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DIFFERENTIAL INDEPENDENCE VIA AN ASSOCIATIVE PRODUCT OF INFINITELY MANY LINEAR FUNCTIONALS

BY

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Abstract. We generalize the infinitesimal independence appearing in free probability of type B in two directions: to higher order derivatives and other natural independences: tensor, monotone and Boolean. Such generalized infinitesimal independences can be defined by using associative products of infinitely many linear functionals, and therefore the associated cumulants can be defined. These products can be seen as the usual natural products of linear maps with values in formal power series.

1. Introduction. Free probability theory [24] was initiated by Voiculescu to solve problems in operator algebras. Many concepts in probability theory have analogues in free probability by replacing the concept of independence of random variables with free independence. For instance, cumulants which appear in probability theory are replaced by free cumulants in free probability. There are other examples such as the central limit theorem and infinitely divisible distributions [23].

In probability theory, a standard model of independent random variables is constructed by taking the direct product of probability spaces. In the algebraic description, the usual independence is realized on the tensor product of algebras equipped with the tensor product of states. By analogy, free independent random variables can be realized in the free product of algebras equipped with the free product of states [24]. An important point is that the tensor and free products of states are associative; this property enables us to define unique cumulants associated to free independence.

There are other associative products of states on the free product of algebras (with or without unit): Boolean [4, 22], monotone or anti-monotone [16, 18] products. Under natural conditions, Schürmann, Speicher, Ben Ghorbal and Muraki [2, 18, 17, 19, 21] proved that an associative product of states is necessarily one of the tensor, free, Boolean, monotone or anti-monotone products. Among them, only monotone and anti-monotone products are not symmetric, that is, independence of random variables X

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and Y does not imply independence of Y and X. This makes it impossible to define cumulants in the usual sense (a unified treatment of cumulants in the "usual sense" can be found in [15]). In [14] and [13], however, it is proved that there exist uniquely defined cumulants in a generalized sense. We will use this generalized framework later.

There are also several attempts to construct an associative product of more than one state. A conditionally free (or c-free for simplicity) product was introduced by Bożejko, Leinert and Speicher in [6, 5] as a product of two states. This product is important since it interpolates free and Boolean products, preserving the associative laws. Moreover, in [10] Franz found that the c-free product also unifies monotone and anti-monotone products, not preserving the associative laws (the reader is referred to [12] for details). Then a c-monotone product of two states was introduced in [11] to unify monotone and Boolean products. Furthermore, an ordered free product and an indented product were defined in [12]; the former is defined for two states and unifies free and monotone products, and the latter is defined for three states unifying free, Boolean, monotone and anti-monotone products (and also c-free and c-monotone products).

In this paper, we construct an associative product of infinitely many linear functionals for a given natural independence. The idea of a product of infinitely many states or linear functionals was proposed by Cabanal-Duvillard and Ionescu in [7]. They constructed a product of infinitely many states as an extension of the c-free product. This product, however, is not associative.

The product defined in this paper extends the infinitesimal aspect of free probability of type B. Free probability of type B was originally defined by Biane, Goodman and Nica [3] to find a free probability associated to type B non-crossing partitions. Later, in [1] Belinschi and Shlyakhtenko found that free probability of type B appears as an infinitesimal property of free probability. More precisely, let $\{\varphi_t\}_{0\leq t<\varepsilon}$ be a family of states on an algebra $\mathcal{A}$ for $\varepsilon > 0$. If $\mathcal{A}_j$ are free independent with respect to φ_t for all t, then free probability of type B can be realized with respect to the pair (φ_0, φ_0'), where φ_0' is defined by the linear functional $X \mapsto \frac{d}{dt}\varphi_t(X)|_{t=0}$. Février and Nica investigated combinatorial properties of infinitesimal free probability in [9].

We extend free probability of type B from two viewpoints: higher derivatives and natural independence. This is the content of Section 2. This extension can be understood as an associative product of infinitely many linear functionals. By using associativity, we define cumulants in Section 3. These cumulants are obtained by formally differentiating the usual cumulants for natural independence. In Section 4, we study two examples: one is formal multi-variate Lévy processes, and the other is dual derivation systems introduced in [9]. Many results of Sections 2–4 can easily be extended to more general settings. We will mention such generalizations in Section 5.

NOTE. The author has learned that Février also extended free probability of type B to higher order derivatives, independently of this paper. He also clarified combinatorics of higher order derivatives; see [8].

2. Differential independence. In this section we extend the infinitesimal realization of free probability of type B to higher derivatives, and also extend free independence to other natural independences: tensor, monotone and Boolean.

Throughout this paper, symbols such as $\mathcal{A}$ and $\mathcal{A}_i$ always denote algebras over $\mathbb{C}$. We denote by $\mathcal{A}^*$ the set of linear functionals from $\mathcal{A}$ to $\mathbb{C}$, and by $\mathcal{A}_1 * \mathcal{A}_2$ the free product of $\mathcal{A}_1$ and $\mathcal{A}_2$ without identification of units. Let $\star$ be any one of the natural products of linear functionals, i.e., a product

$$\star : \mathcal{A}_1^* \times \mathcal{A}_2^* \to (\mathcal{A}_1 * \mathcal{A}_2)^*,$$

defined for arbitrary algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, satisfying some natural conditions in terms of category theory. Natural products were classified into five types: tensor, free, Boolean, monotone and anti-monotone ones. It is known that natural products of linear functionals preserve positivity, and therefore a natural product of states is again a state.

Let $\mathbb{C}[\![t]\!]$ be the unital ring of formal power series. We introduce $\mathbb{C}[\![t]\!]$-valued linear maps to treat all the objects algebraically. First we consider a natural product of $\mathbb{C}[\![t]\!]$-valued linear maps $\varphi^t : \mathcal{A} \to \mathbb{C}[\![t]\!]$ which is defined essentially in the same way as in the $\mathbb{C}$-valued case. For instance, the concept of joint moments is almost the same as for $\mathbb{C}$-valued independence.

In this paper we do not discuss anti-monotone independence since what we prove for monotone independence can easily be translated into anti-monotone independence. This is so only within the scope of this paper and we are not claiming that every property of anti-monotone independence always translates into one of monotone independence.

