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EXAMPLE OF A MEAN ERGODIC L^1 OPERATOR WITH THE LINEAR RATE OF GROWTH

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Abstract. The rate of growth of an operator T satisfying the mean ergodic theorem (MET) cannot be faster than linear. It was recently shown (Kornfeld–Kosek, Colloq. Math. 98 (2003)) that for every $\gamma > 0$, there are positive $L^1[0,1]$ operators T satisfying MET with $\lim_{n\to\infty} \|T^n\|/n^{1-\gamma} = \infty$. In the class of positive L^1 operators this is the most one can hope for in the sense that for every such operator T, there exists a $\gamma_0 > 0$ such that $\limsup \|T^n\|/n^{1-\gamma_0} = 0$. In this note we construct an example of a nonpositive L^1 operator with the highest possible rate of growth, that is, $\limsup_{n\to\infty} \|T^n\|/n > 0$.

1. Introduction. We say that a bounded linear operator T in a Banach space X satisfies the *MET* (or is *mean ergodic*) if $\lim_{n\to\infty} n^{-1}\sum_{k=1}^{n} T^k f$ exists for all $f \in X$. The mean ergodic theorem was originally proved in the 1930's by von Neumann for unitary operators in a Hilbert space. Since then, the theorem has been extended to many different classes of spaces and operators (see, e.g., [9]).

For a given operator T, define the nth average operator by $A_n = A_n(T) = n^{-1}\sum_{k=1}^{n} T^k$. It follows from the Banach–Steinhaus theorem that if T is mean ergodic, then it must be *Cesàro bounded*, i.e. $\sup_{n\ge 1}\|A_n\| < \infty$. It is easy to see that the rate of growth of a Cesàro bounded operator T can be at most linear, since $\|T^n\| = \|nA_n - (n-1)A_n\| \le 2n\sup\|A_n\|$. However, it is not necessary for T to be power bounded, i.e. we may have $\sup_{n\ge 1}\|T^n\| = \infty$. The first example in that direction was given in 1945 by E. Hille [5]. He showed that the operator T defined on $L^1[0,1]$ by $Tf(x) = f(x) - \int_0^x f(y)\,dy$ is mean ergodic, with the norms of the T^n growing as $n^{1/4}$. In Hille's example, a rather nontrivial one, the rate of growth of $\|T^n\|$ is $n^{1/4}$, and it is related to the asymptotics of the Laguerre polynomials, which appear in the kernels of the iterations of the integral operator T. However, to quote Hille's own words ([5, p. 247]), "it is still a far cry from $O(n^{1/4})$ to $o(n)$". Cesàro bounded operators (but not necessarily mean ergodic) which are not power bounded have also been studied in [11] and [10]. Recently, a mean ergodic operator T in L^2 with the highest

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possible rate of growth, namely $\limsup n^{-1}\|T^n\| > 0$, was constructed by Y. Derriennic [2].

For any fixed p, $1 \le p \le \infty$, an L^p operator T is called *positive* if T preserves the cone $L^p_+ = \{f \in L^p : f \ge 0\}$. For $1 < p < \infty$ (but not for $p = 1$), the existence of positive mean ergodic L^p operators which are not power bounded follows from the results of R. Émilion [4]. It was shown in [1, Lemma 5] that the rate of growth for a positive Cesàro bounded L^1 operator T satisfies $\lim_{n\to\infty} \|T^n\|/n = 0$. This estimate was further strengthened in [6]:

Theorem 1. *Let T be a positive Cesàro bounded L^1 operator such that* $\sup_n \|A_n(T)\| = K$. *Then* $\limsup \|T^n\|/n^{1-1/K} < \infty$.

It was also shown in [6] that for any $\gamma > 0$, the rate of growth of $\|T^n\|$ for positive mean ergodic L^1 operators T can actually be higher than $n^{1-\gamma}$. In other words, we now know exactly how fast a positive, mean ergodic L^1 can grow (in n).

This leaves us with a natural question: is there a (nonpositive) mean ergodic L^1 operator with $\limsup \|T^n\|/n = \infty$? An operator with this property is constructed in Section 2.

Examples of L^1 operators constructed in [6] can be realized in a class of operators which appear naturally in dynamics, in particular, in questions about cocycles for nonsingular transformations (see, e.g., [8], [7] and [12]). To improve the rate of growth, a new "twist" in the construction appears necessary.

