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CHARACTERIZING CHAINABLE, TREE-LIKE, AND CIRCLE-LIKE CONTINUA

BY

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Abstract. We prove that a continuum X is tree-like (resp. circle-like, chainable) if and only if for each open cover $\mathcal{U}_4=\{U_1,U_2,U_3,U_4\}$ of X there is a $\mathcal{U}_4$-map $f\colon X\to Y$ onto a tree (resp. onto the circle, onto the interval). A continuum X is an acyclic curve if and only if for each open cover $\mathcal{U}_3=\{U_1,U_2,U_3\}$ of X there is a $\mathcal{U}_3$-map $f\colon X\to Y$ onto a tree (or the interval $[0,1]$).

1. Main results. In this paper we characterize chainable, tree-like and circle-like continua in the spirit of the following characterization of covering dimension due to Hemmingsen (see [6, 1.6.9]).

Theorem 1 (Hemmingsen). *For a compact Hausdorff space X the following conditions are equivalent:*

(1) $\dim X\le n$, *which means that any open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ of order $\le n+1$;*
(2) *each open cover $\mathcal{U}$ of X with cardinality $|\mathcal{U}|\le n+2$ has an open refinement $\mathcal{V}$ of order $\le n+1$;*
(3) *each open cover $\{U_i\}_{i=1}^{n+2}$ of X has an open refinement $\{V_i\}_{i=1}^{n+2}$ with $\bigcap_{i=1}^{n+2}V_i=\emptyset$.*

We say that a cover $\mathcal{V}$ of X is a *refinement* of a cover $\mathcal{U}$ if each $V\in\mathcal{V}$ lies in some $U\in\mathcal{U}$. The *order* of a cover $\mathcal{U}$ is defined as the cardinal

$$\mathrm{ord}(\mathcal{U})=\sup\{|\mathcal{F}|:\mathcal{F}\subseteq\mathcal{U}\text{ with }\textstyle\bigcap\mathcal{F}\ne\emptyset\}.$$

A family $\mathcal{U}$ of subsets of a set X is called

- *chain-like* if for $\mathcal{U}$ there is an enumeration $\mathcal{U}=\{U_1,\dots,U_n\}$ such that $U_i\cap U_j\ne\emptyset$ if and only if $|i-j|\le 1$ for all $1\le i,j\le n$;
- *circle-like* if there is an enumeration $\mathcal{U}=\{U_1,\dots,U_n\}$ such that $U_i\cap U_j\ne\emptyset$ if and only if $|i-j|\le 1$ or $\{i,j\}=\{1,n\}$;
- *tree-like* if $\mathcal{U}$ contains no circle-like subfamily $\mathcal{V}\subseteq\mathcal{U}$ of cardinality $|\mathcal{V}|\ge 3$.

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We recall that a continuum X is called *chainable* (resp. *tree-like*, *circle-like*) if each open cover of X has a chain-like (resp. tree-like, circle-like) open refinement. By a *continuum* we understand a connected compact Hausdorff space.

The following characterization of chainable, tree-like and circle-like continua is the main result of this paper. For chainable and tree-like continua this characterization was announced (but not proved) in [1].

THEOREM 2. *A continuum X is chainable* (*resp. tree-like, circle-like*) *if and only if any open cover $\mathcal{U}$ of X of cardinality $|\mathcal{U}| \le 4$ has a chain-like* (*resp. tree-like, circle-like*) *open refinement.*

In fact, this theorem will be derived from a more general theorem concerning K-like continua.

DEFINITION 1. Let K be a class of continua and n be a cardinal number. A continuum X is called K-*like* (resp. n-K-*like*) if for any open cover $\mathcal{U}$ of X (of cardinality $|\mathcal{U}| \le n$) there is a $\mathcal{U}$-map $f\colon X \to K$ onto some space $K \in \mathsf{K}$.

We recall that a map $f : X \to Y$ between two topological spaces is called a $\mathcal{U}$-*map*, where $\mathcal{U}$ is an open cover of X, if there is an open cover $\mathcal{V}$ of Y such that the cover $f^{-1}(\mathcal{V}) = \{f^{-1}(V) : V \in \mathcal{V}\}$ refines $\mathcal{U}$. It is worth mentioning that a closed map $f\colon X \to Y$ is a $\mathcal{U}$-map if and only if the family $\{f^{-1}(y)\colon y \in Y\}$ refines $\mathcal{U}$.

It is clear that a continuum X is tree-like (resp. chainable, circle-like) if and only if it is K-like for the class K of all trees (resp. for $\mathsf{K} = \{[0, 1]\}$, $\mathsf{K} = \{S^1\}$). Here $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ stands for the circle.

It turns out that each 4-K-like continuum is $\widehat{\mathsf{K}}$-like for some extension $\widehat{\mathsf{K}}$ of K. This extension is defined with the help of locally injective maps.

A map $f\colon X \to Y$ between topological spaces is called *locally injective* if each point $x \in X$ has a neighborhood $O(x) \subseteq X$ such that the restriction $f{\restriction}O(x)$ is injective. For a class of continua K let $\widehat{\mathsf{K}}$ be the class of all continua X that admit a locally injective map $f\colon X \to Y$ onto some $Y \in \mathsf{K}$.

THEOREM 3. *Let K be a class of* 1-*dimensional continua. If a continuum X is* 4-K-*like, then X is* $\widehat{\mathsf{K}}$-*like.*

In Proposition 1 we shall prove that each locally injective map $f\colon X \to Y$ from a continuum X onto a tree-like continuum Y is a homeomorphism. Consequently, $\widehat{\mathsf{K}} = \mathsf{K}$ for any class K of tree-like continua. This fact combined with Theorem 3 implies the following characterization:

THEOREM 4. *Let K be a class of tree-like continua. A continuum X is* K-*like if and only if it is* 4-K-*like.*

One may ask if the number 4 in this theorem can be lowered to 3 as in Hemmingsen's characterization of 1-dimensional compacta. It turns out that this cannot be done: 3-K-likeness is equivalent to being an acyclic curve. A continuum X is called a *curve* if $\dim X \leq 1$. It is *acyclic* if each map $f : X \to S^1$ to the circle is null-homotopic.

