COLLOQUIUM MATHEMATICUM

VOL. 124 2011 NO. 1

ALMOST EVERYWHERE CONVERGENCE OF GENERALIZED ERGODIC TRANSFORMS FOR INVERTIBLE POWER-BOUNDED OPERATORS IN L^p

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Abstract. We show that some results of Gaposhkin about a.e. convergence of series associated to a unitary operator U acting on $L^2(X,\Sigma,\mu)$ (μ is a σ-finite measure) may be extended to the case where U is an invertible power-bounded operator acting on $L^p(X,\Sigma,\mu)$, $p>1$. The proofs make use of the spectral integration initiated by Berkson–Gillespie and, more specifically, of recent results of the author.

1. Introduction. In a series of papers (see [8], [9] or [10]), Gaposhkin studied conditions, expressed in terms of $\|f+\cdots+U^{n-1}f\|_2$, ensuring the a.e. convergence of averages $\big\{\frac{1}{\varphi(n)}(f+\cdots+U^{n-1}f)\big\}$ or of series $\sum_n a_nU^nf$, where U is a unitary operator acting on $L^2(X,\Sigma,\mu)$ and $f\in L^2(X,\mu)$. His proofs were based on the use of spectral theory and on the dyadic chaining to obtain maximal inequalities.

It has been realised recently (see Theorem 6.3 of [6] and the remark below) that when the weight $\varphi(n)$ is smaller than and not "too close" to n, the results of Gaposhkin concerning averages remain valid for power-bounded operators T acting on $L^p(X,\mu)$ ($p>1$). Hence, the use of spectral theory does not seem to be relevant in this specific situation. However, concerning the a.e. convergence of series, it seems that the results of Gaposhkin for unitary operators do not pass to power-bounded operators on $L^p(X,\mu)$ (that are not invertible).

In this paper, we prove that conditions analogous to Gaposhkin's are sufficient for the a.e. convergence of the above series when U is assumed to be an invertible operator on $L^p(X,\mu)$ ($p>1$ fixed) which is *doubly power-bounded*, i.e. $\sup_{n\in\mathbb{Z}}\|T^n\|<\infty$.

In particular, we obtain the following, where $\log_k n := (\log\circ\cdots\circ\log)(n)$ (k-fold composition) is defined for n large enough. Throughout the paper, log will denote the logarithm to base e.

2010 *Mathematics Subject Classification*: Primary 47B40, 37A30.

Key words and phrases: Berkson–Gillespie spectral integral, L^p spaces, ergodic transforms, almost everywhere convergence.

DOI: 10.4064/cm124-1-5

THEOREM 1.1. *Let $1 < p \le 2$. Let U be an invertible operator on $L^p(X,\Sigma,\mu)$, doubly power-bounded (i.e. $\sup_{n\in\mathbb{Z}}\|U^n\|<\infty$). Let $f\in L^p(X,\mu)$ and $\alpha\in[1/p,1]$, $\beta\in\mathbb{R}$. Assume one of the following:*

$$
\begin{aligned}
&\sum_n \frac{\|f+\dots+U^{n-1}f\|_p^p}{n^2(\log n)^{p(\beta-1)}}<\infty && \text{if } \alpha=1/p \text{ and } \beta>1/p,\\
&\sum_n \frac{\|f+\dots+U^{n-1}f\|_p^p}{n^{1+p\alpha}(\log n)^{p\beta}}(\log_2 n)^p<\infty && \text{if } 1/p<\alpha<1,\\
&\sum_n \frac{\|f+\dots+U^{n-1}f\|_p^p}{n^{1+p}(\log n)^{p(\beta-1)+1}}(\log_3 n)^p<\infty && \text{if } \alpha=1 \text{ and } \beta<1,\\
&\sum_n \frac{\|f+\dots+U^{n-1}f\|_p^p}{n^{1+p}\log n}(\log_2 n)^{p-1}(\log_4 n)^p<\infty && \text{if } \alpha=\beta=1.
\end{aligned}
\tag{1}
$$

Then $\sum_{n\ge3}\frac{U^nf}{n^\alpha(\log n)^\beta}$ converges μ-a.e.

For $p>2$, we obtain the following.

THEOREM 1.2. *Let $p>2$. Let U be an invertible operator on $L^p(X,\Sigma,\mu)$, doubly power-bounded. Let $f\in L^p(X,\mu)$ and $\alpha\in[1/p,1]$, $\beta\in\mathbb{R}$. Assume that*

$$
\begin{aligned}
&\sum_n \frac{\|f+\dots+U^{n-1}f\|_p^p}{n^2(\log n)^{p(\beta-1)}}<\infty && \text{if } \alpha=1/p \text{ and } \beta>1/p,\\
&\sum_{n\ge3} \frac{\|f+\dots+U^{n-1}f\|_p^2}{n^{1+2\alpha}(\log n)^{2\beta}}<\infty && \text{if } 1/p<\alpha<1,\\
&\sum_{n\ge3} \frac{\|f+\dots+U^{n-1}f\|_p^2}{n^{3}(\log n)^{2(\beta-1)+1}}<\infty && \text{if } \alpha=1 \text{ and } \beta<1,\\
&\sum_{n\ge3} \frac{\|f+\dots+U^{n-1}f\|_p^2}{n^{3}\log n}(\log_2 n)<\infty && \text{if } \alpha=\beta=1.
\end{aligned}
$$

Then $\sum_{n\ge3}\frac{U^nf}{n^\alpha(\log n)^\beta}$ converges μ-a.e.

In [3], Cohen–Cuny–Lin proved that when U is induced by a measure preserving transformation and $0<\alpha<1$, $\beta=0$, some weaker conditions are sufficient.

Finally, for $p\ge2$, we obtain a sufficient condition for the convergence of the Cesàro averages associated to a doubly power-bounded operator. When $p=2$, we recover Gaposhkin's condition (see e.g. [8]). We have not succeeded in obtaining a similar condition for $p<2$.

THEOREM 1.3. *Let $p \geq 2$. Let $f \in L^p(X,\mu)$ and U be an invertible power-bounded operator on $L^p(X,\Sigma,\mu)$. Assume that*

$$\sum_{n\geq 1} \frac{\|f+\cdots+U^{n-1}f\|_p^2}{n^3\log n}\log\log n < \infty.$$

Then $\lim_{n\to\infty} \frac{f+\cdots+U^{n-1}f}{n} = 0$ *μ-a.e. and* $\sum_{k=1}^n \frac{U^k f - U^{-k} f}{k}$ *converges μ-a.e.*

2. Almost everywhere convergence of averages. We now give some results about a.e. convergence of averages. These results are essentially known (see e.g. Wu [13], Cohen–Lin [4] or Weber [12]), but we state them in a somewhat more elegant form, which is needed later and does not seem to appear in the literature. In this section, T is a power-bounded operator on $L^p(X,\mu)$ which is *not necessarily invertible.*

Let us recall the following maximal inequality, which is a straightforward application of Proposition 1(i) of [13].

