \mathcal{A}$, $\mu(A) < \delta$ implies $\lambda(A) < \varepsilon$. This notion is, of course, stronger than $\lambda \ll_w \mu$. Note that for a $\{0,1\}$-valued p-charge λ, $\lambda \ll \mu$ holds precisely when $b(\lambda) > 0$, where

$$b(\lambda) = \inf\{\mu(A) : A \in \mathcal{A} \text{ and } \lambda(A) = 1\}.$$

We will start with two claims which do not involve semigroup actions.

CLAIM 1. *If μ is not strongly continuous, then there exists a $\{0,1\}$-valued p-charge $\lambda \ll \mu$.*

Proof of Claim 1. We say that $A \in \mathcal{A}$ is *ε-covered* if A can be covered by finitely many sets from $\mathcal{A}$ of μ-measure $< \varepsilon$. By assumption for some $\varepsilon > 0$, which we fix, X is not ε-covered so the infimum in the following definition makes sense. Let $b = \inf\{\mu(A) : A \in \mathcal{A}$ is not ε-covered$\}$. Note that $b \ge \varepsilon > 0$. Let $A_0 \in \mathcal{A}$ be such that it cannot be ε-covered and $\mu(A_0) < 2b$. Define

$$\mathcal{I} = \{A \in \mathcal{A} : A \cap A_0 \text{ is } \varepsilon\text{-covered}\}.$$

Now $X \notin \mathcal{I}$, and $\mathcal{I}$ is closed under taking subsets which are in $\mathcal{A}$ and under finite unions. If $A \in \mathcal{A}$, then, since $\mu(A_0) < 2b$, we have either $\mu(A \cap A_0) < b$ or $\mu((X \setminus A) \cap A_0) < b$. Thus, either $A \cap A_0$ or $(X \setminus A) \cap A_0$ is ε-covered, that is, either $A \in \mathcal{I}$ or $X \setminus A \in \mathcal{I}$. This shows that λ defined by $\lambda(A) = 0$ if $A \in \mathcal{I}$ and $\lambda(A) = 1$ if $A \in \mathcal{A} \setminus \mathcal{I}$ is a $\{0,1\}$-valued p-charge. Since no set in $\mathcal{A} \setminus \mathcal{I}$ is ε-covered, $b(\lambda) \geq b > 0$.

CLAIM 2. *For any $b > 0$ there are at most finitely many $\{0,1\}$-valued p-charges λ with $b(\lambda) > b$.*

Proof of Claim 2. Given $b > 0$, let n_b be the smallest natural number with $n_b b > 1$. Then, obviously, there is a small enough $\delta > 0$ such that $1 < n_b b - (n_b(n_b+1)/2)\delta$. Therefore, using the formula

$$\mu\Big(\bigcup_{i\leq n} A_i\Big) \geq \sum_{i\leq n} \mu(A_i) - \sum_{i<j\leq n} \mu(A_i \cap A_j),$$

we can find a $\delta > 0$ such that for any sequence $A_n \in \mathcal{A}$, $n \in \mathbb{N}$, with $\mu(A_n) > b$ there are $i < j$ with $\mu(A_i \cap A_j) \geq \delta$.

Let λ_n, $n \in \mathbb{N}$, be pairwise distinct $\{0,1\}$-valued p-charges with $b(\lambda_n) > b$. We can assume that for for some $b' \geq b$ and all n, $b' < b(\lambda_n) < b' + \delta$. Now, for each i and n with $i < n$, we can find $A_{i,n} \in \mathcal{A}$ with $\lambda_i(A_{i,n}) = 0$ and $\lambda_n(A_{i,n}) = 1$. If we let $A_n = \bigcap_{i<n} A_{i,n}$, then $A_n \in \mathcal{A}$ with $\lambda_n(A_n) = 1$ and $\lambda_i(A_n) = 0$ for all $i < n$. Since $b(\lambda_n) < b' + \delta$, we can assume that $\mu(A_n) < b' + \delta$ by intersecting A_n with an $A \in \mathcal{A}$ such that $\lambda_n(A) = 1$ and $\mu(A) < b' + \delta$. Note that $\mu(A_n) \geq b(\lambda_n) > b'$ for all n, which, by the choice of δ, gives two natural numbers $i < j$ such that $\mu(A_i \cap A_j) > \delta$. Then