THEOREM 5. *Let $\mathsf{K} \ni [0,1]$ be a class of tree-like continua. A continuum X is 3-K-like if and only if it is an acyclic curve.*

It is known that each tree-like continuum is an acyclic curve, but there are acyclic curves which are not tree-like [3]. On the other hand, each locally connected acyclic curve is tree-like (moreover, it is a dendrite [10, Chapter X]). Therefore, for any continuum X and a class $\mathsf{K} \ni [0,1]$ of tree-like continua we get the following chain of equivalences and implications (in which the dotted implication holds under the additional assumption that X is locally connected):

$$\begin{tikzcd}
\text{4-chainable} \arrow[r] \arrow[d, leftrightarrow] & \text{4-}\mathsf{K}\text{-like} \arrow[r] \arrow[d, leftrightarrow] & \text{4-tree-like} \arrow[r] \arrow[d, leftrightarrow] & \text{3-}\mathsf{K}\text{-like} \arrow[d, leftrightarrow] \\
\text{chainable} \arrow[r] & \mathsf{K}\text{-like} \arrow[r] & \text{tree-like} \arrow[r] & \text{acyclic curve} \arrow[l, dotted, bend right]
\end{tikzcd}$$

Finally, let us present a factorization theorem that reduces the problem of studying n-K-like continua to the metrizable case. It will play an important role in the proof of the "circle-like" part of Theorem 2.

THEOREM 6. *Let $n \in \mathbb{N} \cup \{\omega\}$ and K be a family of metrizable continua. A continuum X is n-K-like if and only if any map $f\colon X \to Y$ to a metrizable compact space Y can be written as the composition $f = g \circ \pi$ of a continuous map $\pi\colon X \to Z$ onto a metrizable n-K-like continuum Z and a continuous map $g\colon Z \to Y$.*

2. Proof of Theorem 5. Let $\mathsf{K} \ni [0,1]$ be a class of tree-like continua. We need to prove that a continuum X is 3-K-like if and only if it is an acyclic curve.

To prove the "if" part, assume that X is an acyclic curve. By Theorem 2.1 of [1], X is 3-chainable. Since $[0,1] \in \mathsf{K}$, the continuum X is 3-K-like and we are done.

Now assume conversely that a continuum X is 3-K-like. First, using Hemmingsen's Theorem 1, we shall show $\dim X \leq 1$. Let $\mathcal{V} = \{V_1, V_2, V_3\}$ be an open cover of X. Since X is 3-K-like, we can find a $\mathcal{V}$-map $f\colon X \to T$ onto a tree-like continuum T. Using the 1-dimensionality of tree-like continua, we find an open cover $\mathcal{W}$ of T order ≤ 2 such that the cover $f^{-1}(\mathcal{W}) = \{f^{-1}(W) : W \in \mathcal{W}\}$ is a refinement of $\mathcal{V}$. The continuum X is 1-dimensional by the implication (2)$\Rightarrow$(1) of Hemmingsen's theorem.

It remains to prove that X is acyclic. Let $f\colon X \to S^1$ be a continuous map. Let $\mathcal{U} = \{U_1, U_2, U_3\}$ be a cover of the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$

by three open arcs U_1, U_2, U_3, each of length $< \pi$. Such a cover necessarily has $\mathrm{ord}(\mathcal{U}) = 2$. By our assumption there is an $f^{-1}(\mathcal{U})$-map $g\colon X \to T$ onto a tree-like continuum T. From tree-likeness of T it follows that $f^{-1}(\mathcal{U})$ has a tree-like refinement $\mathcal{V}$, and we can assume that T is a tree. It is well-known (see e.g. [3]) that there exists a continuous map $h\colon T \to S^1$ such that $h \circ g$ is homotopic to f. But each map from a tree to the circle is null-homotopic. Hence $h \circ g$ as well as f are null-homotopic too.

3. Proof of Theorem 3. We shall use some terminology from graph theory. First we recall some definitions.

By a (*combinatorial*) *graph* we understand a pair $G = (V, E)$ consisting of a finite set V of vertices and a set $E \subseteq \{\{a, b\} : a, b \in V,\ a \neq b\}$ of unordered pairs of vertices, called *edges*. A graph $G = (V, E)$ is *connected* if any two distinct vertices $u, v \in V$ can be linked by a path $(v_0, v_1, \dots, v_n)$ with $v_0 = u$, $v_n = v$ and $\{v_{i-1}, v_i\} \in E$ for $i \leq n$. The number n is called the *length* of the path (and equal to the number of edges involved). Each connected graph possesses a natural path-metric on the set of vertices: the distance $d(u, v)$ between two distinct vertices $u, v \in V$ equals the smallest length of a path linking them.

Two vertices $u, v \in V$ of a graph are *adjacent* if $\{u, v\} \in E$ is an edge. The *degree* $\deg(v)$ of a vertex $v \in V$ is the number of vertices $u \in V$ adjacent to v in the graph. The number $\deg(G) = \max_{v \in V} \deg(v)$ is called the *degree* of the graph. By an *r-coloring* of the graph we understand any map $\chi\colon V \to r = \{0, \dots, r-1\}$. In this case the value $\chi(v)$ is called the *color* of $v \in V$.