PROPOSITION 2.1 (Wu [13]). *Let T be a power-bounded operator on $L^p(X,\Sigma,\mu)$, $p>1$. Then, for every $f\in L^p(X,\mu)$ and $n\geq 0$,*

$$\Big\| \max_{1\leq k\leq 2^n} |f+\cdots+T^{k-1}f| \Big\|_p \leq K\sum_{l=0}^n 2^{(n-l)/p}\|f+\cdots+T^{2^l-1}f\|_p,$$

where $K=\sup_{n\geq 1}\|T^n\|$.

Let ψ be a positive non-decreasing function on $[1,\infty[$. When $\sum_{n\geq 1} 1/\psi^p(n) < \infty$, we write $M_m = M_m(\psi,p) := \sum_{n\geq m} 1/\psi^p(n)$.

PROPOSITION 2.2. *Let $\psi : [0,\infty[\ \to\ ]0,\infty[$ be non-decreasing and such that there exists $C>0$ with $\psi(u)\geq C\psi(2u)$ for every $u>0$. Let T be a power-bounded operator on $L^p(X,\Sigma,\mu)$, $p>1$. Let $f\in L^p(X,\mu)$ be such that*

$$\sum_{n\geq 1}\frac{(\sum_{l=0}^n 2^{(n-l)/p}\|f+\cdots+T^{2^l-1}f\|_p)^p}{\psi^p(2^n)} < \infty. \tag{2}$$

Then

$$\frac{1}{\psi(n)}\sum_{k=0}^{n-1}T^k f \xrightarrow[n\to\infty]{} 0 \ \mu\text{-a.e.} \quad \text{and} \quad \sup_{n\geq 1}\frac{1}{\psi(n)}\Big|\sum_{k=0}^{n-1}T^k f\Big| \in L^p(X,\mu).$$

In particular, if $\sum_{n\geq 1} 1/\psi^p(n) < \infty$ and

$$\sum_n \frac{\|f+\cdots+T^{n-1}f\|_p^p}{n\psi^p(n)}\Big(\frac{M_n\psi^p(n)}{n}\Big)^p < \infty, \tag{3}$$

then (2) is satisfied.

Proof. See Appendix A.

From this proposition, it is not difficult to recover Theorem B and Theorem 3(iii)–(vii) of [9], as in the corollary below.

We will need the following application.

COROLLARY 2.3. *Let T be a power-bounded operator on $L^p(X,\mu)$, $p>1$. Let $\alpha\in[1/p,1]$ and $\beta,\gamma\in\mathbb{R}$. Let $f\in L^p(X,\mu)$ be such that*

$$\sum_n \frac{\|f+\cdots+T^{n-1}f\|_p^p}{n^{1+p\alpha}(\log n)^{p\beta}(\log\log n)^{p\gamma}}<\infty \qquad \text{if } 1/p<\alpha\le 1, \tag{4}$$

$$\sum_n \frac{\|f+\cdots+T^{n-1}f\|_p^p}{n^{2}(\log n)^{p(\beta-1)}(\log\log n)^{p\gamma}}<\infty \qquad \text{if } \alpha=1/p \text{ and } \beta>1/p. \tag{5}$$

Then

$$\frac{1}{n^\alpha(\log n)^\beta(\log\log n)^\gamma}\sum_{k=0}^{n-1}T^kf \xrightarrow[n\to\infty]{} 0 \qquad \mu\text{-a.e.}$$

3. Almost everywhere convergence of series. We start by recalling the definition and properties of the spectral integral of Berkson–Gillespie (see [2] or [6] for more details).

Let (X,Σ,μ) be a σ-finite measure space. Let $p>1$ fixed. Let U be an invertible operator on $L^p(X,\Sigma,\mu)$, doubly power-bounded, i.e. such that

$$c:=\sup_{n\in\mathbb{Z}}\|U^n\|<\infty.$$

It has been proved in [2] that there exists a unique projection-valued function $E:[0,2\pi]\to\mathcal{B}(L^p(X,\mu))$, where $\mathcal{B}(L^p(X,\mu))$ stands for the Banach algebra of bounded operators on $L^p(X,\mu)$, with the following properties.

The function E is right continuous on $[0,2\pi]$ in the strong operator topology (SOT), admits at each point $s\in(0,2\pi]$ a SOT left-hand limit, and further satisfies

(i) $E(s)E(t)=E(\min(s,t))$ for all $s,t\in[0,2\pi]$;
(ii) $\lim_{s\to0}E(s)=0$ (SOT);
(iii) $\lim_{s\to2\pi}E(s)=I=E(2\pi)$ (SOT),

where I denotes the identity operator on $L^p(X,\mu)$. Those properties allow one to define an integral (of Riemann–Stieltjes type) with respect to E. More precisely, there exists a map $g\mapsto\int_{[0,2\pi]}g(t)\,dE(t)$ defined on the Banach algebra $BV[0,2\pi]$ of functions with bounded variation on $[0,2\pi]$, that is an identity preserving algebra homomorphism into $\mathcal{B}(L^p(X,\mu))$. Moreover

$$\int_{[0,2\pi]}e^{it}\,dE(t)f=Uf,$$

and there exists a constant C_p, depending only on p, such that

$$\sup_{t\in[0,2\pi[} \|E(t)\| \le c^2 C_p. \tag{6}$$

We want to obtain conditions on $f \in L^p(X,\mu)$ that are sufficient for the a.e. convergence of series $\sum_n a_n U^n f$. Write $S_n = S_n(f) = Uf + \cdots + U^n f$. By Abel summation, the above convergence is equivalent to the fact that the series $\sum_n (a_n - a_{n+1})S_n$ converges a.e. and $a_n S_n \to 0$ a.e. In what follows, the sequence $\{\alpha_n\}$ that we use will play the role of $\{a_n - a_{n+1}\}$.

In [6], we were interested in the norm convergence of $\sum_n a_n U^n f$ and, for technical reasons, we already had to decompose the problem in two steps, while in the case $p = 2$, Gaposhkin is doing everything at once.

Denote by $\mathcal{L}$ the set of positive functions $\varphi \in C^1(]0,\pi])$ such that there exists $\varepsilon \in]0,1[$ such that

(C1) φ and $-\varphi'$ are non-increasing.
(C2) $t \mapsto t^{1-\varepsilon}\varphi(t)$ and $t \mapsto t^{2-\varepsilon}\varphi'(t)$ are non-decreasing.