2. Main construction. For simplicity of notation we will construct a discrete version first, for the space $L^1(\mathbb{Z})$. With minor modifications, the same can be done in the continuous case, for $L^1([0,1])$. However, the exposition and notation is simpler in the discrete case. The example can then be transferred to $L^1([0,1])$.

Let $X = L^1(\mathbb{Z})$ be the space of all doubly infinite sequences $\{x_j\}_{j=-\infty}^{\infty}$ of real numbers with $\|x\| = \sum_{j=-\infty}^{\infty} |x_j| < \infty$. For any two bounded sequences $w = \{w_j\}_{j=-\infty}^{\infty}$, $v = \{v_j\}_{j=-\infty}^{\infty}$ one can define a bounded linear operator $T = T_{w,v}$ in X in the following way: for $x \in X$ we put $Tx = y$, where

$$y_j = w_j x_{j+1} + v_j x_{j+2}$$

for all j. (One could think of T as a sum of two weighted shift operators, where one of them shifts by 2 notches.) Define

$$w_j = \begin{cases} 0 & \text{if } j \le 0, \\ 0 & \text{if } j = 2^p,\ p = 0, 1, \ldots, \\ -1 & \text{if } j = 2^p - 2^q,\ p = 3, 4, \ldots, \text{ and } q = 1, \ldots, p-2, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$v_j = \begin{cases} 1 & \text{if } j = 2^p - 2^q - 1,\ p = 3, 4, \ldots, \text{ and } q = 1, \ldots, p-2, \\ 0 & \text{otherwise.} \end{cases}$$

We want to show that this will guarantee the linear growth of the norms of T^n. We will also verify that

$$\lim_{n\to\infty} A_n x = 0 \quad \text{for all } x \in X, \tag{1}$$

where as before, A_n is the nth average operator, i.e. $n^{-1}\sum_{k=1}^{n} T^k$. This makes the operator T mean ergodic.

Before making any formal arguments, let us take an informal look at what the operator T does. It is convenient to think of intervals $J_p = \{2^{p-1}+1, \ldots, 2^p\}$, which are further divided into subintervals $J_{p,q} = \{2^p - 2^q + 1, \ldots, 2^p - 2^{q-1}\}$, for $p = 3, 4, \ldots,$ and $q = 2, \ldots, p-1$ (and $J_{p,1} = \{2^p - 1, 2^p\}$). When the transformation T is applied, the coordinates inside every $J_{p,q}$ are simply shifted by one to the left, except when they are at the left end point of $J_{p,q}$, i.e. for $i \in J_{p,q}$ and $0 \le k \le i - \min J_{p,q}$ we have $T^k(e_i) = e_{i-k}$. At the left end of every subinterval $J_{p,q}$, the operator T has built-in "splitters", except for $J_{p,p-1}$, where the "filters" are located. It is convenient to visualize x as a signal which gets transmitted. When the signal encounters a splitter ($w_j = -1$, $v_{j-1} = 1$), it becomes "split": in addition to the original signal we now have its opposite following it. At the left end point of every interval J_p the signal is "filtered" to 0. In addition, after passing through at least one splitter, the average values at each coordinate do not get very large. This is what makes the norm of the average operator $\|A_n\|$ bounded. At the same time, the norm ("strength of the signal") is doubled every time a splitter is encountered.

For example, let us see what happens when the transformation T is applied to the vector $x = e_{31} = (\ldots, 0, 0, 1, 0, 0, \ldots)$, where $x_{31} = 1$. We get $Tx = (\ldots, 0, 1, -1, 0, 0, 0, \ldots)$, where $(Tx)_{29} = 1$ and $(Tx)_{30} = -1$, in other words $Tx = e_{29} - e_{30}$. Applying T twice more we get $T^3 x = e_{26} - 2e_{27} + e_{28}$. The nonzero coordinates of $T^7 x$ are: $1, -3, 3, -1$. A pattern of alternating binomial coefficients emerges, until the signal is "filtered" by $\nu_{15} = w_{16} = 0$.

This observation is stated formally in the next lemma. In spite of a somewhat uninviting appearance, it simply describes the image of e_i, where $i = \max J_{p,q}$ for $q \ge 2$ (as well as for $i = \min J_{p,1}$), after passing through exactly m splitters.

Lemma 2. *Let $i = 2^p - 2^{q-1}$ and $k = 2^{q-1} + 2^q + \cdots + 2^{q+m-2}$ for some $p \ge 3$, $1 \le q \le p-2$ and $1 \le m \le p-q-1$. Then*

$$T^k(e_i) = \sum_{s=0}^{m} (-1)^s \binom{m}{s} e_{j(s)}, \tag{2}$$

where $j(s) = i - k - m + s$.