LEMMA 1. *Let $G = (V, E)$ be a connected graph with $\deg(G) \leq 3$ such that $d(u, v) \geq 6$ for any two vertices $u, v \in V$ of degree 3. Then there is a 4-coloring $\chi\colon V \to 4$ such that no two distinct vertices $u, v \in V$ with $d(u, v) \leq 2$ have the same color.*

Proof. Let $V_3 = \{v \in V : \deg(v) = 3\}$ and let $\bar{B}(v) = \{v\} \cup \{u \in V\colon \{u, v\} \in E\}$ be the unit ball centered at $v \in V$. It follows from $\deg(G) \leq 3$ that $|\bar{B}(v)| \leq 4$ for each $v \in V$. Moreover, for any distinct $v, u \in V_3$ the balls $\bar{B}(v)$ and $\bar{B}(u)$ are disjoint (because $d(v, u) \geq 6 > 2$). Hence we can define a 4-coloring χ on $\bigcup_{v \in V_3} \bar{B}(v)$ so that χ is injective on each $\bar{B}(u)$ and $\chi(v) = \chi(w)$ for each $v, w \in V_3$. Next, it remains to color the remaining vertices, all of order ≤ 2, by four colors so that $\chi(x) \neq \chi(y)$ if $d(x, y) \leq 2$. It is easy to check that this can always be done. ∎

Each graph $G = (V, E)$ can also be thought of as a topological object: just embed the set of vertices V as a linearly independent subset into a suitable Euclidean space and consider the union $|G| = \bigcup_{\{u,v\} \in E} [u, v]$ of intervals corresponding to the edges of G. Assuming that each interval $[u, v] \subseteq |G|$ is isometric to the unit interval $[0, 1]$, we can extend the path-metric of G to a path-metric d on the geometric realization $|G|$ of G. For $x \in |G|$ we shall denote by $B(x) = \{y \in |G| : d(x,y) < 1\}$ and $\bar{B}(x) = \{y \in |G| : d(x,y) \le 1\}$ respectively the open and closed unit balls centered at x. More generally, $B_r(x) = \{y \in |G| : d(x,y) < r\}$ will denote the open ball of radius r with center at x in $|G|$.

By a *topological graph* we shall understand a topological space Γ homeomorphic to the geometric realization $|G|$ of some combinatorial graph G. In this case G is called the *triangulation* of Γ. The degree of $\Gamma = |G|$ will be defined as the degree of the combinatorial graph G (it does not depend on the choice of a triangulation).

It turns out that any graph can be transformed by a small deformation into a graph of degree ≤ 3.

LEMMA 2. *For any open cover $\mathcal{U}$ of a topological graph Γ there is a $\mathcal{U}$-map $f\colon \Gamma \to G$ onto a topological graph G of degree ≤ 3.*

This lemma (possibly folklore) can be easily proved by induction. Figure 1 illustrates how to decrease the degree of a selected vertex of a graph.

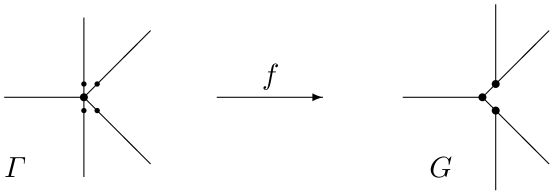

Fig. 1

Now we have all the tools for the proof of Theorem 3. So, take a class K of 1-dimensional continua and assume that X is a 4-K-like continuum. We should prove that X is $\widehat{\mathsf{K}}$-like.

First, we show that X is 1-dimensional. This will follow from Hemmingsen's Theorem 1 as soon as we check that each open cover $\mathcal{U}$ of X of cardinality $|\mathcal{U}| \le 3$ has an open refinement $\mathcal{V}$ of order ≤ 2. Since $|\mathcal{U}| \le 4$ and X is 4-K-like, there is a $\mathcal{U}$-map $f\colon X \to K$ onto a $K \in \mathsf{K}$. It follows that for some open cover $\mathcal{V}$ of K the cover $f^{-1}(\mathcal{V})$ refines $\mathcal{U}$. Since K is 1-dimensional, $\mathcal{V}$ has an open refinement $\mathcal{W}$ of order ≤ 2. Then the cover $f^{-1}(\mathcal{W})$ is an open refinement of $\mathcal{U}$ having order ≤ 2.

To prove that X is $\widehat{\mathsf{K}}$-like, fix any open cover $\mathcal{U}$ of X. By the compactness of X, we can assume that $\mathcal{U}$ is finite. Being 1-dimensional, X admits a $\mathcal{U}$-map $f\colon X \to \Gamma$ onto a topological graph Γ. By Lemma 2, we can assume that $\deg(\Gamma) \le 3$. Adding vertices on edges of Γ, we can find a triangulation (V_Γ, E_Γ) of Γ so fine that

- the path-distance between any vertices of degree 3 in Γ is ≥ 6;
- the cover $\{f^{-1}(B_2(v))\colon v \in V_\Gamma\}$ of X is a refinement of $\mathcal{U}$.

Lemma 1 yields a 4-coloring $\chi\colon V_\Gamma \to 4$ of V_Γ such that any two distinct vertices $u, v \in V_\Gamma$ with $d(u,v) \le 2$ have distinct colors. For each color $i \in 4$ consider the open 1-neighborhood $U_i = \bigcup_{v \in \chi^{-1}(i)} B(v)$ of the monochrome set $\chi^{-1}(i) \subseteq V_\Gamma$ in Γ. Since open 1-balls centered at vertices $v \in V_\Gamma$ cover the graph Γ, the 4-element family $\{U_i : i \in 4\}$ is an open cover of Γ. Then for the 4-element cover $\mathcal{U}_4 = \{f^{-1}(U_i)\colon i \in 4\}$ of the 4-K-like continuum X we can find a $\mathcal{U}_4$-map $g\colon X \to Y$ to a $Y \in \mathsf{K}$. Let $\mathcal{W}$ be a finite open cover of Y such that the cover $g^{-1}(\mathcal{W})$ refines $\mathcal{U}_4$. Since Y is 1-dimensional, we can assume that $\mathrm{ord}(\mathcal{W}) \le 2$. For every $W \in \mathcal{W}$ there is a $\xi(W) \in 4$ such that $g^{-1}(W) \subseteq f^{-1}(U_{\xi(W)})$.