Now we define $\mathbb{C}[\![t]\!]$-valued natural products of linear maps on the free product of algebras. In the following definitions, $(\lambda_1, \dots, \lambda_n)$ is any alternating sequence of 1 and 2 of arbitrary length n, that is, $(\lambda_1, \lambda_2, \dots) = (1, 2, 1, \dots)$ or $(2, 1, 2, \dots)$. Moreover, X_k is any element in $\mathcal{A}_{\lambda_k}$ for $1 \le k \le n$.

DEFINITION 2.1. The *tensor product* $\varphi_1^t \otimes \varphi_2^t$ on $\mathcal{A}_1 * \mathcal{A}_2$ is defined by

$$\varphi_1^t \otimes \varphi_2^t(X_1 \cdots X_n) = \varphi_1^t\Big(\overrightarrow{\prod_{i:\, X_i \in \mathcal{A}_1}} X_i\Big)\varphi_2^t\Big(\overrightarrow{\prod_{i:\, X_i \in \mathcal{A}_2}} X_i\Big).$$

$\overrightarrow{\prod}_{j \in V} X_j$ denotes the ordered product $X_{j_1} \cdots X_{j_k}$ for $V = \{j_1, \dots, j_k\}$, $j_1 < \cdots < j_k$.

DEFINITION 2.2. We assume that $\mathcal{A}_i$ is unital for $i = 1, 2$ and $\mathbb{C}[\![t]\!]$ is contained in the center of $\mathcal{A}_i$, and moreover that the unit of $\mathcal{A}_i$ is the same as the unit of $\mathbb{C}[\![t]\!]$. We assume that $\varphi^t(1) = 1$. Let $\mathcal{A}_1 *_{\mathbb{C}[\![t]\!]} \mathcal{A}_2$ be the amalgamated free product of algebras over $\mathbb{C}[\![t]\!]$. The *free product* $\varphi_1^t * \varphi_2^t$ on $\mathcal{A}_1 *_{\mathbb{C}[\![t]\!]} \mathcal{A}_2$ is defined by the following rule:

$$\varphi_1^t * \varphi_2^t(X_1 \cdots X_n) = 0 \quad \text{whenever} \quad \varphi_{\lambda_i}^t(X_i) = 0 \text{ for all } i.$$

DEFINITION 2.3. The *Boolean product* $\varphi_1^t \diamond \varphi_2^t$ on $\mathcal{A}_1 * \mathcal{A}_2$ is defined by

$$\varphi_1^t \diamond \varphi_2^t(X_1 \cdots X_n) = \varphi_{\lambda_1}^t(X_1)\varphi_{\lambda_2}^t(X_2) \cdots \varphi_{\lambda_n}^t(X_n).$$

DEFINITION 2.4. The *monotone product* $\varphi_1^t \triangleright \varphi_2^t$ on $\mathcal{A}_1 * \mathcal{A}_2$ is defined by

$$\varphi_1^t \triangleright \varphi_2^t(X_1 \cdots X_n) = \varphi_1^t\Big(\overrightarrow{\prod_{X_i \in \mathcal{A}_1}} X_i\Big)\Big(\prod_{X_i \in \mathcal{A}_2} \varphi_2^t(X_i)\Big).$$

The tensor product can be defined on both $\mathcal{A}_1 * \mathcal{A}_2$ and $\mathcal{A}_1 *_{\mathbb{C}[\![t]\!]} \mathcal{A}_2$. By contrast, the Boolean and monotone products cannot be defined on $\mathcal{A}_1 *_{\mathbb{C}[\![t]\!]} \mathcal{A}_2$. We defined the free product of $\mathbb{C}[\![t]\!]$-valued linear maps on $\mathcal{A}_1 *_{\mathbb{C}[\![t]\!]} \mathcal{A}_2$ under special assumptions on the algebras. It is however also possible to define the free product on $\mathcal{A}_1 * \mathcal{A}_2$ for arbitrary algebras $\mathcal{A}_i$. To do so, let $\widetilde{\mathcal{A}}_i$ be the unitization of $\mathcal{A}_i$ defined by $\widetilde{\mathcal{A}}_i = \mathbb{C} \oplus \mathcal{A}_i$. We consider $\widetilde{\mathcal{A}}_i$-valued formal power series $\mathbb{C}[\![t; \widetilde{\mathcal{A}}_i]\!]$. Then $\mathbb{C}[\![t]\!]$ is embedded into the center of $\mathbb{C}[\![t; \widetilde{\mathcal{A}}_i]\!]$. $\mathbb{C}[\![t]\!]$-valued linear maps on $\mathcal{A}_i$ can be naturally extended to unit-preserving ones on $\mathbb{C}[\![t; \widetilde{\mathcal{A}}_i]\!]$. Then we can define the free product by Definition 2.2, and restrict it to $\mathcal{A}_1 * \mathcal{A}_2$, to obtain the free product on $\mathcal{A}_1 * \mathcal{A}_2$.

We can define *universal calculation rules* [18] ([1]) for tensor, free, Boolean and monotone products of $\mathbb{C}[\![t]\!]$-valued linear maps. Then we can easily define independence as follows.

DEFINITION 2.5. We fix an arbitrary natural independence. Let $(\mathcal{A}, \varphi^t)$ be a pair of an algebra and a $\mathbb{C}[\![t]\!]$-valued linear map on $\mathcal{A}$. Subalgebras $(\mathcal{A}_i)_{i \geq 1}$ of $\mathcal{A}$ are called *independent* if for any elements $X_k \in \mathcal{A}_{i_k}$ and indices $i_1 \neq \cdots \neq i_n$, the universal calculation rule holds.

An important property of a natural product is the *associative law*. A product $\star : \mathcal{A}_1^* \times \mathcal{A}_2^* \to (\mathcal{A}_1 * \mathcal{A}_2)^*$ defined for any $\mathcal{A}_1, \mathcal{A}_2$ is said to be *associative* if $(\varphi_1 \star \varphi_2) \star \varphi_3 = \varphi_1 \star (\varphi_2 \star \varphi_3)$ under the natural identification $(\mathcal{A}_1 * \mathcal{A}_2) * \mathcal{A}_3 \cong \mathcal{A}_1 * (\mathcal{A}_2 * \mathcal{A}_3)$. Associativity can be defined for any family of linear functionals as follows. Let $(\mathcal{A}^*)^{\Lambda}$ be the set of functions from a set Λ with values in $\mathcal{A}^*$. In most cases we take Λ to be $\mathbb{N} = \{0, 1, 2, \ldots\}$. We consider a pair $(\mathcal{A}, (\varphi^{(\lambda)})_{\lambda \in \Lambda})$ of an algebra and a function in $(\mathcal{A}^*)^{\Lambda}$.