$$\mu(A_i \setminus (A_i \cap A_j)) < (b' + \delta) - \delta < b(\lambda_i).$$

Thus, $\lambda_i(A_i \setminus (A_i \cap A_j)) = 0$, whence $\lambda_i(A_i \cap A_j) = 1$, contradicting $\lambda_i(A_j) = 0$.

To prove Theorem 1 from the above claims, we proceed as follows. If a semigroup G acts on X with $\mathcal{A}$ being G-invariant and μ being G-invariant and not strongly continuous, then by Claim 1 there exists a $\{0,1\}$-valued p-charge $\lambda \ll \mu$. Obviously, we then have $\lambda \ll_w \mu$. Moreover, $b(\lambda) > 0$ and, by G-invariance of μ, $b(g\lambda) \geq b(\lambda)$ for each $g \in G$. Thus, $G\lambda$ is finite by Claim 2. ∎

Under a suitable condition on (the action of) the semigroup G, infinity of the orbits $G\lambda$, $\lambda \in H$, is also a necessary condition for strong continuity of all G-invariant p-charges on $\mathcal{A}$. (See, however, Remark 1(a) below.) The equivalence of statements (i) and (ii) of the following theorem is due to Adler [1]. We give a different proof of it based on the measure ϱ_μ. The following condition will be used:

(C) $$gG\lambda = G\lambda \quad \text{for every } \lambda \in H,\ g \in G.$$

THEOREM 2. *The following statements are equivalent.*

(i) *Every G-invariant p-charge on $\mathcal{A}$ is strongly continuous.*

(ii) *There is no G-invariant p-charge on $\mathcal{A}$ of the form $n^{-1}\sum_{i=1}^{n}\lambda_i$ with $n\in\mathbb{N}$, $\lambda_1,\dots,\lambda_n\in H$.*

If condition (C) *is satisfied or if G is left amenable* (*as discrete semigroup*), *then* (i) *is also equivalent to*

(iii) *$G\lambda$ is infinite for every $\lambda\in H$.*

Proof. (i)⇒(ii) is obvious and (iii)⇒(ii) follows from Theorem 1.

(ii)⇒(i). Assume that there exists a G-invariant p-charge μ on $\mathcal{A}$ which is not strongly continuous. Since ϱ_μ is G-invariant by Lemma 1, we have

$$\varrho_\mu(gH_0)=\varrho_\mu(g^{-1}(gH_0))\ge\varrho_\mu(H_0)$$

for every finite subset H_0 of H and $g\in G$. Let $a=\max\{\varrho_\mu(\{\lambda\}):\lambda\in H\}$ and

$$H_0=\{\lambda\in H:\varrho_\mu(\{\lambda\})=a\}.$$

By Lemma 2(c), $a>0$ and thus H_0 is a nonempty finite subset of H. Let $g\in G$. Since

$$a\ge\varrho_\mu(\{g\lambda\})\ge\varrho_\mu(\{\lambda\})=a$$

for every $\lambda\in H_0$, one obtains $gH_0\subset H_0$. In view of $\varrho_\mu(gH_0)\ge\varrho_\mu(H_0)$, this implies $\varrho_\mu(gH_0)=\varrho_\mu(H_0)$, which gives $gH_0=H_0$. It follows that the p-charge $|H_0|^{-1}\sum_{\lambda\in H_0}\lambda$ on $\mathcal{A}$ is G-invariant.