It follows from (C2) that

$$\varphi(t) \ge -t\varphi'(t) \qquad \forall t \in]0,\pi] \tag{7}$$

and

$$\lim_{t\to 0} t\varphi(t) = \lim_{t\to 0} t^2\varphi'(t) = 0. \tag{8}$$

Given $\varphi \in \mathcal{L}$, we define $\mathcal{L}_\varphi$ to be the set of positive sequences $\{\alpha_n\}$ such that $\{n\alpha_n\}$ is non-increasing and there exists $C > 0$ for which

$$\sum_{k=1}^{n} k^2 \alpha_k \le -C\varphi'(\pi/n) \qquad \forall n \ge 1. \tag{9}$$

Then, by (7), we have

$$\sum_{k=1}^{n} k^2 \alpha_k \le C' n\varphi(\pi/n) \qquad \forall n \ge 1. \tag{10}$$

Notice that, using (C2) and Abel summation, (9) implies

$$\sum_{k\ge n} \alpha_k \le -C'' \frac{\varphi'(\pi/n)}{n^2} \qquad \forall n \ge 1. \tag{11}$$

In the following we will take $\alpha_n := \frac{1}{n^{1+\alpha}(\log n)^\beta}$ and

(a) $\varphi(x) = \dfrac{1}{x^{1-\alpha}|\log x|^\beta}$ if $1/p < \alpha < 1$,

(b) $\varphi(x) = |\log x|^{1-\beta}$ if $\alpha = 1$, $\beta < 1$,

(c) $\varphi(x) = \log|\log x|$ if $\alpha = 1$, $\beta = 1$.

Write, for every $n \geq 1$, $A_n = A_n(f) = (f + \cdots + U^{n-1}f)/n$, and for every $t \in \mathbb{R}$, $\sigma_n(t) = 1 + \cdots + e^{i(n-1)t}$. For $\{\alpha_n\} \in \mathcal{L}_\varphi$, define $W(t) = \sum_{n\geq 1} \alpha_n \sigma_n(t)$, which converges uniformly on every compact subset of $]0, 2\pi[$, since $\{\alpha_n\}$ is decreasing to 0.

Proposition 3.1. *Let $\varphi \in \mathcal{L}$ and $\{\alpha_n\} \in \mathcal{L}_\varphi$. Let $\{n_k\}$ be an increasing sequence of positive integers. Let $f \in L^p(X,\mu)$ be such that $E(0)f = 0$ and*

$$\sum_{k\geq 0} (\varphi(\pi/2^{n_k})\|A_{2^{n_k}}\|_p)^{\min(p,2)} < \infty. \tag{12}$$

Then there exists $\{g_m\} \subset L^p(\mu)$ with $\lim_{k\to\infty} g_{2^{n_k}} = 0$ μ-a.e. and in L^p, such that, for every $m \geq 1$, taking $n = \left[\frac{\log m}{\log 2}\right]$, we have

$$\sum_{k=1}^{m} \alpha_k S_k(f) = g_m + \int_{]\pi/2^n, 2\pi-\pi/2^n[} W(t)\, dE(t)f. \tag{13}$$

Proof. Let $g_m = \sum_{k=1}^m \alpha_k S_k(f) - \int_{]\pi/2^n,2\pi-\pi/2^n[} W(t)\,dE(t)f$. It follows from Lemmas 4.5 and 4.6 and Theorem 3.3 of [6] that there exists $K > 0$ such that

$$\begin{aligned} \|g_{2^n}\|_p^p \leq K\bigg(&\frac{\varphi^p(\pi/2^n)}{2^{pn}}\bigg\|\Big(\sum_{l=0}^{n-1} 2^{2l}|A_{2^l} - A_{2^{l+1}}|^2\Big)^{1/2}\bigg\|_p^p \\ &+ \bigg\|\varphi(\pi/2^n)\Big(\sum_{l\geq n} |A_{2^l} - A_{2^{l+1}}|^2\Big)^{1/2}\bigg\|_p^p\bigg). \end{aligned} \tag{14}$$

To prove the proposition, it suffices to show that $\sum_{k\geq 0} \|g_{2^{n_k}}\|_p^p < \infty$.

Define $Q_n^2 = Q_n^2(f) = \sum_{l\geq n} |A_{2^l} - A_{2^{l+1}}|^2$. It has been proved in [6, (10)] that there exists $C_p > 0$ (depending only on p) such that $\|Q_n\|_p \leq c^2 C_p \|A_{2^n}\|_p$, where $c = \sup_{n\in\mathbb{Z}} \|U^n\|$.

Hence, by (12),

$$\begin{aligned} \sum_{k\geq 0} \bigg\|\varphi(\pi/2^{n_k})\Big(\sum_{l\geq n_k} |A_{2^l} - A_{2^{l+1}}|^2\Big)^{1/2}\bigg\|_p^p& \\ \leq c^{2p} C_p^p \sum_{k\geq 0} (\varphi(\pi/2^{n_k})\|A_{2^{n_k}}\|_p)^p < \infty.& \end{aligned}$$

For $k \geq 0$, we have $\sum_{l=0}^{n_k-1} 2^{2l}|A_{2^l} - A_{2^{l+1}}|^2 \leq \sum_{j=1}^k 2^{2n_j} Q_{n_{j-1}}^2$.

Assume that $p \geq 2$. We have, using (C2) and the fact that $x \mapsto x^{p/2}$ is superadditive on $[0, \infty[$,

$$\sum_{k\geq 0}\frac{\varphi^p(\pi/2^{n_k})}{2^{pn_k}}\Big\|\Big(\sum_{l=0}^{n_k-1}2^{2l}|A_{2^l}-A_{2^{l+1}}|^2\Big)^{1/2}\Big\|_p^p$$
$$\leq\Big\|\Big(\sum_{k\geq 0}\frac{\varphi^2(\pi/2^{n_k})}{2^{2n_k}}\sum_{j=1}^{k}2^{2n_j}Q_{n_{j-1}}^2\Big)^{1/2}\Big\|_p^p\leq C\Big\|\Big(\sum_{j\geq 1}\varphi^2(\pi/2^{n_j})Q_{n_{j-1}}\Big)^{1/2}\Big\|_p^p$$
$$\leq C'\Big(\sum_{k\geq 0}\varphi^2(\pi/2^{n_k})\|A_{n_k}\|_p^2\Big)^{p/2}.$$

If $p\in\,]1,2[$, we can use

$$\Big\|\Big(\sum_{l=0}^{n_k-1}2^{2l}|A_{2^l}(f)-A_{2^{l+1}}(f)|^2\Big)^{1/2}\Big\|_p^p\leq\sum_{j=1}^{k}2^{pn_j}\|Q_{n_{j-1}}\|_p^p$$
$$\leq\sum_{j=1}^{k}2^{pn_j}\|A_{2^{n_{j-1}}}\|_p^p,$$

and then conclude as above. ■

Proposition 3.2. *Let $\varphi\in\mathcal{L}$ and $\{\alpha_n\}\in\mathcal{L}_\varphi$. Let $\{n_k\}$ be an increasing sequence of positive integers. Let $f\in L^p(X,\mu)$ be such that $E(0)f=0$ and*

$$\sum_{k\geq 0}(\varphi(\pi/2^{n_k})\|A_{2^{n_k}}\|_p\log k)^p<\infty\qquad\text{if } p\in\,]1,2], \tag{15}$$

$$\sum_{k\geq 0}(\varphi(\pi/2^{n_k})\|A_{2^{n_k}}\|_p)^2<\infty\qquad\text{if } p>2. \tag{16}$$

Then $\{\int_{]\pi/2^{n_k},2\pi-\pi/2^{n_k}[}W(t)\,dE(t)f\}$ converges μ-a.e.