([1]) Muraki called these rules *quasi-universal calculation rules.*

A product $\star : (\mathcal{A}_1^*)^\Lambda \times (\mathcal{A}_2^*)^\Lambda \to ((\mathcal{A}_1 * \mathcal{A}_2)^*)^\Lambda$, defined for any $\mathcal{A}_1, \mathcal{A}_2$, is said to be associative if $((\varphi_1^{(\lambda)}) \star (\varphi_2^{(\lambda)})) \star (\varphi_3^{(\lambda)}) = (\varphi_1^{(\lambda)}) \star ((\varphi_2^{(\lambda)}) \star (\varphi_3^{(\lambda)}))$ under the natural isomorphism $(\mathcal{A}_1 * \mathcal{A}_2) * \mathcal{A}_3 \cong \mathcal{A}_1 * (\mathcal{A}_2 * \mathcal{A}_3)$.

Associativity can also be defined for linear maps with values in a common algebra. In particular, we often consider associative products defined for linear maps with values in $\mathbb{C}[\![t]\!]$.

We have defined products of algebraic probability spaces $(\mathcal{A}, \varphi^t)$, where φ^t are $\mathbb{C}[\![t]\!]$-valued linear maps. By the way, φ^t can be identified with an infinite sequence $(\varphi^{(n)})_{n\geq 0}$ of linear functionals by

$$\varphi^t(X) = \varphi^{(0)}(X) + \varphi^{(1)}(X)t + \varphi^{(2)}(X)\frac{t^2}{2!} + \cdots .$$

Formally, $\varphi^{(n)}(X) = \frac{d^n}{dt^n}\varphi^t(X)\big|_{t=0}$. Let $\mathrm{LiMap}(\mathcal{A}, \mathbb{C}[\![t]\!])$ be the set of all linear maps from $\mathcal{A}$ to $\mathbb{C}[\![t]\!]$. Then the map $F : \varphi^t \mapsto (\varphi^{(n)})_{n\geq 0}$ is bijective from $\mathrm{LiMap}(\mathcal{A}, \mathbb{C}[\![t]\!])$ to $(\mathcal{A}^*)^{\mathbb{N}}$. Let $\star$ be any one of the natural products of $\mathbb{C}[\![t]\!]$-valued linear maps. We can define an associative product

$$(\mathcal{A}_1, (\varphi_1^{(n)})_{n\geq 0}) \star^D (\mathcal{A}_2, (\varphi_2^{(n)})_{n\geq 0}) = (\mathcal{A}_1 * \mathcal{A}_2, (\varphi^{(n)})_{n\geq 0})$$

by

$$(\varphi^{(n)})_{n\geq 0} := F\big(F^{-1}((\varphi_1^{(n)})_{n\geq 0}) \star F^{-1}((\varphi_2^{(n)})_{n\geq 0})\big).$$

By definition this product is associative. In the case of the tensor and free products, we can replace $\mathcal{A}_1 * \mathcal{A}_2$ by $\mathcal{A}_1 *_{\mathbb{C}[\![t]\!]} \mathcal{A}_2$. The sequence $(\varphi^{(n)})$ contains information on infinitesimal properties of φ^t. Therefore, the following terminology is reasonable.

Definition 2.6. We call $\star^D$ a *differential product* associated to a natural product $\star$. More concretely, if $\star = \otimes, *, \diamond, \triangleright$, we call the products $\otimes^D, *^D, \diamond^D, \triangleright^D$ respectively the *differentially tensor product, differentially free product, differentially Boolean product* and *differentially monotone product.*

We can also define a product up to the nth derivative. That is, let $\mathbb{C}[\![t]\!]_n$ be the unital ring of power series with the relation $t^{n+1} = 0$. This ring can be realized by upper triangular matrices of the form

$$\mathbb{G}_n := \left\{ A \in M_{n+1}(\mathbb{C}); A = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ 0 & a_0 & a_1 & \dots & a_{n-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & a_0 & a_1 \\ 0 & 0 & \dots & 0 & a_0 \end{pmatrix} \right\}.$$

$\mathbb{G}_1$ is used in free probability of type B (see [3]). Let $\mathrm{LiMap}(\mathcal{A}, \mathbb{C}[\![t]\!]_n)$ be the set of linear maps from $\mathcal{A}$ to $\mathbb{C}[\![t]\!]_n$. Then each $\varphi^t \in \mathrm{LiMap}(\mathcal{A}, \mathbb{C}[\![t]\!]_n)$ is of the form

$$\varphi^t(X) = \sum_{k=0}^{n} \varphi^{(k)}(X)\frac{t^k}{k!}.$$

We can define natural products of $\mathbb{C}[\![t]\!]_n$-valued linear maps analogously to Definitions 2.1–2.4. Let $F_n : \mathrm{LiMap}(\mathcal{A}, \mathbb{C}[\![t]\!]_n) \to (\mathcal{A}^*)^n$ be the map defined by $F_n(\varphi^t) = (\varphi^{(k)})_{k=0}^n$. Then we define a product

$$(\mathcal{A}_1, (\varphi_1^{(k)})_{k=0}^n) \star^D (\mathcal{A}_2, (\varphi_2^{(k)})_{k=0}^n) := (\mathcal{A}_1 * \mathcal{A}_2, (\varphi^{(k)})_{k=0}^n)$$

by

$$(\varphi^{(k)})_{k=0}^n := F_n(F_n^{-1}((\varphi_1^{(k)})_{k=0}^n) \star F_n^{-1}((\varphi_2^{(k)})_{k=0}^n)).$$

We call this product the *n-differential product.* All results in this paper can easily be proved for n-differential products; therefore, we only focus on infinitely many linear functionals.

Now we show how to calculate joint moments. By definition, we can calculate the joint moments $\varphi^{(n)}(X_1 \cdots X_n)$ by taking derivatives of $\varphi^t(X_1 \cdots X_n) = \sum_{n=0}^\infty \varphi^{(n)}(X_1 \cdots X_n)\frac{t^n}{n!}$.