Since Y is a continuum, in particular, a normal Hausdorff space, we may find a partition of unity subordinated to $\mathcal{W}$. This is a family $\{\lambda_W\colon W \in \mathcal{W}\}$ of continuous functions $\lambda_W\colon Y \to [0,1]$ such that

(a) $\lambda_W(y) = 0$ for $y \in Y \setminus W$;
(b) $\sum_{W \in \mathcal{W}} \lambda_W(y) = 1$ for all $y \in Y$.

For every $W \in \mathcal{W}$ consider the "vertical" family of rectangles

$$\mathcal{R}_W = \{W \times B(v) : v \in V_\Gamma,\ \chi(v) = \xi(W)\}$$

in $Y \times \Gamma$ and let $\mathcal{R} = \bigcup_{W \in \mathcal{W}} \mathcal{R}_W$. For every $R \in \mathcal{R}$ choose $W_R \in \mathcal{W}$ and $v_R \in V_\Gamma$ such that $R = W_R \times B(v_R)$. Also let $\mathcal{R}_R = \{S \in \mathcal{R} : R \cap S \ne \emptyset\}$.

Claim 1. *For any $R \in \mathcal{R}$ and $y \in W_R$ the set $\mathcal{R}_{R,y} = \{S \in \mathcal{R}_R : y \in W_S\}$ contains at most two distinct rectangles.*

Proof. Assume that besides R the set $\mathcal{R}_{R,y}$ contains two other distinct rectangles $S_1 = W_{S_1} \times B(v_{S_1})$ and $S_2 = W_{S_2} \times B(v_{S_2})$. Taking into account that $y \in W_R \cap W_{S_1} \cap W_{S_2}$ and $\mathrm{ord}(\mathcal{W}) \le 2$, we conclude that either $W_{S_1} = W_{S_2}$ or $W_R = W_{S_1}$ or $W_R = W_{S_2}$. If $W_{S_1} = W_{S_2}$, then

$$\chi(v_{S_1}) = \xi(W_{S_1}) = \xi(W_{S_2}) = \chi(v_{S_2}).$$

Since $B(v_R) \cap B(v_{S_1}) \ne \emptyset \ne B(v_R) \cap B(v_{S_2})$ the property of the 4-coloring χ implies that $v_{S_1} = v_{S_2}$ and hence $S_1 = S_2$. Analogously we can prove that $W_R = W_{S_1}$ implies $R = S_1$ and $W_R = W_{S_2}$ implies $R = S_2$, which contradicts the choice of $S_1, S_2 \in \mathcal{R}_{R,y} \setminus \{R\}$. ∎

Claim 1 implies that for every rectangle $R = W_R \times B(v_R)$ the function $\lambda_R : W_R \to \bar{B}(v_R) \subseteq \Gamma$ defined by

$$\lambda_R(y) = \begin{cases} \lambda_{W_R}(y)v_R + \lambda_{W_S}(y)v_S & \text{if } \mathcal{R}_{R,y} = \{R, S\} \text{ for some } S \ne R, \\ v_R & \text{if } \mathcal{R}_{R,y} = \{R\} \end{cases}$$

is well-defined and continuous. Let $\pi_R\colon R \to W_R \times \bar{B}(v_R) \subseteq \overline{R}$ be defined by $\pi_R(y,t) = (y, \lambda_R(y))$.

The graphs of λ_R and λ_S for two intersecting rectangles $R, S \in \mathcal{R}$ are drawn in Figure 2.

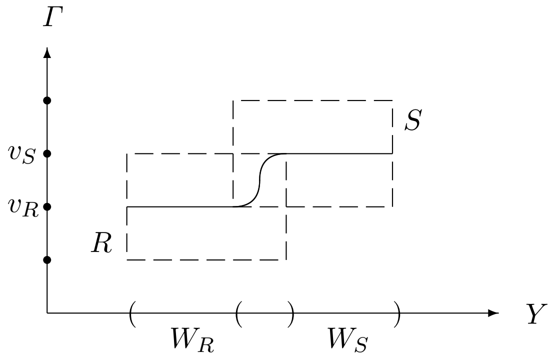

Fig. 2

It follows that for any $R, S \in \mathcal{R}$ we get $\pi_R{\restriction}R\cap S = \pi_S{\restriction}R\cap S$, which implies that $\pi = \bigcup_{R\in\mathcal{R}} \pi_R\colon \bigcup\mathcal{R} \to \bigcup\mathcal{R}$ is a well-defined continuous function. It is easy to check that for every $R = W \times B(v) \in \mathcal{R}$ we get

$$\pi^{-1}(W \times B(v)) \subseteq W \times B_2(v).$$

Consider the diagonal product $g \triangle f\colon X \to Y \times \Gamma$. It is easy to check that $(g\triangle f)(X) \subseteq \bigcup\mathcal{R}$, which implies that the composition $h = \pi\circ(g \triangle f)\colon X \to \bigcup\mathcal{R}$ is well-defined. We claim that h is a $\mathcal{U}$-map onto the continuum $L = h(X)$, which belongs to the class $\widehat{\mathsf{K}}$.