Proof. We show the μ-a.e. convergence of $\{\int_{]\pi/2^{n_k},\pi[}W(t)\,dE(t)f\}$, the proof for the second part being entirely the same.

Define $w_k:=\int_{]\pi/2^{n_{k+1}},\pi/2^{n_k}]}W(t)\,dE(t)f$. Hence we have to prove the μ-a.e. convergence of the series $\sum_{k\geq 0}w_k$. It follows from [5] that, for every $s>r\geq 0$,

$$\Big\|\sum_{k=r}^{s}w_k\Big\|_p^p\leq C\Big\|\Big(\sum_{l=2^{n_r}}^{2^{n_s}}\varphi^2(\pi/2^l)|A_{2^l}-A_{2^{l+1}}|^2\Big)^{1/2}\Big\|_p^p$$
$$\leq C\Big\|\Big(\sum_{j=r}^{s}\varphi^2(\pi/2^{n_j})Q_{n_j}^2\Big)^{1/2}\Big\|_p^p.$$

Hence, we obtain

$$\Big\|\sum_{k=r}^{s} w_k\Big\|_p^p \le \begin{cases} C\displaystyle\sum_{j=r}^{s}\varphi^p(\pi/2^{n_j})\|A_{2^{n_j}}\|_p^p & \text{if } p\in\,]1,2], \\ C\Big(\displaystyle\sum_{j=r}^{s}\varphi^2(\pi/2^{n_j})\|A_{2^{n_j}}\|_p^2\Big)^{p/2} & \text{if } p>2. \end{cases}$$

Then the result follows by applying Theorem 2.4 of [4] when $p\in\,]1,2]$, and Theorem 2.5 of [4] when $p>2$. ∎

By Propositions 3.1 and 3.2, we see that, under conditions (15) or (16), the series $\sum_{k\ge 0}\sum_{n=2^{n_k}}^{2^{n_{k+1}}-1}\alpha_n S_n$ converges μ-a.e. Hence, to prove the μ-a.e. convergence of $\sum_n \alpha_n S_n$, it suffices to prove that $\sum_{l=2^{n_k}}^{2^{n_{k+1}}-1}\alpha_l|S_l|\to 0$ μ-a.e.

Before proving this last fact, we explain how to choose the sequence $\{n_k\}$, depending on φ.

Let $\{n_k\}$ be an increasing sequence of positive integers. We say that $\varphi\in\mathcal{L}$ is $\{n_k\}$-*adapted* if there exists $C>0$ such that

$$\varphi(\pi/2^{n_{s+1}})\le C\varphi(\pi/2^{n_s})\qquad\forall s\ge 0. \tag{17}$$

Let $\{\alpha_n\}\in\mathcal{L}_\varphi$. We say that $(\{\alpha_n\},\varphi)$ is $\{n_k\}$-*adapted* if φ is $\{n_k\}$-adapted and there exists $C'>0$ such that

$$\sum_{k=2^{n_{s-1}}}^{2^{n_s}-1} k\alpha_k \ge C'\varphi(\pi/2^{n_{s+1}})\qquad\forall s\ge 1. \tag{18}$$

For instance, as before one may take $\alpha_n := \frac{1}{n^{1+\alpha}(\log n)^\beta}$ and

(a) $\varphi(x)=\dfrac{1}{x^{1-\alpha}|\log x|^\beta}$, $n_k=k$, if $1/p<\alpha<1$,

(b) $\varphi(x)=|\log x|^{1-\beta}$, $n_k=2^k$, if $\alpha=1$, $\beta<1$,

(c) $\varphi(x)=\log|\log x|$, $n_k=2^{2^k}$, if $\alpha=1$, $\beta=1$.

We will need the following technical lemmata. The proofs are left to the appendix.

LEMMA 3.3. *Let $r>1$, $\varphi\in\mathcal{L}$ and $\{\alpha_n\}\in\mathcal{L}_\varphi$. Let $\{n_k\}$ be an increasing sequence of positive integers. Let $\{w_n\}$ be a subadditive sequence of positive numbers. Consider the following conditions:*

(i) $\displaystyle\sum_{k\ge 0}\Big(\varphi(\pi/2^{n_k})\frac{w_{2^{n_k}}}{2^{n_k}}\Big)^r<\infty$,

(ii) $\displaystyle\sum_{n\ge 1}\varphi(\pi/n)^{r-1}n\alpha_n\Big(\frac{w_n}{n}\Big)^r<\infty$.

If φ is $\{n_k\}$-adapted, then (i)⇒(ii), *and if $(\{\alpha_n\},\varphi)$ is $\{n_k\}$-adapted, then* (ii)⇒(i).

REMARK. When T is a power-bounded operator on L^p, then $N_p(f) := \sup_{n\geq 0} \|T^n f\|_p$ defines a norm on L^p, equivalent to $\|\cdot\|_p$, for which T is a contraction. In particular, $\{N_p(f+\cdots+T^{n-1}f)\}_n$ is subadditive. Since N_p and $\|\cdot\|_p$ are equivalent, the lemma applies with $\{\|f+\cdots+T^{n-1}f\|_p\}_n$ in place of $\{w_n\}_n$.

LEMMA 3.4. *Let $\{n_k\}$ be any increasing sequence of integers. Let $\varphi \in \mathcal{L}$ be $\{n_k\}$-adapted and let $\{\alpha_n\}_{n\in\mathbb{N}} \in \mathcal{L}_\varphi$. Let $f \in L^p(X,\mu)$ be such that $E(0)f = 0$. Assume that*

$$\sum_{k\geq 0} (\varphi(\pi/2^{n_k})\|A_{2^{n_k}}\|_p)^{\min(p,2)} < \infty. \tag{19}$$

Then

$$\sum_{l=2^{n_k}}^{2^{n_{k+1}}-1} |\alpha_l S_l| \xrightarrow[n\to\infty]{} 0 \qquad \mu\text{-a.e.}$$

PROPOSITION 3.5. *Let $\varphi \in \mathcal{L}$ and $\{\alpha_n\} \in \mathcal{L}_\varphi$. Let $\{n_k\}$ be an increasing sequence of positive integers such that φ is $\{n_k\}$-adapted. Let $f \in L^p(X,\mu)$ be such that $E(0)f = 0$ and*

$$\sum_{k\geq 0} (\varphi(\pi/2^{n_k})\|A_{2^{n_k}}\|_p \log k)^p < \infty \qquad \text{if } p \in]1,2], \tag{20}$$

$$\sum_{k\geq 0} (\varphi(\pi/2^{n_k})\|A_{2^{n_k}}\|_p)^2 < \infty \qquad \text{if } p > 2. \tag{21}$$

Then $\sum_{n\geq 1} \alpha_n S_n$ converges μ-a.e.