THEOREM 2.7. *Let $(\mathcal{A}, (\varphi^{(n)})_{n\geq 0}) := \star^D_{k\geq 1}(\mathcal{A}_k, (\varphi_k^{(n)})_{n\geq 0})$ be a differential product. For any elements $X_k \in \mathcal{A}_{i_k}$ and indices $i_1 \neq \cdots \neq i_n$, we have the following.*

(1) *Let $\star$ be the tensor product. Then for $m \geq 0$,*

$$\varphi^{(m)}(X_1 \cdots X_n) = \sum_{k_1+k_2+\cdots=m;\, k_i\geq 0} \frac{m!}{k_1!k_2!\cdots}\prod_{i=1}^{\infty}\varphi^{(k_i)}\Big(\overrightarrow{\prod_{j:\, X_j\in\mathcal{A}_i}} X_j\Big).$$

(2) *Let $\star$ be the monotone product. If j satisfies $i_{j-1} < i_j > i_{j+1}$ then*

$$\begin{aligned}&\varphi^{(m)}(X_1 \cdots X_n)\\ &\quad = \sum_{k=0}^{n} \frac{m!}{k!(m-k)!}\varphi^{(k)}(X_j)\varphi^{(m-k)}(X_1 \cdots X_{j-1}X_{j+1}\cdots X_n).\end{aligned}$$

(3) *Let $\star$ be the Boolean product. Then*

$$\varphi^{(m)}(X_1 \cdots X_n) = \sum_{k_1+\cdots+k_n=m;\, k_i\geq 0} \frac{m!}{k_1!\cdots k_n!}\prod_{i=1}^{n}\varphi^{(k_i)}(X_i).$$

Thus we obtained the calculation rules of higher order derivatives for tensor, monotone and Boolean cases. The 1-differential free product is an important aspect of free probability of type B, and a calculation rule for it was clarified in [3]. However, higher order cases are complicated and we do not calculate them in this paper. Anyway, if we use a universal calculation rule [21] for free independent algebras, we can write down the calculations of moments for higher order derivatives in an abstract way.

The above calculation rules for differentially tensor, free, monotone and Boolean products are called *differential universal calculation rules* in this paper. Now we can define differential independence of subalgebras.

DEFINITION 2.8. We fix an arbitrary natural independence and let $(\mathcal{A}, (\varphi^{(n)})_{n\geq 0})$ be a pair of an algebra and a sequence of linear functionals. We say that subalgebras $(\mathcal{A}_i)_{i\geq 1}$ of $\mathcal{A}$ are *differentially independent* if for any $a_k \in \mathcal{A}_{i_k}$ and $i_1 \neq \cdots \neq i_n$, the differential universal calculation rule holds.

EXAMPLE 2.9. (1) Let $\{X, X'\}$ and $\{Y, Y'\}$ be differentially tensor independent. Then

$$\begin{aligned}\varphi^{(0)}(XYX'Y') &= \varphi^{(0)}(XX')\varphi^{(0)}(YY'),\\ \varphi^{(1)}(XYX'Y') &= \varphi^{(1)}(XX')\varphi^{(0)}(YY') + \varphi^{(0)}(XX')\varphi^{(1)}(YY'),\\ \varphi^{(2)}(XYX'Y') &= \varphi^{(2)}(XX')\varphi^{(0)}(YY') + 2\varphi^{(1)}(XX')\varphi^{(1)}(YY')\\ &\qquad + \varphi^{(0)}(XX')\varphi^{(2)}(YY').\end{aligned}$$

(2) Let $\{X, X'\}$ and Y be differentially free independent. Then

$$\begin{aligned}\varphi^{(0)}(XYX') &= \varphi^{(0)}(XX')\varphi^{(0)}(Y),\\ \varphi^{(1)}(XYX') &= \varphi^{(1)}(XX')\varphi^{(0)}(Y) + \varphi^{(0)}(XX')\varphi^{(1)}(Y),\\ \varphi^{(2)}(XYX') &= \varphi^{(2)}(XX')\varphi^{(0)}(Y) + 2\varphi^{(1)}(XX')\varphi^{(1)}(Y) + \varphi^{(0)}(XX')\varphi^{(2)}(Y).\end{aligned}$$

(3) Let $\{X, X'\}$ and $\{Y, Y'\}$ be differentially monotone independent. Then

$$\begin{aligned}\varphi^{(0)}(XYX'Y') &= \varphi^{(0)}(XX')\varphi^{(0)}(Y)\varphi^{(0)}(Y'),\\ \varphi^{(1)}(XYX'Y') &= \varphi^{(1)}(XX')\varphi^{(0)}(Y)\varphi^{(0)}(Y') + \varphi^{(0)}(XX')\varphi^{(1)}(Y)\varphi^{(0)}(Y')\\ &\qquad + \varphi^{(0)}(XX')\varphi^{(0)}(Y)\varphi^{(1)}(Y'),\\ \varphi^{(2)}(XYX'Y') &= \varphi^{(2)}(XX')\varphi^{(0)}(Y)\varphi^{(0)}(Y') + \varphi^{(0)}(XX')\varphi^{(2)}(Y)\varphi^{(0)}(Y')\\ &\qquad + \varphi^{(0)}(XX')\varphi^{(0)}(Y)\varphi^{(2)}(Y') + 2\varphi^{(1)}(XX')\varphi^{(1)}(Y)\varphi^{(0)}(Y')\\ &\qquad + 2\varphi^{(1)}(XX')\varphi^{(0)}(Y)\varphi^{(1)}(Y')\\ &\qquad + 2\varphi^{(0)}(XX')\varphi^{(1)}(Y)\varphi^{(1)}(Y').\end{aligned}$$

REMARK 2.10. Let us consider only the zeroth and first derivatives. In this section, we have constructed an associative product

$$\big((\mathcal{A}_1, \varphi_1^{(0)}, \varphi_1^{(1)}), (\mathcal{A}_2, \varphi_2^{(0)}, \varphi_2^{(1)})\big) \mapsto (\mathcal{A}_1 * \mathcal{A}_2, \varphi^{(0)}, \varphi^{(1)}).$$

The product for the second components is the usual natural product. The marginal distribution of $\varphi^{(1)}$ on $\mathcal{A}_i$ is $\varphi_i^{(1)}$. Therefore the situation is similar to c-free and c-monotone products. However, the calculation rule of second moments is different from the c-free and c-monotone cases since $\varphi^{(1)}(ab) = \varphi^{(1)}(ba) = \varphi_1^{(1)}(a)\varphi_2^{(0)}(b) + \varphi_1^{(0)}(a)\varphi_2^{(1)}(b)$ for $a \in \mathcal{A}_1$ and $b \in \mathcal{A}_2$.