Given $R = W \times B(v) \in \mathcal{R}$, observe that

$$\begin{aligned} h^{-1}(R) &= (g \triangle f)^{-1}(\pi^{-1}(W \times B(v))) \subseteq (g \triangle f)^{-1}(W \times B_2(v)) \\ &= g^{-1}(W) \cap f^{-1}(B_2(v)) \subseteq f^{-1}(B_2(v)) \subseteq U \end{aligned}$$

for some $U \in \mathcal{U}$. Hence h is a $\mathcal{U}$-map.

The projection $\mathrm{pr}_Y\colon L \to Y$ is locally injective because $L \subseteq \bigcup\mathcal{R}$ and for every $R \in \mathcal{R}$ the restriction $\mathrm{pr}_Y{\restriction}R\cap L\colon R\cap L \to Y$ is injective. As $Y \in \mathsf{K}$, we conclude that $L \in \widehat{\mathsf{K}}$, by the definition of the class $\widehat{\mathsf{K}}$.

4. Locally injective maps onto tree-like continua and circle. The following theorem is known for metrizable continua [7].

PROPOSITION 1. *Each locally injective map $f : X \to Y$ from a continuum X onto a tree-like continuum Y is a homeomorphism.*

Proof. By the local injectivity of f, there is an open cover $\mathcal{U}'$ such that $f{\restriction}U$ is injective for every $U \in \mathcal{U}'$. Let $\mathcal{U}$ be an open cover of X whose star $\mathcal{S}t(\mathcal{U})$ refines $\mathcal{U}'$. Here $\mathcal{S}t(U,\mathcal{U}) = \bigcup\{U' \in \mathcal{U} : U \cap U' \neq \emptyset\}$ and $\mathcal{S}t(\mathcal{U}) = \{\mathcal{S}t(U,\mathcal{U}) : U \in \mathcal{U}\}$.

For every $x \in X$ choose a set $U_x \in \mathcal{U}$ that contains x. Observe that for distinct points $x, x' \in X$ with $f(x) = f(x')$ the sets $U_x, U_{x'}$ are disjoint. In the opposite case $x, x' \in U_x \cup U_{x'} \subseteq \mathcal{S}t(U_x, \mathcal{U}) \subseteq U$ for some $U \in \mathcal{U}'$, which is not possible as $f{\restriction}U$ is injective.

Hence for every $y \in Y$ the family $\mathcal{U}_y = \{U_x : x \in f^{-1}(y)\}$ is disjoint. Since f is closed and surjective, the set $V_y = Y \setminus f(X \setminus \bigcup \mathcal{U}_y)$ is an open neighborhood of y in Y such that $f^{-1}(V_y) \subseteq \bigcup \mathcal{U}_y$.

Since the continuum Y is tree-like, the cover $\mathcal{V} = \{V_y : y \in Y\}$ has a finite tree-like refinement $\mathcal{W}$. For every $W \in \mathcal{W}$ find $y_W \in Y$ with $W \subseteq V_{y_W}$ and consider the disjoint family $\mathcal{U}_W = \{U \cap f^{-1}(W) : U \in \mathcal{U}_{y_W}\}$. It follows that $f^{-1}(W) = \bigcup \mathcal{U}_W$ and so $\mathcal{U}_\mathcal{W} = \bigcup_{W \in \mathcal{W}} \mathcal{U}_W$ is an open cover of X.

Now we are able to show that the map f is injective. Assuming the converse, pick a point $y \in Y$ and two distinct points $a, b \in f^{-1}(y)$. Since X is connected, there is a chain $G_1, \dots, G_n \in \mathcal{U}_\mathcal{W}$ such that $a \in G_1$ and $b \in G_n$. We can assume that the length n of this chain is the smallest possible. In this case all sets $G_1, \dots, G_n$ are pairwise distinct.

Let us show that $n \geq 3$. In the opposite case $a \in G_1 = U_1 \cap f^{-1}(W_1) \in \mathcal{U}_\mathcal{W}$, $b \in G_2 = U_2 \cap f^{-1}(W_2) \in \mathcal{U}_\mathcal{W}$ and $G_1 \cap G_2 \neq \emptyset$. So, $a, b \in U_1 \cup U_2 \subseteq \mathcal{S}t(U_1, \mathcal{U}) \subseteq U$ for some $U \in \mathcal{U}'$ and then $f{\restriction}U$ is not injective. Therefore $n \geq 3$.

For every $i \leq n$ consider the point $y_i = y_{W_i}$ and find $W_i \in \mathcal{W}$ and $U_i \in \mathcal{U}_{y_i}$ such that $G_i = U_i \cap f^{-1}(W_i) \in \mathcal{U}_{W_i}$. Then $(W_1, \dots, W_n)$ is a sequence of elements of the tree-like cover $\mathcal{W}$ such that $y \in W_1 \cap W_n$ and $W_i \cap W_{i+1} \neq \emptyset$ for all $i < n$. Since the tree-like cover $\mathcal{W}$ does not contain circle-like subfamilies of length ≥ 3 there are two numbers $1 \leq i < j \leq n$ such that $W_i \cap W_j \neq \emptyset$, $|j - i| > 1$ and $\{i, j\} \neq \{1, n\}$. We can assume that the difference $k = j - i$ is the smallest possible. In this case $k = 2$. Otherwise, $W_i, W_{i+1}, \dots, W_j$ is a circle-like subfamily of length ≥ 3 in $\mathcal{W}$, which is forbidden. Therefore, $j = i + 2$ and the family $\{W_i, W_{i+1}, W_{i+2}\}$ contains at most two distinct sets (in the opposite case this family is circle-like, which is forbidden). If $W_i = W_{i+1}$, then $U_i = U_{i+1}$ as the family $\mathcal{U}_{W_i}$ is disjoint. The assumption $W_{i+1} = W_{i+2}$ leads to a similar contradiction. It remains to consider the case $W_i = W_{i+2} \neq W_{i+1}$. Since $U_i, U_{i+2} \in \mathcal{U}_{y_i}$ are distinct, there are distinct $x_i, x_{i+2} \in f^{-1}(y_i)$ such that $x_i \in U_i$ and $x_{i+2} \in U_{i+2}$. Since $x_i, x_{i+2} \in U_i \cup U_{i+2} \subseteq \mathcal{S}t(U_{i+1}, \mathcal{U}) \subseteq U$ for some $U \in \mathcal{U}'$, the restriction $f{\restriction}U$ is not injective. This contradiction completes the proof. ∎