Proof. Combine Lemma 3.4 and Propositions 3.1 and 3.2. ∎

Proof of Theorem 1.1. We want to prove that $\sum U^n f/(n^\alpha(\log n)^\beta)$ converges a.e. Using Abel summation by parts, it is sufficient (actually equivalent) to prove that $S_n/(n^\alpha(\log n)^\beta) \to 0$ μ-a.e. as $n\to\infty$ and that

$$\sum_n \left(\frac{1}{n^\alpha(\log n)^\beta} - \frac{1}{(n+1)^\alpha(\log(n+1))^\beta}\right) S_n$$

converges μ-a.e.

There exist $K, L > 0$ such that

$$\frac{1}{n^\alpha(\log n)^\beta} - \frac{1}{(n+1)^\alpha(\log(n+1))^\beta} = \frac{K}{n^{\alpha+1}(\log n)^\beta} + \frac{L}{n^{\alpha+1}(\log n)^{\beta+1}} + O\left(\frac{1}{n^{\alpha+2}(\log n)^\beta}\right).$$

The series $\sum S_n/(n^{2+\alpha}(\log n)^\beta)$ clearly converges (absolutely) μ-a.e. On the other hand, if $\sum_{n\geq 3} S_n/(n^{1+\alpha}(\log n)^\beta)$ converges μ-a.e., then so does

$\sum_{n\geq 3} S_n/(n^{1+\alpha}(\log n)^{\beta+1})$ (by Abel summation by parts, using that $1/\log n$ decreases to 0).

Hence, we have to prove that $S_n/(n^\alpha(\log n)^\beta) \to 0$ μ-a.e. and that the series $\sum_{n\geq 3} S_n/(n^{1+\alpha}(\log n)^\beta)$ converges μ-a.e.

Assume that one of the conditions of Theorem 1.1 is satisfied. Then the conditions of Corollary 2.3 with the corresponding choice of α, β are satisfied and $S_n/(n^\alpha(\log n)^\beta) \to 0$ μ-a.e.

Now, we want to apply Proposition 3.5 with $\alpha_n = 1/(n^{\alpha+1}(\log n)^\beta)$. We choose φ and n_k as in cases (a), (b) and (c). Then we have

$$\sum \varphi^{p-1}(\pi/n) n\alpha_n (\|S_n\|_p/n)^p \log_l n < \infty,$$

where $l = 2$ if $\alpha < 1$, $l = 3$ if $\alpha = 1$ and $\beta < 1$, and $l = 4$ if $\alpha = \beta = 1$.

An adaptation of the proof of Lemma 3.3 shows that (20) holds, which finishes the proof of Theorem 1.1.

Proof of Theorem 1.2. Assume first that $\alpha > 1/p$. We follow the same steps as in the proof of Theorem 1.1. Assume that one of the conditions of Theorem 1.2 is satisfied.

We use the previous notations $(\alpha_n, \varphi, \ldots)$. By Lemma 3.3 with $r = 2$ (see also the remark after the lemma), we find that (21) is satisfied, and Proposition 3.5 applies.

Now, by (21), $\varphi(\pi/2^{n_k})\|A_{2^{n_k}}\|_p$ goes to 0 and we easily derive that

$$\sum_{k\geq 0} (\varphi(\pi/2^{n_k})\|A_{2^{n_k}}\|_p)^p < \infty,$$

which implies by Lemma 3.3 that

$$\sum_{n\geq 1} \varphi^{p-1}(\pi/n) n\alpha_n (\|S_n\|_p/n)^p < \infty.$$

Then one can check that conditions (4) and (5) of Corollary 2.3 are satisfied, which finishes the proof.

For $\alpha = 1/p$, we first notice that condition (5) is satisfied and we apply Corollary 2.3. Then by Hölder's inequality with $p_0 := p/2$ and $1/p_0 + 1/q_0 = 1$ (i.e. $q_0 = p/(p-2)$) we find

$$\begin{aligned}\sum_{n\geq 3} \frac{\|f+\cdots+U^{n-1}f\|_p^2}{n^{1+2\alpha}(\log n)^{2\beta}} &= \sum_{n\geq 3} \left(\frac{\|f+\cdots+U^{n-1}f\|_p^2}{n^{1/p_0+2\alpha}(\log n)^{2(\beta-1)}}\right)\left(\frac{1}{n^{1/q_0}(\log n)^2}\right) \\ &\leq \left(\sum_{n\geq 3} \frac{\|f+\cdots+U^{n-1}f\|_p^p}{n^{1+p\alpha}(\log n)^{(p-1)\beta}}\right)^{1/p_0} \left(\sum_{n\geq 3} \frac{1}{n(\log n)^{2p/(p-2)}}\right)^{1/q_0} < \infty,\end{aligned}$$

by assumption and since $2p/(p-2) > 2 > 1$.

Then we can conclude as above, using Proposition 3.5.

4. Proof of Theorem 1.3. Let U be an invertible operator on $L^p(X,\Sigma,\mu)$ with $\sup_{n\in\mathbb{Z}}\|U^n\|<\infty$, where we assume $p\ge 2$. Let $f\in L^p(X,\mu)$ and define $K_nf=\int_{]0,\pi/2^n]}dE(t)f$ and $L_nf=\int_{]2\pi-\pi/2^n,2\pi]}dE(t)f$. Generalizing a result of Gaposhkin for unitary operators, Berkson, Bourgain and Gillespie [1] proved that $\{(f+\cdots+U^{n-1}f)/n\}$ converges μ-a.e. if and only if $\{K_n(f)+L_n(f)\}$ converges μ-a.e. They also proved that $\sum_{k=1}^n(U^kf-U^{-k}f)/k$ converges μ-a.e. if and only if $\{K_n(f)-L_n(f)\}$ does, generalizing, that time, a result of Jajte [11].

We now give a sufficient condition for the convergence μ-a.e. of $K_n(f)$ and $L_n(f)$. We will deal only with $\{K_n(f)\}$, the proof for $\{L_n(f)\}$ being the same.

Define $w_n=\int_{]\pi/2^{n+1},\pi/2^n]}dE(t)f$. Then, by an analogue of the Littlewood–Paley theorem (see e.g. Theorem 2.4 of [6]), $\int_{]0,\pi/2^n]}dE(t)f=\sum_{k\ge n}w_k$ with convergence in $L^p(X,\mu)$, and for every $n>m$,

$$\Big\|\sum_{k=m}^{n-1}w_k\Big\|_p^p\le(c^2C_p)^p\Big\|\Big(\sum_{k=m}^{n-1}|w_k|^2\Big)^{1/2}\Big\|_p^p.$$

Hence, we have to find a condition for the μ-a.e. convergence of $\sum_{n\ge1}w_n$.