(i)⇒(iii). Suppose that $G\lambda$ is finite for some $\lambda\in H$. Under condition (C), the p-charge $|G\lambda|^{-1}\sum_{\nu\in G\lambda}\nu$ on $\mathcal{A}$ is G-invariant and not strongly continuous. Now let G be a left amenable semigroup. The convex hull $\mathrm{co}(G\lambda)$ of the finite orbit $G\lambda$ is a weak*-compact subset of K. Furthermore, $\mathrm{co}(G\lambda)$ is G-invariant, i.e. $g\,\mathrm{co}(G\lambda)\subset\mathrm{co}(G\lambda)$ for every $g\in G$ and the left action of G on K is weak*-continuous. Since G is left amenable, Day's fixed point theorem (cf. [12, 1.14]) yields a G-invariant charge in $\mathrm{co}(G\lambda)$. ∎

REMARK 1. (a) It is not possible to replace (iii) in Theorem 2 with the weaker condition saying that all orbits of the action of G on X are infinite. To see this, consider $G=\mathbb{Z}$, the integers with addition, acting on $X=\mathbb{Z}$ by translations, $\mathcal{A}$ consisting of all finite and all cofinite subsets of $\mathbb{Z}$, and μ defined on $\mathcal{A}$ to be 0 on finite sets and 1 on cofinite sets. Then orbits of the $\mathbb{Z}$-action on X are infinite and yet (i) fails as shown by μ.

(b) Condition (C) or amenability cannot be dispensed with in the proof of (i)⇒(ii) in Theorem 2. This is shown by the semigroup G consisting of the two constant maps from $\{0,1\}$ to $\{0,1\}$ taken with composition and acting on $X=\{0,1\}$ with $\mathcal{A}=\mathcal{P}(\{0,1\})$ in the natural way. For this action (i) holds (as there are no G-invariant p-charges) and (iii) fails.

(c) If G is a right simple semigroup, i.e. $gG = G$ for every $g \in G$, and in particular, if G is a group, then condition (C) is satisfied.

3. Applications. In case $\mathcal{A} = \mathcal{P}(X)$, the following lemma [4, Theorem 3.3] provides an important tool to verify that orbits $G\lambda$ are infinite. For $g \in G$, let X_g denote the set of g-fixed points in X, i.e. $X_g = \{x \in X : gx = x\}$. (This lemma is proved in [4] for a countable set X but the argument there works for an arbitrary X.)

LEMMA 3. *Let λ be a $\{0,1\}$-valued p-charge on $\mathcal{P}(X)$ and $g \in G$. Then $g\lambda = \lambda$ if and only if $\lambda(X_g) = 1$.*

A first application is as follows. Let X be a right cancellative semigroup (i.e. $x_1 y = x_2 y$ implies $x_1 = x_2$) and $\mathcal{A} = \mathcal{P}(X)$. Then left multiplication by $g \in X$ has no fixed point if and only if g is not idempotent. Let $g \in X$ be of infinite order and $G = \{g^n : n \in \mathbb{N}\}$. Then G acts on X by left multiplication. By Lemma 3, $g^n\lambda = g^{n-m}g^m\lambda \neq g^m\lambda$ for every $\lambda \in H$, $n > m \geq 1$. Therefore, $G\lambda$ is infinite for every $\lambda \in H$. It follows from Theorem 1 that every G-left invariant p-charge on $\mathcal{P}(X)$ is strongly continuous (see Granirer [8, p. 387] for a related result). This extends a corresponding result for groups by Francke *et al.* [7].

The following two corollaries provide extensions of a result of Chou [6]; see also Stroetman [15]. In case G is a group, a G-invariant p-charge μ on $\mathcal{P}(X)$ is called *aperiodic* if $\mu(X_g) = 0$ for every $g \in G$, $g \neq e$.