Proof. Consider p fixed. We will use induction on m. For $m=1$, and for all $1 \le q \le p-2$, the result is obvious, as e_i is first shifted inside $J_{p,q}$ by $2^{q-1}-1$ to the left and then "split": $T^k(e_i) = e_{i-k-1} - e_{i-k}$.

Assume that the conclusion is true for all values of m up to n, and for all permissible q's, that is, $1 \le q \le p-m-1$. For $m = n+1$, we have $k = 2^{q-1} + 2^q + \cdots + 2^{q+n-1}$. Clearly $T^k(e_i) = T^{k-\widetilde{k}} T^{\widetilde{k}}(e_i)$, where $\widetilde{k} = 2^{q-1}$. Applying formula (2) with $m=1$ we get

$$T^{\widetilde{k}}(e_i) = e_{\widehat{\imath}-1} - e_{\widehat{\imath}},$$

where $\widehat{\imath} = i - \widetilde{k} = \max J_{p,q+1} = 2^p - 2^q$ and $\widehat{k} = k - \widetilde{k} = 2^q + \cdots + 2^{q+n-1}$. We can now apply (2) to $e_{\widehat{\imath}}$, with $\widehat{m} = n$ (and with the appropriate value of $\widehat{q} = q+1$). We have

$$T^{\widehat{k}}(e_{\widehat{\imath}}) = \sum_{s=0}^{n} (-1)^s \binom{n}{s} e_{\widehat{\imath}-\widehat{k}-n+s} = \sum_{s=0}^{n} (-1)^s \binom{n}{s} e_{i-k-n+s}.$$

Also, T is just a plain shift at $\widehat{\imath}$, i.e. $T(e_{\widehat{\imath}}) = e_{\widehat{\imath}-1}$. Hence

$$\begin{aligned} T^{\widehat{k}}(e_{\widehat{\imath}-1}) &= T^{k-\widetilde{k}}(T(e_{\widehat{\imath}})) = T(T^{k-\widetilde{k}}(e_{\widehat{\imath}})) = T\left(\sum_{s=0}^{n} (-1)^s \binom{n}{s} e_{i-k-n+s}\right) \\ &= \sum_{s=0}^{n} (-1)^s \binom{n}{s} T e_{i-k-n+s} = \sum_{s=0}^{n} (-1)^s \binom{n}{s} e_{i-k-n+s-1}, \end{aligned}$$

as T is also a plain shift inside the $J_{p,q+n+1}$. We have

$$\begin{aligned} T^k(e_i) &= T^{k-\widetilde{k}}(T^{\widetilde{k}}(e_i)) = T^{\widehat{k}}(e_{\widehat{\imath}-1} - e_{\widehat{\imath}}) \\ &= \sum_{s=0}^{n} (-1)^s \binom{n}{s} e_{j(s)} - \sum_{s=0}^{n} (-1)^s \binom{n}{s} e_{j(s+1)}, \end{aligned}$$

where $j(s) = i - k - n - 1 + s$. Next,

$$\begin{aligned} T^k(e_i) &= e_{j(0)} + \sum_{s=1}^{n} (-1)^s \binom{n}{s} e_{j(s)} - \sum_{s=0}^{n} (-1)^s \binom{n}{s} e_{j(s+1)} \\ &= e_{j(0)} + \sum_{s=1}^{n} (-1)^s \binom{n}{s} e_{j(s)} - \sum_{s=1}^{n+1} (-1)^{s-1} \binom{n}{s-1} e_{j(s)} \\ &= e_{j(0)} + (-1)^{n+1} e_{j(n+1)} + \sum_{s=1}^{n} (-1)^s \left[\binom{n}{s} + \binom{n}{s-1}\right] e_{j(s)} \end{aligned}$$

$$= e_{j(0)} + (-1)^{n+1} e_{j(n+1)} + \sum_{s=1}^{n} (-1)^s \binom{n+1}{s} e_{j(s)}$$
$$= \sum_{s=0}^{n+1} (-1)^s \binom{n+1}{s} e_{j(s)}. \ \blacksquare$$