By definition, we have the following.

PROPOSITION 2.11. *Let $(\mathcal{A}, (\varphi^{(n)})_{n\geq 0})$ be a pair of an algebra and a sequence of linear functionals. For each concept of natural independence, subalgebras $(\mathcal{A}_i)_{i\geq 1}$ of $\mathcal{A}$ are differentially independent if and only if they are independent (in the sense of Definition 2.5) in the algebraic probability space $(\mathcal{A}, \varphi^t)$, where φ^t is defined by $\varphi^t(X) = \sum_{k=0}^n \varphi^{(k)}(X)\frac{t^k}{k!}$.*

3. Cumulants associated with differential independence. Cumulants can be defined for a natural independence along the lines of [13]. Now we are considering infinitely many linear functionals, but the idea for a single linear functional can be extended easily. We note that Lehner's approach [15] is applicable to all natural independences except for monotone independence. We outline how to define cumulants. Proofs are the same as in [13] if we use Proposition 2.11.

DEFINITION 3.1. Let $(\mathcal{A}, (\varphi^{(n)})_{n\geq 0})$ be a pair of an algebra and a sequence of linear functionals. We take copies $\{X^{(j)}\}_{j\geq 1}$ (in an algebraic probability space) of every $X \in \mathcal{A}$ which satisfy

(1) $\varphi^{(k)}(X_1^{(j)} \cdots X_n^{(j)}) = \varphi^{(k)}(X_1 \cdots X_n)$ for any $X_i \in \mathcal{A}$, $j, n \geq 1$, $k \geq 0$;
(2) the subalgebras $\mathcal{A}^{(j)} := \{X^{(j)} : X \in \mathcal{A}\}$, $j \geq 1$, are differentially independent.

We define a dot operation $N.X$ by the sum of i.i.d. random variables:

$$N.X = X^{(1)} + \cdots + X^{(N)}$$

for $X \in \mathcal{A}$ and $N \in \mathbb{N}$; $0.X$ is defined to be 0.

LEMMA 3.2. *The dot operation is associative:*

$$\varphi^{(k)}(N.(M.X_1) \cdots N.(M.X_n)) = \varphi^{(k)}((MN).X_1 \cdots (MN).X_n) \tag{3.1}$$

for any $X_i \in \mathcal{A}$, $n \geq 1$, $k \geq 0$.

LEMMA 3.3. *The expression $\varphi^{(k)}(N.X_1 \cdots N.X_n)$ is a polynomial in N and $\varphi^{(m)}(X_{i_1} \cdots X_{i_p})$ for $i_1 < \cdots < i_p$, $1 \leq p \leq n$, $m \leq k$, with no constant term with respect to N.*

DEFINITION 3.4. For each natural independence, the $(k;n)$-*differential cumulant* $K_n^{(k)}(X_1, \ldots, X_n)$ is defined to be the coefficient of N in $\varphi^{(k)}(N.X_1 \cdots N.X_n)$.

PROPOSITION 3.5. *We have the following properties:*

(1) (Multilinearity) $K_n^{(k)}$ *are multilinear.*
(2) (Polynomiality) *There exist polynomials* $P_n^{(k)}$ *such that*

$$K_n^{(k)}(X_1, \dots, X_n) \\ = \varphi^{(k)}(X_1 \cdots X_n) + P_n^{(k)}(\{\varphi^{(m)}(X_{i_1} \cdots X_{i_p})\}_{\substack{1 \le p \le n-1;\, m \le k; \\ i_1 < \cdots < i_p}}).$$

(3) (Extensivity)

$$K_n^{(k)}(N.X_1, \dots, N.X_n) = N K_n^{(k)}(X_1, \dots, X_n).$$

Moreover, differential cumulants satisfying the above three properties are unique.

Cumulants can be calculated from the $\mathbb{C}[\![t]\!]$-valued usual cumulants thanks to Proposition 2.11. We know the $\mathbb{C}[\![t]\!]$-valued moment-cumulant formula for each natural independence since the proof needs no changes. Let ${}^T K_n^t(X_1, \dots, X_n)$, ${}^F K_n^t(X_1, \dots, X_n)$, ${}^M K_n^t(X_1, \dots, X_n)$ and ${}^B K_n^t(X_1, \dots, X_n)$ be the $\mathbb{C}[\![t]\!]$-valued tensor, free, monotone and Boolean cumulants, respectively. For every p-multilinear functional f_p, set $f_p(X_V) := f_n(X_{i_1}, \dots, X_{i_p})$ for any subset $V = \{i_1, \dots, i_p\} \subset \{1, \dots, n\}$, $i_1 < \cdots < i_p$. Then

$$\varphi^t(X_1 \cdots X_n) = \sum_{\pi \in \mathcal{P}(n)} \prod_{V \in \pi} {}^T K_{|V|}^t(X_V), \tag{3.2}$$

$$\varphi^t(X_1 \cdots X_n) = \sum_{\pi \in \mathcal{NC}(n)} \prod_{V \in \pi} {}^F K_{|V|}^t(X_V), \tag{3.3}$$

$$\varphi^t(X_1 \cdots X_n) = \sum_{\pi \in \mathcal{M}(n)} \prod_{V \in \bar{\pi}} \frac{1}{|\pi|!} {}^M K_{|V|}^t(X_V), \tag{3.4}$$

$$\varphi^t(X_1 \cdots X_n) = \sum_{\pi \in \mathcal{I}(n)} \prod_{V \in \pi} {}^B K_{|V|}^t(X_V), \tag{3.5}$$

where $\mathcal{P}(n)$, $\mathcal{NC}(n)$, $\mathcal{M}(n)$ and $\mathcal{I}(n)$ are the sets of partitions, non-crossing partitions, monotone partitions and interval partitions, respectively, and $\bar{\pi}$ denotes the projection from $\mathcal{M}(n)$ onto $\mathcal{NC}(n)$. (3.3) was proved in [20], (3.4) in [13] and (3.5) in [15]. The reader is referred to [13, 15] for the definitions of the above partitions.