PROPOSITION 2. *If $f\colon X \to S^1$ is a locally injective map from a continuum X onto the circle S^1, then X is an arc or a circle.*

Proof. The compact space X has a finite cover by compact subsets that embed into the circle. Consequently, X is metrizable and 1-dimensional. We claim that X is locally connected. Assuming the converse and applying Theorem 1 of [9, §49.VI] (or [10, 5.22(b) and 5.12]), we could find a convergence continuum $K \subseteq X$. This a non-trivial continuum, and it is the limit of a sequence $(K_n)_{n\in\omega}$ of continua that lie in $X \setminus K$.

By the local injectivity of f, the continuum K meets some open set $U \subseteq X$ such that $f{\restriction}U : U \to S^1$ is a topological embedding. The intersection $U \cap K$, being a non-empty open subset of the continuum K, is not zero-dimensional. Consequently, its image $f(U \cap K) \subseteq S^1$ is not zero-dimensional either and hence contains a non-empty open subset V of S^1. Choose any point $x \in U \cap K$ with $f(x) \in V$. The convergence $K_n \to K$ implies the existence of a sequence of points $x_n \in K_n$, $n \in \omega$, that converge to x. By the continuity of f, the sequence $(f(x_n))_{n\in\omega}$ converges to $f(x) \in V$. So, there is n such that $f(x_n) \in V \subseteq f(U \cap K)$ and $x_n \in U$. The injectivity of $f{\restriction}U$ guarantees that $x_n \in U \cap K$, which is not possible as $x_n \in K_n \subset X \setminus K$.

Therefore, the continuum X is locally connected. By the local injectivity, each point $x \in X$ has an open connected neighborhood V homeomorphic to a (connected) subset of S^1. Now we see that the space X is a compact 1-dimensional manifold (possibly with boundary). So, X is homeomorphic either to the arc or to the circle. ∎

5. Proof of Theorem 6. In the proof we shall use the technique of inverse spectra described in [5, §2.5] or [4, Ch. 1]. Given a continuum X embed it into a Tikhonov cube $[0,1]^\kappa$ of weight $\kappa \ge \aleph_0$.

Let A be the set of all countable subsets of κ, partially ordered by the inclusion relation: $\alpha \le \beta$ iff $\alpha \subseteq \beta$. For a countable subset $\alpha \subseteq \kappa$ let $X_\alpha = \mathrm{pr}_\alpha(X)$ be the projection of X onto the face $[0,1]^\alpha$ of the cube $[0,1]^\kappa$ and $p_\alpha : X \to X_\alpha$ be the projection map. For any countable subsets $\alpha \subseteq \beta$ of κ let $p^\beta_\alpha : X_\beta \to X_\alpha$ be the restriction of the natural projection $[0,1]^\beta \to [0,1]^\alpha$. In such a way we have defined an inverse spectrum $\mathcal{S} = \{X_\alpha, p^\beta_\alpha, A\}$ over the index set A, which is ω-complete in the sense that any countable subset $B \subseteq A$ has the smallest upper bound $\sup B = \bigcup B$ and for any increasing sequence $(\alpha_i)_{i\in\omega}$ in A with supremum $\alpha = \bigcup_{i\in\omega} \alpha_i$ the space X_α is the limit of the inverse sequence $\{X_{\alpha_i}, p^{\alpha_{i+1}}_{\alpha_i}, \omega\}$. The spectrum $\mathcal{S}$ consists of metrizable compacta X_α, $\alpha \in A$, and its inverse limit $\varprojlim \mathcal{S}$ can be identified with the space X. By Corollary 1.3.2 of [4], the spectrum $\mathcal{S}$ is factorizing in the sense that any continuous map $f : X \to Y$ to a second countable space Y can be written as the composition $f = f_\alpha \circ p_\alpha$ for some index $\alpha \in A$ and some continuous map $f_\alpha : X_\alpha \to Y$.

Now we are able to prove the "if" and "only if" parts of Theorem 6. To prove the "if" part, assume that each map $f : X \to Y$ factorizes through a metrizable n-K-like continuum, where $n \in \mathbb{N} \cup \{\omega\}$. To show that X is n-K-like, fix any open cover $\mathcal{U}$ of X with $k = |\mathcal{U}| \le n$. By compactness of X

we can assume that k is finite and $\mathcal{U} = \{U_1, \dots, U_k\}$. By Theorem 1.7.8 of [6], there is a closed cover $\{F_1, \dots, F_k\}$ of X such that $F_i \subseteq U_i$ for all $i \le k$. Since F_i and $X \setminus U_i$ are disjoint closed subsets of the compact space $X = \varprojlim \mathcal{S}$, there is an index $\alpha \in A$ such that for every $i \le k$ the images $p_\alpha(X \setminus U_i)$ and $p_\alpha(F_i)$ are disjoint and hence $W_i = X_\alpha \setminus p_\alpha(X \setminus U_\alpha)$ is an open neighborhood of $p_\alpha(F_i)$. Then $\{W_1, \dots, W_k\}$ is an open cover of X_α such that $p_\alpha^{-1}(W_i) \subseteq U_i$ for all $i \le k$.