In particular, for every $p \in \mathbb{N}$, and for $q = 1$ and $m = p - 2$, we get $i = 2^p - 1$ and $k = 2^0 + 2^1 + \cdots + 2^{p-3} = 2^{p-2} - 1$. Informally speaking, there are $p - 2$ splitters between i and $i - k$, and the norm is doubled every time a splitter is encountered. To be precise, it follows from Lemma 2 that

$$\frac{1}{k}\|T^k(e_i)\| = \frac{1}{2^{p-2} - 1} \sum_{s=0}^{p-2} \binom{p-2}{s} = \frac{2^{p-2}}{2^{p-2} - 1} > 1.$$

Therefore, we have

Corollary 3. *The operator T has a linear rate of growth, that is,*

$$\limsup_{n\to\infty} \frac{1}{n}\|T^n\| > 0.$$

Remark 4. With not much effort one could modify this example and ensure that the limit $\lim_{n\to\infty} n^{-1}\|T^n\|$ exists. We will not pursue this here, in order to keep the exposition short and simple.

Let us now notice that since $w_j = v_j = 0$ for $j \leq 0$, in order to prove that $A_n(x) \to 0$ for every $x \in X$, it is enough to show that the norms of $\|A_n\|$ are uniformly bounded. To see this, we cut x into two parts, one of which has finitely many nonzero terms and the norm of the other is small. The averages of the first part disappear with time, and the averages of the second are controlled by the uniformly bounded norms of A_n. Specifically, assume that $\|A_n\| \leq K$, take $x \in X$ and fix $\varepsilon > 0$. Choose N such that $\sum_{j>N} |x_j| < \varepsilon/(2K)$. Let

$$x_n^{(1)} = \begin{cases} x_n & \text{for } 1 \leq n \leq N, \\ 0 & \text{otherwise,} \end{cases}$$

and $x^{(2)} = x - x^{(1)}$. Since $w_j = v_j = 0$ for $j \leq 0$, we have $T^n(x^{(1)}) = 0$ for $n \geq N$. Let $S = \max_{0\leq n<N} \|T^n(x^{(1)})\|$. Then for every $n \geq 2NS/\varepsilon$ we have

$$\|A_n(x)\| \leq \|A_n(x^{(1)})\| + \|A_n(x^{(2)})\| \leq \frac{\varepsilon}{2} + K\frac{\varepsilon}{2K} = \varepsilon.$$

In addition, notice that for every (finite) linear combination of the basis vectors we have $\|\sum_{j\in J} \lambda_j e_j\| = \sum_{j\in J} |\lambda_j|\,\|e_j\|$, and all such linear combinations form a dense set in L^1. The discrete case of the example will be finished as soon as we prove Proposition 7. First, let us make a couple of observations, which guarantee that the norms of T^n do not grow too fast.

Of course, Corollary 6 must hold for any mean ergodic operator, but so far mean ergodicity of T has not been established.

LEMMA 5. *If $x \in L^1(\mathbb{Z})$ is such that all its nonzero coordinates are in $J_{p,q}$ for some $p \ge 3$, $1 \le q \le p-2$, then for every $1 \le r < \#J_{p,q+1} = 2^q$ we have $\|T^r(x)\| \le 2\|x\|$.*

Proof. Suppose that $x = \sum_{j\in J_{p,q}} \alpha_j e_j$. For those values of $j \in J_{p,q}$ for which $j - r > \min J_{p,q} = 2^p - 2^q + 1$, we have $T^r(e_j) = e_{j-r}$. On the other hand for those j's for which $j-r \le \min J_{p,q}$, we have $T^r(e_j) = e_{j-r-1} - e_{j-r}$. In either case, $\|T^r(e_j)\| \le 2$. Therefore

$$\begin{aligned}\|T^r(x)\| &= \Big\|T^r\Big(\sum_{j\in J_{p,q}} \alpha_j e_j\Big)\Big\| = \Big\|\sum_{j\in J_{p,q}} \alpha_j T^r(e_j)\Big\| \\ &\le \sum_{j\in J_{p,q}} |\alpha_j|\, \|T^r(e_j)\| \le 2\sum_{j\in J_{p,q}} |\alpha_j| = 2\|x\|. \ \blacksquare\end{aligned}$$