COROLLARY 1. *Let G be an infinite group and $\mathcal{A} = \mathcal{P}(X)$. Then every G-invariant aperiodic p-charge on $\mathcal{P}(X)$ is strongly continuous.*

Proof. Let μ be a G-invariant aperiodic p-charge on $\mathcal{P}(X)$ and let $\lambda \in H$ with $\lambda \ll_w \mu$. Then $\lambda(X_g) = 0$ for every $g \in G$, $g \neq e$, and therefore, by Lemma 3, the isotropy group of λ in G satisfies $G_\lambda = \{e\}$. Since G is infinite, this implies that the orbit $G\lambda$ is infinite. It follows from Theorem 1 that μ is strongly continuous. ■

COROLLARY 2. *Let G be a group and $\mathcal{A} = \mathcal{P}(X)$. Let F denote the subgroup of G generated by $\bigcup_{x \in X} G_x$, where G_x denotes the isotropy group of x in G. Consider the following statements*:

(i) *Every G-invariant p-charge on $\mathcal{P}(X)$ is strongly continuous.*
(ii) *G/F is infinite.*

Then (ii) *implies* (i) *and if F/G_x is finite for some $x \in X$,* (i) *and* (ii) *are equivalent.*

Proof. (ii)⇒(i). Let $g \in G$, $g \notin F$. Then $X_g = \emptyset$ and by Lemma 3, $g \notin \bigcup_{\lambda \in H} G_\lambda$. We obtain $G_\lambda \subset F$ for every $\lambda \in H$. By (ii), this implies that G/G_λ is infinite for every $\lambda \in H$. The assertion follows from Theorem 1.

(i)⇒(ii). By Theorem 2, $G\varepsilon_x$ is infinite for every $x \in X$, where ε_x is the $\{0,1\}$-valued p-charge concentrated on x. Hence G/G_x is infinite for every $x \in X$. Since $|G/G_x| = |G/F| \cdot |F/G_x|$ and F/G_x is finite for some $x \in X$, this implies that G/F is infinite. ∎

REMARK 2. (a) Let G be a group and let μ be a G-invariant p-charge on $\mathcal{A}$. In case $\mathcal{A} = \mathcal{P}(X)$, μ is aperiodic if and only if ϱ_μ is aperiodic, i.e., $\varrho_\mu(H_g) = 0$ for every $g \in G$, $g \neq e$, where $H_g = \{\lambda \in H : g\lambda = \lambda\}$. In fact, by Lemma 3,

$$\mu(X_g) = \varrho_\mu(\{\lambda \in H : \lambda(X_g) = 1\}) = \varrho_\mu(H_g).$$

For $\mathcal{A} \neq \mathcal{P}(X)$ the following version of Corollary 1 holds. Let G be an infinite group. Then every G-invariant p-charge μ on $\mathcal{A}$ with aperiodic measure ϱ_μ is strongly continuous. To see this assume $\varrho_\mu(\{\lambda\}) > 0$ for some $\lambda \in H$. Then $G\lambda$ is finite and hence $G_\lambda \neq \{e\}$. So $\lambda \in H_g$ for some $g \neq e$, which gives $\varrho_\mu(H_g) > 0$.

(b) The example in Remark 1(a) shows that Corollaries 1 and 2 are not valid, in general, for $\mathcal{A} \neq \mathcal{P}(X)$. It is possible to relate the general case $\mathcal{A} \neq \mathcal{P}(X)$ to the case $\mathcal{A} = \mathcal{P}(X)$. Let μ be a p-charge on $\mathcal{A}$. Then μ is strongly continuous if and only if every extension of μ to a p-charge on $\mathcal{P}(X)$ is strongly continuous (cf. [13]). However, for our purposes this result has the disadvantage that for invariant charges μ it is not enough to consider only invariant extensions of μ as is again illustrated by the example in Remark 1(a): the group $G = \mathbb{Z}$ is Abelian, therefore, there are many extensions of μ to a G-invariant p-charge on $\mathcal{P}(\mathbb{Z})$; by Corollary 1 (or Corollary 2), these extensions are strongly continuous and yet μ is not.