By [6, Theorem 3.3, (11)], we have, for every $n>m$,

$$\Big\|\sum_{k=m}^{n-1}w_k\Big\|_p^p\le(c^2C_p)^p\Big\|\Big(\sum_{k=m}^{n-1}|A_{2^{k+1}}(f)-A_{2^k}(f)|^2\Big)^{1/2}\Big\|_p^p. \tag{22}$$

Write

$$\begin{aligned}d(m,n)&=\Big\|\Big(\sum_{k=m}^{n-1}|A_{2^{k+1}}(f)-A_{2^k}(f)|^2\Big)^{1/2}\Big\|_p^p\\&=\int_X\Big(\sum_{k=m}^{n-1}|A_{2^{k+1}}(f)-A_{2^k}(f)|^2\Big)^{p/2}d\mu.\end{aligned}$$

Notice that, since $p\ge2$, we have $\|\ \|_{\ell^{p/2}}\le\|\ \|_{\ell^1}$, and $\{d(m,n)\}$ defines a superadditive array of numbers, that is, $d(m,n)+d(n,r)\le d(m,r)$ whenever $0<m<n<r$. Hence, by Proposition 2.2 of [4], there exists $K>0$ such that, for every $n\ge1$, we have

$$\Big\|\max_{2^n\le k\le 2^{n+1}-1}\Big|\sum_{l=2^n}^k w_k\Big|\Big\|_p^p\le Kn^pd(2^n,2^{n+1}),$$

and so

$$\sum_{n\ge1}\Big\|\max_{2^n\le k\le 2^{n+1}-1}\Big|\sum_{l=2^n}^k w_k\Big|\Big\|_p^p\le K\sum_{n\ge1}n^pd(2^n,2^{n+1}).$$

On the other hand we have, with $1/p+1/q=1$,

$$\sum_{n\geq 1}\Big\|\sum_{l=2^n}^{2^{n+1}-1} w_k\Big\|_p \leq \Big(\sum_{n\geq 1}\frac{1}{n^q}\Big)^{1/q}\Big(\sum_{n\geq 1} n^p\Big\|\sum_{l=2^n}^{2^{n+1}-1} w_k\Big\|_p^p\Big)^{1/p}.$$

Hence, if we show that $\sum_{n\geq 1} n^p d(2^n,2^{n+1})<\infty$, using (22) and the definition of $d(m,n)$, we will obtain

$$\max_{2^n\leq k\leq 2^{n+1}-1}\Big|\sum_{l=2^n}^{k} w_l\Big| \xrightarrow[n\to\infty]{} 0 \qquad \mu\text{-a.e.}$$

and

$$\sum_{n\geq 1}\Big|\sum_{l=2^n}^{2^{n+1}-1} w_k\Big|<\infty \qquad \mu\text{-a.e.},$$

which yields the μ-a.e. convergence of $\sum_{n\geq 1} w_n$.

Recall that $Q_n^2(f)=\sum_{k\geq n}|A_{2^{k+1}}(f)-A_{2^k}(f)|^2$. Using superadditivity again, and Fubini, we obtain

$$\begin{aligned}\sum_{n\geq 1} n^p d(2^n,2^{n+1}) &\leq \int_X \Big(\sum_{n\geq 1} n^2\sum_{k=2^n}^{2^{n+1}-1}|A_{2^{k+1}}(f)-A_{2^k}(f)|^2\Big)^{p/2}d\mu\\ &\leq C\int_X\Big(\sum_{n\geq 1}(Q_{2^n}^2(f)-Q_{2^{n+1}}^2(f))\sum_{l=1}^{n} l\Big)^{p/2}d\mu \leq C\int_X\Big(\sum_{n\geq 1} nQ_{2^n}^2(f)\Big)^{p/2}d\mu.\end{aligned}$$

Recall that, for every $n\geq 0$, $\|Q_n(f)\|_p\leq c^2C_p\|A_{2^n}(f)\|_p$. Hence, using the triangle inequality in $L^{p/2}(\mu)$, we have

$$\Big(\sum_{n\geq 1} n^p d(2^n,2^{n+1}-1)\Big)^{2/p}\leq C\sum_{n\geq 1} n\|A_{2^{2^n}}(f)\|_p^2.$$

The fact that

$$\sum_{n\geq 1} n\|A_{2^{2^n}}(f)\|_p^2<\infty \ \Leftrightarrow\ \sum_{n\geq 2}\frac{\|A_n(f)\|_p^2}{n\log n}\log\log n<\infty$$

follows from Lemma 3.3 applied with $r=2$, $\alpha_n=1/(n^2\log n)$, $\varphi(x)=\log|\log x|$ and $n_k=2^k$. ∎

Appendix A. Proof of Proposition 2.2. By the assumption (2), the series $\sum_n|(f+\cdots+T^{2^n-1}f)/\psi(2^n)|^p$ converges μ-a.e. In particular, we have

$$\frac{1}{\psi(2^n)}\sum_{k=0}^{2^n-1} T^k f \xrightarrow[n\to\infty]{} 0 \qquad \mu\text{-a.e.}$$

Similarly, applying Proposition 2.1 to $T^{2^n}f$, we deduce that

$$\frac{1}{\psi(2^n)} \max_{l=0}^{2^n-1} \Big| \sum_{k=0}^{l} T^k(T^{2^n} f) \Big| \xrightarrow[n\to\infty]{} 0 \qquad \mu\text{-a.e.}$$

Now, since ψ is non-decreasing, for every $2^n \le m \le 2^{n+1}-1$ we have

$$\begin{aligned} &\Big| \frac{f+\cdots+T^{m-1}f}{\psi(m)} - \frac{f+\cdots+T^{2^n-1}f}{\psi(2^n)} \Big| \\ &\qquad\qquad \le \Big| \frac{T^{2^n}f+\cdots+T^{m-1}f}{\psi(m)} \Big| + 2\Big| \frac{f+\cdots+T^{2^n-1}f}{\psi(2^n)} \Big|, \end{aligned}$$

and the result follows.

Assume that ψ satisfies

$$\sum_n \frac{1}{\psi^p(n)} < \infty \quad \text{and} \quad \sum_n \frac{\|f+\cdots+T^{n-1}\|_p^p}{n\psi^p(n)} \Big(\frac{M_n \psi^p(n)}{n}\Big)^p < \infty.$$

Let us prove that (2) holds.