Differential cumulants can be calculated from the above formulae as follows.

PROPOSITION 3.6. *Let* $\varphi^t(X) = \sum_{n=0}^{\infty} \varphi^{(n)}(X) \frac{t^n}{n!}$ *be a* $\mathbb{C}[\![t]\!]$*-valued linear map on an algebra* $\mathcal{A}$*. Then the* $(k;n)$*-differential cumulant* $K_n^{(k)}$ *associated to a natural independence is calculated as the* k*th derivative of the* $\mathbb{C}[\![t]\!]$*-*

valued nth cumulant K_n^t*:*

$$K_n^{(k)}(X_1,\dots,X_n)=\frac{d^k}{dt^k}K_n^t(X_1,\dots,X_n)\Big|_{t=0}.$$

Proof. By Proposition 2.11, the dot operation for differential independence coincides with that for $\mathbb{C}[\![t]\!]$-valued independence. We define

$$L_n^{(k)}(X_1,\dots,X_n):=\frac{d^k}{dt^k}K_n^t(X_1,\dots,X_n)\Big|_{t=0}.$$

We notice that $K_n^t(N.X_1,\dots,N.X_n)=NK_n^t(X_1,\dots,X_n)$ by definition. Therefore, the extensivity $L_n^{(k)}(N.X_1,\dots,N.X_n)=NL_n^{(k)}(X_1,\dots,X_n)$ holds. Multilinearity and polynomiality are easy to prove. By the uniqueness of differential cumulants, $L_n^{(k)}$ coincides with $K_n^{(k)}$. ■

We can now prove the vanishing property of cumulants.

THEOREM 3.7. *Consider any one of tensor, free and Boolean independences. Let* $(\mathcal{A},(\varphi^{(n)})_{n\ge 0})$ *be a pair of an algebra and a sequence of linear functionals. Subalgebras* $\mathcal{A}_i$ $(i\ge 1)$ *of* $\mathcal{A}$ *are differentially independent if and only if the following vanishing property of cumulants holds:*

(Vanishing property) *For any* $i_1,\dots,i_n$ *and* $X_k\in\mathcal{A}_{i_k}$ *for* $1\le k\le n$, $K_n^{(m)}(X_1,\dots,X_n)=0$ *for all* $m\ge 0$ *unless all* i_k*'s are the same number.*

Proof. The proof of the direct implication is the same as in Proposition 3.6 of [13]. The converse implication follows from the formulae (3.2), (3.3) or (3.5). Indeed, these moment-cumulant formulae enable us to calculate the joint moments $\varphi^{(m)}(X_1\cdots X_n)$ for any $n\ge 1$, $i_1,\dots,i_n\ge 1$ and any $X_k\in\mathcal{A}_{i_k}$ only using sums and products of $\varphi(X_{j_1}\cdots X_{j_p})$, where all X_{j_k} belong to the same $\mathcal{A}_r$. ■

4. Examples

4.1. Formal multi-variate convolution semigroups. We introduced the dot operation $N.X$ in Section 2. Let $\mathbb{C}[\![X_1,\dots,X_n]\!]$ be the unital ring generated by n non-commuting indeterminates. Let φ be a linear functional on $\mathbb{C}[\![X_1,\dots,X_n]\!]$ with $\varphi(1)=1$. Then $\varphi((N.X_{j_1})\cdots(N.X_{j_p}))$ is a polynomial in N, and hence we can define a new linear functional φ^t by $\varphi^t(X_{j_1}\cdots X_{j_p}):=\varphi((t.X_{j_1})\cdots(t.X_{j_p}))$ for $t\in\mathbb{R}$, $p\ge 1$ and $1\le j_1,\dots,j_p\le n$. Formally, this corresponds to a multivariate convolution semigroup for each concept of natural independence. In particular, if $n=1$ and the probability distribution μ of X_1 is infinitely divisible, then $\varphi^t(X_1^p)$ becomes the pth moment of the associated Lévy process at time $t\ge 0$.

${}^F K_n^{(1)}$ was calculated in [9]. We can also calculate the other differential cumulants associated to each natural independence.

Proposition 4.1. *Let $\mathcal{P}(n)^{(m)}$, $\mathcal{NC}(n)^{(m)}$, $\mathcal{M}(n)^{(m)}$ and $\mathcal{I}(n)^{(m)}$ be respectively the set of the partitions, non-crossing partitions, monotone partitions and interval partitions of the set $\{1,\dots,n\}$ with m blocks. If the independence under consideration is tensor, free, monotone or Boolean, we respectively obtain*

$$\varphi^{(m)}(X_{j_1}\cdots X_{j_p}) = m! \sum_{\pi\in\mathcal{P}(p)^{(m)}} \prod_{V\in\pi} {}^{T}K_{|V|}(X_{j(V)}), \tag{4.1}$$

$$\varphi^{(m)}(X_{j_1}\cdots X_{j_p}) = m! \sum_{\pi\in\mathcal{NC}(p)^{(m)}} \prod_{V\in\pi} {}^{F}K_{|V|}(X_{j(V)}), \tag{4.2}$$

$$\varphi^{(m)}(X_{j_1}\cdots X_{j_p}) = m! \sum_{\pi\in\mathcal{M}(p)^{(m)}} \prod_{V\in\bar{\pi}} \frac{1}{|\pi|!}{}^{M}K_{|V|}(X_{j(V)}), \tag{4.3}$$

$$\varphi^{(m)}(X_{j_1}\cdots X_{j_p}) = m! \sum_{\pi\in\mathcal{I}(p)^{(m)}} \prod_{V\in\pi} {}^{B}K_{|V|}(X_{j(V)}) \tag{4.4}$$

for $0\le m\le p$, where $K_{|V|}(X_{j(V)})$ is defined to be $K_{|V|}(X_{j_{i_1}},\dots,X_{j_{i_r}})$ for $V=\{i_1,\dots,i_r\}$, $i_1<\cdots<i_r$, and $\varphi^{(m)}(X_{j_1}\cdots X_{j_p})=0$ for $m>p$.

The proof is easy from the extensivity of cumulants.