The following corollary provides an extension of a result of Snell [14]. Recall that a group F equipped with a σ-algebra $\mathcal{B}$ is called *Lusin measurable* if $F \times F \to F$, $(g,h) \mapsto gh^{-1}$, is $(\mathcal{B} \otimes \mathcal{B}, \mathcal{B})$-measurable and the measurable space $(F,\mathcal{B})$ is isomorphic to a Polish space equipped with its Borel σ-algebra.

COROLLARY 3. *Let G be an infinite subsemigroup of a Lusin measurable group $(F,\mathcal{B})$ and let $\mathcal{A} = G \cap \mathcal{B}$. Assume that there exists a non-trivial σ-finite F-left quasi-invariant measure on $\mathcal{B}$. Then every G-left invariant p-charge on $\mathcal{A}$ is strongly continuous.*

Proof. According to a result of Mackey, there is a topology on F such that F becomes a locally compact group and $\mathcal{B}$ is the Borel σ-algebra (cf. [5]). Now it follows from [9, Theorem 3] that there is no G-left invariant p-charge in $\mathrm{co}(H)$. By Theorem 2, this gives the assertion. ∎

REFERENCES

[1] A. Adler, *On the continuity of invariant measures*, Proc. Amer. Math. Soc. 41 (1973), 487–491.

[2] E. M. Alfsen, *Compact Convex Sets and Boundary Integrals*, Springer, New York, 1971.

[3] K. P. S. Bhaskara Rao and M. Bhaskara Rao, *Theory of Charges*, Academic Press, New York, 1983.

[4] D. Booth, *Ultrafilters on a countable set*, Ann. Math. Logic 2 (1970/1971), 1–24.

[5] Q. L. Burrell, *Standard groups with a quasi-invariant measure*, J. London Math. Soc. 2 (1970), 627–634.

[6] C. Chou, *On a conjecture of E. Granirer concerning the range of an invariant mean*, Proc. Amer. Math. Soc. 26 (1970), 105–107.

[7] H. Francke, D. Plachky and W. Thomsen, *A finitely additive version of Poincaré's recurrence theorem*, in: Stochastic Differential Systems, Lecture Notes in Control and Inform. Sci. 78, Springer, New York, 1986, 159–162.

[8] E. Granirer, *On the range of an invariant mean*, Trans. Amer. Math. Soc. 125 (1966), 384–394.

[9] E. Granirer and A. Lau, *Invariant means on locally compact groups*, Illinois J. Math. 15 (1971), 249–257.

[10] Z. Lipecki, *On compactness and extreme points of some sets of quasi-measures and measures*, Manuscripta Math. 86 (1995), 349–365.

[11] H. Luschgy, *Decomposition of charges and measures*, Proc. Amer. Math. Soc. 96 (1986), 121–125.

[12] A. L. T. Paterson, *Amenability*, Amer. Math. Soc., Providence, RI, 1988.

[13] D. Plachky, *Darboux property of measures and contents*, Math. Slovaca 30 (1980), 243–246.

[14] R. C. Snell, *The range of invariant means on locally compact groups and semigroups*, Proc. Amer. Math. Soc. 37 (1972), 441–447.

[15] H. Stroetmann, *On the strong continuity of invariant charges*, Statist. Decisions 6 (1988), 189–193.

[16] S. Wagon, *The Banach–Tarski Paradox*, Cambridge Univ. Press, Cambridge, 1985.

FB IV-Mathematik
Universität Trier
D-54286 Trier, BR Deutschland
E-mail: luschgy@uni-trier.de

Department of Mathematics
University of Illinois
1409 W. Green St.
Urbana, IL 61801, U.S.A.
E-mail: ssolecki@math.uiuc.edu

Received 18 March 2004;
revised 6 October 2004 (3384)