By our assumption the projection $p_\alpha\colon X \to X_\alpha$ can be written as the composition $p_\alpha = g \circ \pi$ of a map $\pi\colon X \to Z$ onto a metrizable n-K-like continuum Z and a map $g\colon Z \to X_\alpha$. For every $i \le k$ consider the open subset $V_i = g^{-1}(W_i)$ of Z. Since Z is n-K-like, for the open cover $\mathcal{V} = \{V_1, \dots, V_k\}$ of Z there is a $\mathcal{V}$-map $h : Z \to K$ onto a space $K \in \mathsf{K}$. Then $h \circ \pi : X \to K$ is a $\mathcal{U}$-map of X onto $K \in \mathsf{K}$ witnessing that X is an n-K-like continuum.

To prove the "only if" part we need the following lemma.

LEMMA 3. *Suppose that X is an n-K-like continuum and $\alpha \in A$. Then there is $\beta \ge \alpha$ in A having the property that for any open cover $\mathcal{V}$ of X_α with $|\mathcal{V}| \le n$, there is a map $f\colon X_\beta \to K$ onto a space $K \in \mathsf{K}$ such that $f \circ p_\beta : X \to K$ is a $p_\alpha^{-1}(\mathcal{V})$-map.*

Proof. Let $\mathcal{B}$ be a countable base of the topology of the compact metrizable space X_α such that $\mathcal{B}$ is closed under unions. Denote by $\mathfrak{U}$ the family of all possible finite k-set covers $\{B_1, \dots, B_k\} \subseteq \mathcal{B}$ of X_α with $k \le n$. It is clear that the family $\mathfrak{U}$ is countable.

Each cover $\mathcal{U} = \{B_1, \dots, B_k\} \in \mathfrak{U}$ induces the open cover $p_\alpha^{-1}(\mathcal{U}) = \{p_\alpha^{-1}(B_i) : 1 \le i \le k\}$ of X. Since the continuum X is n-K-like, there is a $p_\alpha^{-1}(\mathcal{U})$-map $f_{\mathcal{U}} : X \to K_{\mathcal{U}}$ onto a space $K_{\mathcal{U}} \in \mathsf{K}$. By the metrizability of $K_{\mathcal{U}}$ and the factorizing property of the spectrum $\mathcal{S}$, for some $\alpha_{\mathcal{U}} \ge \alpha$ in A there is a map $f_{\alpha_{\mathcal{U}}}\colon X_{\alpha_{\mathcal{U}}} \to K_{\mathcal{U}}$ such that $f_{\mathcal{U}} = f_{\alpha_{\mathcal{U}}} \circ p_{\alpha_{\mathcal{U}}}$. Consider the countable set $\beta = \bigcup_{\mathcal{U} \in \mathfrak{U}} \alpha_{\mathcal{U}}$, which is the smallest lower bound of the set $\{\alpha_{\mathcal{U}} : \mathcal{U} \in \mathfrak{U}\}$ in A. We claim that β has the required property.

Let $\mathcal{V}$ be any open cover of X_α with $k = |\mathcal{V}| \le n$. We can assume that k is finite and $\mathcal{V} = \{V_1, \dots, V_k\}$. By Theorem 1.7.8 of [6], there is a closed cover $\{F_1, \dots, F_k\}$ of X_α such that $F_i \subseteq V_i$ for all $i \le k$. Since $\mathcal{B}$ is the base of the topology of X_α and $\mathcal{B}$ is closed under finite unions, for every $i \le k$ there is a basic set $B_i \in \mathcal{B}$ such that $F_i \subseteq B_i \subseteq V_i$. Then the cover $\mathcal{U} = \{B_1, \dots, B_k\}$ belongs to the family $\mathfrak{U}$ and refines the cover $\mathcal{V}$. Consider the map $f = f_{\alpha_{\mathcal{U}}} \circ p^{\beta}_{\alpha_{\mathcal{U}}} : X_\beta \to K = K_{\mathcal{U}}$ and observe that $f \circ p_\beta = f_{\alpha_{\mathcal{U}}} \circ p_{\alpha_{\mathcal{U}}}$ is a $p_\alpha^{-1}(\mathcal{U})$-map and a $p_\alpha^{-1}(\mathcal{V})$-map. ∎

Now let us return to the proof of the theorem. Assume that the continuum X is n-K-like. Given a map $f\colon X \to Y$ to a second countable space, we need to find a map $\pi\colon X \to Z$ onto a metrizable n-K-like continuum Z and a map $g\colon Z \to Y$ such that $f = g \circ \pi$. Since the spectrum $\mathcal{S}$ is factorizing, there are $\alpha_0 \in A$ and $f_0\colon X_{\alpha_0} \to Y$ such that $f = f_0 \circ p_{\alpha_0}$. Using Lemma 3, by induction we construct an increasing sequence $(\alpha_i)_{i\in\omega}$ in A such that for every $i \in \omega$ and any open cover $\mathcal{V}$ of X_{α_i} with $|\mathcal{V}| \le n$, there is a map $f\colon X_{\alpha_{i+1}} \to K$ onto a space $K \in \mathsf{K}$ such that $f \circ p_{\alpha_{i+1}}$ is a $p^{-1}_{\alpha_i}(\mathcal{V})$-map.