4.2. Dual derivation systems. Février and Nica defined dual derivation systems in [9] which give us many examples of differential cumulants and independence. We basically follow the notation and definitions in [9], except for the range of the operator d_n below. Let $\mathfrak{M}$ be the set of sequences $(f_n)_{n\ge1}$, where f_n is an n-multilinear functional from $\mathcal{A}^n$ to $\mathbb{C}$ for each $n\ge1$. For a partition $\pi=\{V_1,\dots,V_p\}\in\mathcal{P}(n)$ and $f=(f_n)\in\mathfrak{M}$, we define an n-multilinear functional $J_\pi(f)$ by

$$J_\pi(f)(X_1,\dots,X_n)=\prod_{V\in\pi} f_{|V|}(X_V).$$

$J_\pi(f)$ only depends on $f_{|V_1|},\dots,f_{|V_{|\pi|}|}$. Therefore, we may denote $J_\pi(f)$ by $J_\pi(f_{|V_1|},\dots,f_{|V_{|\pi|}|})$. Let Mult_n be the multiplication map from $\mathcal{A}^n$ to $\mathcal{A}$ defined by $\mathrm{Mult}_n(X_1,\dots,X_n)=X_1\cdots X_n$.

Definition 4.2. Let $\mathfrak{M}_n$ be the set of all n-multilinear functionals from $\mathcal{A}^n$ to $\mathbb{C}$. A family of linear maps $(d_n:\mathfrak{D}_n\to\mathfrak{D}_n)_{n\ge1}$, where $\mathfrak{D}_n$ is a subspace of $\mathfrak{M}_n$, is called a *dual derivation system* if it satisfies the following conditions:

(i) Let $\mathfrak{D}$ be the set of sequences $(f_n)_{n\ge1}$ satisfying $f_n\in\mathfrak{D}_n$. Then

$$d_n(J_\pi(f_{|V_1|},\dots,f_{|V_{|\pi|}|}))=\sum_{k=1}^{|\pi|} J_\pi(f_{|V_1|},\dots,d_{|V_k|}f_{|V_k|},\dots,f_{|V_{|\pi|}|})$$

for all $f\in\mathfrak{D}$ and $\pi\in\mathcal{P}(n)$.

(ii) For every $f \in \mathfrak{D}_1$ and every $n \geq 1$, $f \circ \mathrm{Mult}_n \in \mathfrak{D}_n$ and

$$d_n(f \circ \mathrm{Mult}_n) = (d_1 f) \circ \mathrm{Mult}_n.$$

If a linear functional $\varphi \in \mathfrak{D}_1$ and a dual derivation system $(d_n : \mathfrak{D}_n \to \mathfrak{D}_n)_{n\geq 1}$ are given, we can construct a $\mathbb{C}[\![t]\!]$-valued linear map φ^t by

$$\varphi^t = \exp(td_1)\varphi = \sum_{n=0}^{\infty} \frac{t^n}{n!} d_1^n \varphi.$$

In other words, $\varphi^{(k)}$ is defined by $\varphi^{(k)} = d_1^k \varphi$ for $k \geq 0$. In this setting, we can calculate the differential cumulants as follows.

PROPOSITION 4.3. *Fix a natural independence, and let K_n be the associated cumulants. Then the differential cumulants $K_n^{(k)}$ associated to the sequence $(d_1^n \varphi)_{n\geq 0}$ are given by $K_n^{(k)} = d_n^k K_n$. Equivalently, the $\mathbb{C}[\![t]\!]$-valued cumulants K_n^t associated to $\varphi^t := \exp(td_1)\varphi$ are given by $K_n^t = \exp(td_n)K_n$.*

Proof. We prove the last equivalent claim. Moreover, we only prove the claim for tensor independence; the other cases are proved in the same way.

First we notice that $\exp(td_n)$ satisfies

$$\exp(td_n)J_\pi((f_k)_{k\geq 1}) = J_\pi((\exp(td_k)f_k)_{k\geq 1})$$

for every $\pi \in \mathcal{P}(n)$ and every $(f_k)_{k\geq 1} \in \mathfrak{D}$.

The cumulants ${}^T K_n$ for φ are defined by

$$\varphi(X_1 \cdots X_n) = \sum_{\pi \in \mathcal{P}(n)} \prod_{V \in \pi} {}^T K_{|V|}(X_V),$$

which is equivalent to

$$\varphi \circ \mathrm{Mult}_n = \sum_{\pi=(V_1,\dots,V_{|\pi|})\in\mathcal{P}(n)} J_\pi({}^T K_{|V_1|}, \dots, {}^T K_{|V_{|\pi|}|}).$$

Applying $\exp(td_n)$ to the above equality, we obtain

$$\begin{aligned}&(\exp(td_1)\varphi) \circ \mathrm{Mult}_n \\ &\quad = \sum_{\pi=(V_1,\dots,V_{|\pi|})\in\mathcal{P}(n)} J_\pi(\exp(td_{|V_1|})^T K_{|V_1|}, \dots, \exp(td_{|V_{|\pi|}|})^T K_{|V_{|\pi|}|}),\end{aligned}$$

which completes the proof. ■

Let φ be a linear functional on $\mathcal{A}$. If $D : \mathcal{A} \to \mathcal{A}$ is a derivation, then we can define $d_n : \mathfrak{M}_n \to \mathfrak{M}_n$ by

$$d_n f_n(X_1, \dots, X_n) = \sum_{k=1}^{n} f_n(X_1, \dots, DX_k, \dots, X_n).$$

It is easy to check that $(d_n : \mathfrak{M}_n \to \mathfrak{M}_n)_{n\geq 1}$ is a dual derivation system. Let $\{\alpha_t\}_{t\in\mathbb{R}}$ be the formal automorphism group $\alpha_t = \exp(tD)$. Then $\varphi^t := \varphi \circ \alpha_t$ is the $\mathbb{C}[\![t]\!]$-valued linear map associated to D.