By the assumption on ψ, using Cauchy's condensation principle, we see that $\sum_n 2^n/\psi^p(2^n) < \infty$. Denote $R_n = \sum_{m\ge n} 2^m/\psi^p(2^m)$. Let $q = p/(p-1)$, $0 < \varepsilon < q/p$ and define

$$\chi(n) := \Big(\frac{1}{R_n^\varepsilon} - \frac{1}{R_{n-1}^\varepsilon}\Big)^{1/q}.$$

Using Hölder's inequality, we have

$$\sum_{l=0}^{n} \frac{\|S_{2^l}\|_p}{2^{l/p}} \le \Big(\sum_{k=0}^{n} \chi^q(k)\Big)^{1/q} \Big(\sum_{l=0}^{n} \frac{\|S_{2^l}\|_p^p}{\chi^p(l)2^l}\Big)^{1/p}.$$

Hence, by the definition of χ,

$$\sum_{n\ge 1} \frac{(\sum_{l=0}^n 2^{(n-l)/p}\|f+\cdots+T^{2^l-1}f\|_p)^p}{\psi^p(2^n)} \le \sum_{l\ge 0} \frac{\|S_{2^l}\|_p^p}{\chi^p(l)2^l} \sum_{n\ge l} \frac{2^n}{\psi^p(2^n)} R_n^{-\varepsilon p/q}.$$

By elementary considerations,

$$\sum_{n\ge l} \frac{2^n}{\psi^p(2^n)} R_n^{-\varepsilon p/q} \le \int_0^{R_l} \frac{dx}{x^{\varepsilon p/q}} \le C R_l^{1-\varepsilon p/q},$$

and

$$\frac{1}{\chi^p(l)} = \Big(\frac{R_l^\varepsilon R_{l-1}^\varepsilon}{R_{l-1}^\varepsilon - R_l^\varepsilon}\Big)^{p/q} \le C\Big(\frac{R_l^\varepsilon R_{l-1}}{R_{l-1} - R_l}\Big)^{p/q}.$$

Recall that $M_n = \sum_{k\ge n} 1/\psi^p(k)$. Using that ψ is non-decreasing, we see that $2R_n \le M_{2^{n-1}}$ and then

$$\sum_{n\ge 1}\frac{(\sum_{l=0}^{n}2^{(n-l)/p}\|f+\cdots+T^{2^l-1}f\|_p)^p}{\psi^p(2^n)}\le C\sum_{l\ge 0}\|S_{2^l}\|_p^p\frac{R_lR_{l-1}^{p/q}}{(R_{l-1}-R_l)^{p/q}}$$
$$\le C\sum_{l\ge 0}\|S_{2^l}\|_p^p\frac{M_{2^{l-1}}M_{2^{l-2}}^{p/q}\psi^{p^2/q}(2^{l-1})}{2^{(l-1)(1+p/q)}}\le C'\sum_l\|S_{2^l}\|_p^p\frac{M_{2^l}^p\psi^{p^2/q}(2^l)}{2^{pl}},$$

where we have used that $\{M_n\}$ is non-increasing, $\psi(2n)\le C\psi(n)$, and $\|S_{2n}\|_p\le 2\sup_{k\ge 0}\|T^k\|\,\|S_n\|_p$.

It remains to prove that (3) implies that $\sum_l\|S_{2^l}\|_p^pM_{2^l}^p\psi^{p^2/q}(2^l)/2^{pl}<\infty$. This may be proved by exactly the same argument as in the proof of (ii)⇒(i) in Lemma 3.3 (see below), using the remark after Lemma 3.3.

Appendix B. Proof of Lemma 3.3. Assume first that φ is $\{n_k\}$-adapted. Let us prove that (i)⇒(ii).

Since $\varphi\in\mathcal{L}$, there exists $\varepsilon\in\,]0,1[$ such that $t^{1-\varepsilon}\varphi(t)$ is non-decreasing. Let $\eta\in\,]0,\varepsilon[$. Let $k\ge 0$ and $n\in[2^k,2^{k+1}-1]$. Since $\{w_n\}$ is subadditive, using Hölder's inequality we obtain

$$w_n^r\le\Big(\sum_{l=0}^{k}w_{2^l}\Big)^r\le 2^{\eta r(k+1)}\sum_{l=0}^{k}\Big(\frac{w_{2^l}}{2^{\eta l}}\Big)^r.$$

Then

$$\sum_{n\ge 1}\varphi^{r-1}(\pi/n)n\alpha_n\Big(\frac{w_n}{n}\Big)^r\le\sum_{k\ge 0}2^{\eta r(k+1)}\sum_{l=0}^{k}\Big(\frac{w_{2^l}}{2^{\eta l}}\Big)^r\sum_{n=2^k}^{2^{k+1}-1}\varphi^{r-1}(\pi/n)\frac{\alpha_n}{n^{r-1}}$$
$$\le 2^{\eta r}\sum_{l\ge 0}\Big(\frac{w_{2^l}}{2^{\eta l}}\Big)^r\sum_{n\ge 2^l}\varphi^{r-1}(\pi/n)\frac{n\alpha_n}{n^{(1-\eta)r}}$$
$$\le 2^{\eta r}\sum_{k\ge 0}\sum_{l=n_k}^{n_{k+1}}\Big(\frac{w_{2^l}}{2^l}\Big)^r2^{(1-\eta)lr}\sum_{n\ge 2^l}\varphi^{r-1}(\pi/n)\frac{n\alpha_n}{n^{(1-\eta)r}}$$
$$\le 2^{\eta r}\sum_{k\ge 0}\Big(\frac{w_{2^{n_k}}}{2^{n_k}}\Big)^r\sum_{l=n_k}^{n_{k+1}}2^{(1-\eta)lr}\sum_{n\ge 2^l}\varphi^{r-1}\Big(\frac{\pi}{n}\Big)\frac{n\alpha_n}{n^{(1-\eta)r}},$$

where we have used that $\{w_{2^l}/2^l\}$ is non-increasing, by subadditivity.

Iverting the order of summation, using (9), (17) and the fact that φ is non-increasing, we obtain

$$\sum_{l=n_k}^{n_{k+1}}2^{(1-\eta)lr}\sum_{n\ge 2^l}\varphi^{r-1}(\pi/n)\frac{n\alpha_n}{n^{(1-\eta)r}}$$
$$\le\sum_{j=n_k}^{n_{k+1}-1}\sum_{n=2^j}^{2^{j+1}-1}\varphi^{r-1}(\pi/n)\frac{n\alpha_n}{n^{(1-\eta)r}}\sum_{l=n_k}^{j}2^{r(1-\eta)l}$$

$$+\sum_{j\geq n_{k+1}}\sum_{n=2^j}^{2^{j+1}-1}\varphi^{r-1}(\pi/n)\frac{n\alpha_n}{n^{(1-\eta)r}}\sum_{l=n_k}^{n_{k+1}-1}2^{r(1-\eta)l}$$
$$\leq \varphi^r(\pi/2^{n_k})+C2^{n_{k+1}r(1-\eta)}\sum_{l\geq k+1}\left(\frac{\varphi(\pi/2^{n_l})}{2^{n_l(1-\eta)}}\right)^r$$
$$\leq \varphi^r(\pi/2^{n_k})+C\left(\frac{\varphi(\pi/2^{n_{k+1}})}{2^{n_l(\eta-\varepsilon)}}\right)^r\sum_{l\geq k+1}\frac{1}{2^{n_l(\varepsilon-\eta)r}},$$

and the result follows, using (17) again.