Let $\alpha = \sup_{i\in\omega} \alpha_i = \bigcup_{i\in\omega} \alpha_i$. We claim that the metrizable continuum X_α is n-K-like. Given any open finite cover $\mathcal{U} = \{U_1, \dots, U_k\}$ of $X_\alpha = \varprojlim X_{\alpha_i}$, where $k \le n$, we can find $i \in \omega$ such that the sets $W_i = X_{\alpha_i} \setminus p^\alpha_{\alpha_i}(X_\alpha \setminus U_i)$, $i \le k$, form an open cover $\mathcal{W} = \{W_1, \dots, W_n\}$ of X_{α_i} such that the cover $(p^\alpha_{\alpha_i})^{-1}(\mathcal{W})$ refines $\mathcal{U}$. By the choice of α_{i+1}, there is a map $g\colon X_{\alpha_{i+1}} \to K$ onto a space $K \in \mathsf{K}$ such that $g \circ p_{\alpha_{i+1}}\colon X \to K$ is a $p^{-1}_{\alpha_i}(\mathcal{W})$-map. It follows that $g \circ p^\alpha_{\alpha_{i+1}}\colon X_\alpha \to K$ is a $(p^\alpha_{\alpha_i})^{-1}(\mathcal{W})$-map and hence a $\mathcal{U}$-map, witnessing that the continuum X_α is n-K-like.

Now we see that the metrizable n-K-like continuum X_α and the maps $\pi = p_\alpha\colon X \to X_\alpha$ and $g = f_0 \circ p^\alpha_{\alpha_0}\colon X_\alpha \to Y$ satisfy our requirements.

6. Proof of Theorem 2. The "chainable" and "tree-like" parts of Theorem 2 follow immediately from the characterization in Theorem 4. So, it remains to prove the "circle-like" part. Let $\mathsf{K} = \{S^1\}$. We need to prove that each 4-K-like continuum X is K-like. Given an open cover $\mathcal{U}$ of X we need to construct a $\mathcal{U}$-map of X onto the circle. By Theorem 6, there is a $\mathcal{U}$-map f onto a metrizable 4-K-like continuum Y. It follows that for some open cover $\mathcal{V}$ of Y the cover $f^{-1}(\mathcal{V})$ refines $\mathcal{U}$. The proof will be complete as soon as we prove that the continuum Y is circle-like. In this case there is a $\mathcal{V}$-map $g : Y \to S^1$ and the composition $g \circ f\colon X \to S^1$ is a required $\mathcal{U}$-map witnessing that X is circle-like.

By Theorem 3, the metrizable continuum Y is $\widehat{\mathsf{K}}$-like. By Proposition 2, each continuum $K \in \widehat{\mathsf{K}}$ is homeomorphic to S^1 or $[0,1]$. Consequently, the continuum Y is circle-like or chainable. In the first case we are done. So, we assume that Y is chainable.

By [10, Theorem 12.5], the continuum Y is irreducible between some points $p, q \in Y$. This means that each subcontinuum of X that contains p, q coincides with Y. We claim that Y is either indecomposable or the union of two indecomposable subcontinua. For the proof we will use the argument of [10, Exercise 12.50] (cf. also [8, Theorem 3.3]).

Suppose that Y is not indecomposable. This means that there are two proper subcontinua A, B of Y such that $Y = A \cup B$. By the choice of the points p, q, they cannot simultaneously lie in A or in B. So, we can assume that $p \in A$ and $q \in B$.

We claim that the closure of $Y \setminus A$ is connected. Assuming that $\overline{Y \setminus A}$ is disconnected, we can find a proper clopen subset $F \subsetneq \overline{Y \setminus A}$ that contains q

and conclude that $F \cup A$ is a proper subcontinuum of Y that contains both p and q, which is not possible. Replacing B by the closure of $Y \setminus A$, we can assume that $Y \setminus A$ is dense in B. Then $Y \setminus B$ is dense in A.

We claim that the continua A and B are indecomposable. Assuming that A is decomposable, pick two proper subcontinua C, D such that $C \cup D = A$. We can assume that $p \in D$. Then $B \cap D = \emptyset$ (as Y is irreducible between p and q). By Theorem 11.8 of [10], the set $Y \setminus (B \cup D)$ is connected. Let $\mathcal{Z}$ consist of the four open sets $Y \setminus A = Y \setminus (C \cup D)$, $Y \setminus (D \cup \{q\})$, $Y \setminus (B \cup \{p\})$ and $Y \setminus (B \cup C)$. Since $p \notin C$, we see that $\mathcal{Z}$ is a cover of Y and there exists a $\mathcal{Z}$-map $h\colon Y \to S^1$ because Y is 4-$\{S^1\}$-like. Therefore $h^{-1}(h(p)) \subseteq Y \setminus (B \cup C) \subseteq D$, $h^{-1}(h(q)) \subseteq X \setminus A \subseteq B$ and $h(B) \cap h(D) = \emptyset$. Hence $h(Y \setminus (B \cup D)) \subseteq S^1 \setminus \{h(p), h(q)\}$ and $S^1 \setminus (h(B) \cup h(D))$ is the union of two disjoint open intervals, each contained in one of the components of $S^1 \setminus \{h(p), h(q)\}$. This contradicts the connectedness of $h(Y \setminus (B \cup D))$.

Now we know that Y is either indecomposable or the union of two indecomposable subcontinua. Applying Theorem 7 of [2], we conclude that the metrizable chainable continuum Y is circle-like.

7. Open problems

PROBLEM 1. For which families K of connected topological graphs every 4-K-like continuum is K-like? Is it true for the family $\mathsf{K} = \{8\}$, where 8 is the bouquet of two circles?

Also we do not know if Theorem 4 can be generalized to classes of higher-dimensional continua.

PROBLEM 2. Let $k \in \mathbb{N}$ and K be a class of k-dimensional (contractible) continua. Is there a finite number n such that a continuum X is K-like if and only if it is n-K-like?

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