COROLLARY 6. *There exists a uniform bound C such that for any $i, n \in \mathbb{N}$,*

$$\frac{1}{n}\|T^n(e_i)\| \le C.$$

Proof. Since for $i \le 3$, $T(e_i) = \mathbf{0}$, without loss of generality we will assume that $i \in J_p$ for some $p \ge 3$. If $i \in J_{p,p-1}$, then $\|T^n(e_i)\| = 1$ for $n < i - \min J_p$, and $\|T^n(e_i)\| = 0$ for all other values of n. Suppose $i \in J_{p,q}$ for some $p \ge 3$, $1 \le q \le p-2$. The norm of subsequent images of e_i under T does not increase until the left end point of $J_{p,q}$ is reached, that is, $\|T^n(e_i)\| = \|e_{i-j}\| = 1$ for $0 \le n \le i - \min J_{p,q}$. It suffices to verify the claim for $i = \min J_{p,q} = 2^p - 2^q + 1$. In addition, since $T^n(e_i) = \mathbf{0}$ for $n > \min J_{p,q} - \min J_p$, we will also assume that $n \le \min J_{p,q} - \min J_p = 2^{p-1} - 2^q$.

In order to directly apply Lemma 2, consider $\widetilde{i} = i + 2^{q-1} - 1 = 2^p - 2^{q-1}$ and $\widetilde{n} = n + 2^{q-1} - 1$. Clearly $2^{q-1} \le \widetilde{n} < 2^{p-1} - 2^{q-1}$. Let $m \ge 1$ be the largest integer such that $\widetilde{k} = 2^{q-1} + 2^q + \cdots + 2^{q+m-2} \le \widetilde{n}$, and let $\widetilde{r} = \widetilde{n} - \widetilde{k}$. We know from Lemma 2 that $\|T^{\widetilde{k}}(e_{\widetilde{i}})\| = 2^m$ and that all the nonzero coordinates of $T^{\widetilde{k}}(e_{\widetilde{i}})$ are in $J_{p,q+m}$. Lemma 5 implies that

$$\|T^n(e_i)\| = \|T^{\widetilde{n}}(e_{\widetilde{i}})\| = \|T^{\widetilde{r}}T^{\widetilde{k}}(e_{\widetilde{i}})\| \le 2\|T^{\widetilde{k}}(e_{\widetilde{i}})\| = 2^{m+1}.$$

Without striving to obtain the best possible constant C, it suffices to notice that $n = \widetilde{n} - 2^{q-1} + 1 \ge \widetilde{k} - 2^{q-1} + 1 \ge 2^{m-1}$, hence

$$\frac{1}{n}\|T^n(e_i)\| \le \frac{2^{m+1}}{n} \le 4. \ \blacksquare$$

PROPOSITION 7. *There exists a uniform bound K such that for all $i \in \mathbb{Z}$ and $n \in \mathbb{N}$, $\|A_n(e_i)\| \le K$.*

Proof. Let us assume again that $i \in J_p$ for some $p \geq 3$. If $i \in J_{p,p-1}$, then $\|T^n(e_i)\| = 1$ for $n \leq i - \min J_p$, and $\|T^n(e_i)\| = 0$ for all other values of n, thus $\|A_n(e_i)\| \leq 1$. Assume that $i \in J_{p,q}$ for some $1 \leq q \leq p-2$, i.e. $i = \min J_{p,q} + r$ for some $0 \leq r \leq 2^{q-1} - 1$ (r can also be 1 for $q = 1$). We have

$$T^{r+1}(e_i) = e_{\widetilde{i}-1} - e_{\widetilde{i}},$$

where $\widetilde{i} = \max J_{p,q+1} = 2^p - 2^q$. Next (the sum may be vacuous for $n \leq r$),

$$\begin{aligned}\sum_{k=r+1}^{n} T^k(e_i) &= \sum_{k=0}^{n-r-1} (T^k(e_{\widetilde{i}-1}) - T^k(e_{\widetilde{i}})) = \sum_{k=0}^{n-r-1} (T^{k+1}(e_{\widetilde{i}}) - T^k(e_{\widetilde{i}})) \\ &= T^{n-r}(e_{\widetilde{i}}) - e_{\widetilde{i}},\end{aligned}$$

therefore $\|A_n(e_i)\| = 1$ for $n \leq r$, and for every $n \geq r+1$ we have

$$\begin{aligned}\|A_n(e_i)\| &= \left\| \frac{1}{n} \sum_{k=1}^{n} T^k(e_i) \right\| \leq \frac{r}{n} + \left\| \frac{1}{n} \sum_{k=r+1}^{n} T^k(e_i) \right\| \\ &\leq \frac{r}{n} + \frac{1}{n} \|T^{n-r}(e_{\widetilde{i}}) - e_{\widetilde{i}}\| \leq \frac{r}{n} + \frac{1}{n} + \frac{1}{n} \|T^{n-r}(e_{\widetilde{i}})\| \\ &\leq \frac{r+1}{n} + \frac{n-r}{n} \cdot C \leq 1 + C,\end{aligned}$$