Assume now that $\{n_k\}$ is $(\{\alpha_n\},\varphi)$-adapted. Let us prove that (ii)$\Rightarrow$(i).

By subadditivity, for every $k\in\{4^{n-1},\dots 4^n/2-1\}$, we have $w_{4^n}^r\leq 2^{r-1}(w_k^r+w_{4^n-k}^r)$. Hence, using (C2) and the fact that $\{n\alpha_n\}$ is non-increasing, we obtain

$$4^{n-1}\varphi^{r-1}(\pi/4^n)4^n\alpha_{4^n}\left(\frac{w_{4^n}}{4^n}\right)^r\leq 2^{r-1}\sum_{k=4^{n-1}}^{4^n/2-1}\left(\varphi^{r-1}(\pi/k)k\alpha_k\left(\frac{w_k}{k}\right)^r\right.$$
$$\left.+\varphi^{r-1}(\pi/4^n-k)(4^n-k)\alpha_{4^n-k}\left(\frac{w_{4^n-k}}{4^n-k}\right)^r\right).$$

Hence, by (ii),

$$\sum_{n\geq 1}\varphi^{r-1}(\pi/2^{2n})(2^{2n})^2\alpha_{2^{2n}}\left(\frac{w_{2^{2n}}}{2^{2n}}\right)^r<\infty,$$

and the same series along odd indices converges too, using monotonicity properties and the fact that $\{w_{2^n}/2^n\}$ is non-increasing. Hence, using (17), we have

$$\sum_{k\geq 1}\varphi^{r-1}(\pi/2^{n_k})\left(\frac{w_{2^{n_k}}}{2^{n_k}}\right)^r\sum_{n=n_{k-1}}^{n_k-1}2^{2n}\alpha_{2^n}$$
$$\leq C^{r-1}\sum_{n\geq 1}\varphi^{r-1}(\pi/2^n)(2^n)^2\alpha_{2^n}\left(\frac{w_{2^n}}{2^n}\right)^r<\infty.$$

Then, using (18) and the fact that $\{n\alpha_n\}$ is non-increasing, we obtain

$$\sum_{k\geq 1}\varphi^r(\pi/2^{n_k})\left(\frac{w_{2^{n_k}}}{2^{n_k}}\right)^r\leq\sum_{k\geq 1}\varphi^{r-1}(\pi/2^{n_k})\left(\frac{w_{2^{n_k}}}{2^{n_k}}\right)^r\sum_{n=2^{n_{k-1}}}^{2^{n_k-1}}n\alpha_n$$
$$\leq\sum_{k\geq 1}\varphi^{r-1}(\pi/2^{n_k})\left(\frac{w_{2^{n_k}}}{2^{n_k}}\right)^r\sum_{n=n_{k-1}}^{n_k-1}2^{2n}\alpha_{2^n}<\infty.$$

Appendix C. Proof of Lemma 3.4. It suffices to prove that $\sum_{k\geq 1}\|\sum_{l=2^{n_k}}^{2^{n_{k+1}}-1}|\alpha_l S_l|\,\|_p^p<\infty$. By (9), there exists $K>0$ such that for every $n \geq 1$,

$$\sum_{k=1}^{n} k\alpha_k \leq K\varphi(\pi/n) \tag{23}$$

(for a proof, see the beginning of Appendix C of [6]).

Let $r, q > 1$ be such that $1/r + 1/q = 1$. By Hölder's inequality, using (17) and the monotonicity of φ, we have

$$\begin{aligned} \sum_{l=2^{n_k}}^{2^{n_{k+1}}-1} |\alpha_l S_l| &\leq \Big(\sum_{l=2^{n_k}}^{2^{n_{k+1}}-1} l\alpha_l\Big)^{1/q} \bigg(\sum_{l=2^{n_k}}^{2^{n_{k+1}}-1} l\alpha_l \Big(\frac{|S_l|}{l}\Big)^r\bigg)^{1/r} \\ &\leq \varphi^{(r-1)/r}(\pi/2^{n_{k+1}}) \bigg(\sum_{l=2^{n_k}}^{2^{n_{k+1}}-1} l\alpha_l \Big(\frac{|S_l|}{l}\Big)^r\bigg)^{1/r} \\ &\leq \bigg(\sum_{l=2^{n_k}}^{2^{n_{k+1}}-1} \varphi^{r-1}(\pi/l) l\alpha_l \Big(\frac{|S_l|}{l}\Big)^r\bigg)^{1/r}. \end{aligned} \tag{24}$$

Assume that $p \in]1, 2[$. Then, taking $r = p$, we obtain

$$\Big\| \sum_{l=2^{n_k}}^{2^{n_{k+1}}-1} |\alpha_l S_l| \Big\|_p^p \leq \sum_{l=2^{n_k}}^{2^{n_{k+1}}-1} \varphi^{p-1}(\pi/l) l\alpha_l \Big(\frac{\|S_l\|_p}{l}\Big)^p.$$

Since T is power-bounded, $N(h) = \sup_{n\geq 0} \|T^n h\|_p$ defines an equivalent norm for which T is a contraction. Hence $\{N(S_n)\}$ is subadditive and we can apply Lemma 3.3 with $r = p$.

Assume that $p \geq 2$. Then, taking $r = 2$, we obtain

$$\Big\| \sum_{l=2^{n_k}}^{2^{n_{k+1}}-1} |\alpha_l S_l| \Big\|_p^p \leq \Big(\sum_{l=2^{n_k}}^{2^{n_{k+1}}-1} \varphi(\pi/l) l\alpha_l \frac{\|S_l\|_p^2}{l^2}\Big)^{p/2}$$

and

$$\sum_{k\geq 0} \Big\| \sum_{l=2^{n_k}}^{2^{n_{k+1}}-1} |\alpha_l S_l| \Big\|_p^p \leq \sum_{k\geq 0} \bigg(\sum_{l=2^{n_k}}^{2^{n_{k+1}}-1} \varphi(\pi/l) l\alpha_l \Big(\frac{\|S_l\|_p}{l}\Big)^2\bigg)^{p/2},$$

and we can apply Lemma 3.3 with $r = 2$. We conclude as above.

Acknowledgments. Many thanks to Guy Cohen for his interest in this work and useful suggestions. This work was completed while the author was a member of the Équipe ERIM from the University of New Caledonia.

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Received 12 December 2010;
revised 25 May 2011 (5456)