where C is the constant from Corollary 6. ∎

Let us now consider the continuous case, $X = L^1[0,1]$. This was done more or less the same way in [6], but we repeat it here to make this note self-contained. Choose an increasing sequence $\{t_j\}_{j=-\infty}^{\infty}$ of points in $(0,1)$ with $\lim_{j\to-\infty} t_j = 0$, $\lim_{j\to\infty} t_j = 1$, and represent $[0,1]$ (modulo a countable set) as the disjoint union of the intervals $I_j = (t_j, t_{j+1})$, $-\infty < j < \infty$. Let $\lambda_j = t_{j+1} - t_j$ be the length of I_j. Let $\tau : [0,1] \to [0,1]$ be the piecewise linear transformation which maps each I_j to the next one, I_{j+1}, linearly. Define $w, v : [0,1] \to \mathbb{R}_+$ by $w(x) = w_j$, $v(x) = v_j$ for $x \in I_j$, where $\{w_j\}$ and $\{v_j\}$ are the same sequences as before. Similarly to the discrete case, we define a bounded linear operator $\tilde{T} = \tilde{T}_{w,v}$ in the following way: for $f \in X$ we put $\tilde{T}f = g$, where

$$g(x) = w(x)\frac{\lambda_{j+1}}{\lambda_j} f(\tau x) + v(x)\frac{\lambda_{j+2}}{\lambda_j} f(\tau^2 x) \quad \text{for } x \in I_j.$$

To show that $\tilde{T}$ has the desired properties one can repeat, with simple modifications, the argument in the discrete case. A shorter way of dealing with the continuous case is to introduce a map $\theta : L^1[0,1] \to L^1(\mathbb{Z})$ which takes $\tilde{T}$ to T. Namely, for $f \in L^1[0,1]$ we put $\theta f = x$, where $x = \{x_j\}$ with $x_j = \int_{I_j} f$. It is clear that θ is an isometry and that θ conjugates $\tilde{T}$ and T, i.e., $T \circ \theta = \theta \circ \tilde{T}$. These properties imply that $\|\tilde{T}^n f\| = \|T^n(\theta f)\|$ for all $f \in L^1[0,1]$ and all $n \geq 0$.

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REFERENCES

[1] A. Brunel, S. Horowitz and M. Lin, *On subinvariant measures for positive operators in L_1*, Ann. Inst. H. Poincaré Probab. Statist. 29 (1993), 105–117.

[2] Y. Derriennic, *On the mean ergodic theorem for Cesàro bounded operators*, Colloq. Math. 84/85 (2000), 443–455.

[3] Y. Derriennic and M. Lin, *On invariant measures and ergodic theorems for positive operators*, J. Funct. Anal. 13 (1973), 252–267.

[4] R. Émilion, *Mean-bounded operators and mean ergodic theorems*, ibid. 61 (1985), 1–14.

[5] E. Hille, *Remarks on ergodic theorems*, Trans. Amer. Math. Soc. 57 (1945), 246–269.

[6] I. Kornfeld and W. Kosek, *Positive L^1 operators associated with nonsingular mappings and an example of E. Hille*, Colloq. Math. 98 (2003), 63–77.

[7] I. Kornfeld and A. Krygin, *On the cohomology equivalence classes for non-singular mappings*, Ergodic Theory Dynam. Systems 18 (1998), 1385–1397.

[8] I. Kornfeld and M. Lin, *Weak almost periodicity of L_1 contractions and coboundaries of non-singular transformations*, Studia Math. 138 (2000), 225–240.

[9] U. Krengel, *Ergodic Theorems*, de Gruyter Stud. Math. 6, de Gruyter, Berlin, 1985.

[10] M. Lorente Domínguez and F. J. Martín-Reyes, *The ergodic Hilbert transform for Cesàro bounded flows*, Tohoku Math. J. 46 (1994), 541–556.

[11] N. K. Nikol'skiĭ, *Selected Problems of Weighted Approximation and Spectral Analysis*, Proc. Steklov Inst. Math. 120, Amer. Math. Soc., Providence, RI, 1976; Russian original: Nauka, Leningrad, 1974.

[12] K. Schmidt, *Cocycles on Ergodic Transformation Groups*, Macmillan Lectures in Math. 1, Macmillan of India, Delhi, 1977.

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