EXAMPLE 4.4. Let $\mathcal{A} = \mathbb{C}[\![x]\!]$ be the unital ring generated by one indeterminate and let D be the derivation $Dx^n = nx^{n-1}$. For a probability measure μ with finite moments of all orders, we denote by φ_μ the state $\varphi_\mu(x^n) = \int_{\mathbb{R}} x^n\,\mu(dx)$. Then $\varphi^t = \varphi \circ e^{tD}$ has the moments of $x+t$:

$$\varphi^t(x^n) = \varphi_\mu((x+t)^n)$$

for all $n \geq 0$. Let $K_n(x) := K_n(x,\dots,x)$ be the nth cumulant of x for any one of tensor, free or Boolean independences. Then $(e^{tD}K_n)(x) = K_n(x+t, \dots, x+t) = K_n(x) + t\delta_{1n}$. We excluded monotone independence since the above calculation does not hold.

5. Further generalizations. It is easy to extend many results in this paper to a more general setting. Let $\mathcal{R}$ be a commutative algebra over $\mathbb{C}$. We take a basis $\{e_\lambda\}_{\lambda\in\Lambda}$ of the vector space $\mathcal{R}$ over $\mathbb{C}$. We would like to consider infinite sums by identifying the formal expression $\sum_{\lambda\in\Lambda} c_\lambda e_\lambda$ ($c_\lambda \in \mathbb{C}$) with the map $\lambda \mapsto c_\lambda$ which is denoted by $(c_\lambda)_{\lambda\in\Lambda}$. For this purpose, we assume that the multiplication $(\sum_{\lambda\in\Lambda} c_\lambda e_\lambda)(\sum_{\lambda\in\Lambda} d_\lambda e_\lambda)$ is well-defined for any $c_\lambda, d_\lambda \in \mathbb{C}$. That is, we assume that the coefficient of each e_λ, appearing after multiplication, is a finite sum. For instance, this is true for a graded algebra with finite-dimensional homogeneous components. An $\mathcal{R}$-valued linear map φ on an algebra $\mathcal{A}$ can be written as

$$\varphi(X) = \sum_{\lambda\in\Lambda} \varphi^{(\lambda)}(X) e_\lambda$$

for some linear functionals $\varphi^{(\lambda)}$ from $\mathcal{A}$ to $\mathbb{C}$. By this correspondence, we identify φ with $(\varphi^{(\lambda)})_{\lambda\in\Lambda}$. Along the lines of Section 2, we can define an associative product of linear functionals indexed by Λ for each natural product. We give some examples.

(1) Let $\mathbb{C}[\![t_1,\dots,t_n]\!]$ be the unital and commutative algebra over $\mathbb{C}$ generated by $t_1,\dots,t_n$. We introduce the set A_n of multi-indices $\alpha = (\alpha_1,\dots,\alpha_n)$ satisfying $\alpha_k \in \mathbb{Z}$, $\alpha_k \geq 0$ for each k. We define $|\alpha| := \alpha_1 + \cdots + \alpha_n$. A $\mathbb{C}[\![t_1,\dots,t_n]\!]$-valued linear map $\varphi^{t_1,\dots,t_n}$ can be identified with $(\varphi^{(\alpha)})_{\alpha\in A_n}$ by

$$\begin{aligned}\varphi^{(\alpha)}(X) &= \frac{\partial^{|\alpha|}}{\partial t^\alpha}\varphi^{t_1,\dots,t_n}(X)\bigg|_{(t_1,\dots,t_n)=(0,\dots,0)} \\ &= \frac{\partial^{|\alpha|}}{\partial t_1^{\alpha_1}\cdots\partial t_n^{\alpha_n}}\varphi^{t_1,\dots,t_n}(X)\bigg|_{(t_1,\dots,t_n)=(0,\dots,0)}.\end{aligned}$$

Using this identification, we can define an associative product for multi-indexed linear functionals $(\varphi^{(\alpha)})_{\alpha\in A_n}$.

(2) Let G be a finite abelian group and $\mathbb{C}G$ be its group algebra. It is known that a finite abelian group is the direct product of cyclic groups. $\mathbb{C}G$ consists of elements of the form $(c_g)_{g\in G}$, $c_g \in \mathbb{C}$. We note that the example (1) corresponds to the semigroup algebra of $\mathbb{N}^n$, where $\mathbb{N} = \{0, 1, \ldots\}$. More concretely, let G consist of the unit e and g with the relation $g^2 = e$. A $\mathbb{C}G$-valued linear map φ can be identified with a pair $(\varphi^{(e)}, \varphi^{(g)})$ by $\varphi(X) = \varphi^{(e)}(X)e + \varphi^{(g)}(X)g$. If $\{X, X'\}$ and $\{Y, Y'\}$ are free independent under the $\mathbb{C}G$-valued linear map φ, we recall that

$$\begin{aligned}
\varphi(XY) &= \varphi(X)\varphi(Y),\\
\varphi(XYX') &= \varphi(XX')\varphi(Y),\\
\varphi(XYX'Y') &= \varphi(XX')\varphi(Y)\varphi(Y') + \varphi(X)\varphi(X')\varphi(YY')\\
&\quad - \varphi(X)\varphi(X')\varphi(Y)\varphi(Y').
\end{aligned}$$

Therefore, we obtain the following:

$$\begin{aligned}
&\varphi^{(e)}(XY) = \varphi^{(e)}(X)\varphi^{(e)}(Y) + \varphi^{(g)}(X)\varphi^{(g)}(Y),\\
&\varphi^{(g)}(XY) = \varphi^{(e)}(X)\varphi^{(g)}(Y) + \varphi^{(g)}(X)\varphi^{(e)}(Y),\\
&\varphi^{(e)}(XYX') = \varphi^{(e)}(XX')\varphi^{(e)}(Y) + \varphi^{(g)}(XX')\varphi^{(g)}(Y),\\
&\varphi^{(g)}(XYX') = \varphi^{(e)}(XX')\varphi^{(g)}(Y) + \varphi^{(g)}(XX')\varphi^{(e)}(Y),\\
&\varphi^{(e)}(XYX'Y')\\
&\quad = \sum\nolimits_1 \big(\varphi^{(a)}(XX')\varphi^{(b)}(Y)\varphi^{(c)}(Y') + \varphi^{(a)}(X)\varphi^{(b)}(X')\varphi^{(c)}(YY')\big)\\
&\qquad - \sum\nolimits_2 \varphi^{(a)}(X)\varphi^{(b)}(X')\varphi^{(c)}(Y)\varphi^{(d)}(Y'),\\
&\varphi^{(g)}(XYX'Y')\\
&\quad = \sum\nolimits_3 \big(\varphi^{(a)}(XX')\varphi^{(b)}(Y)\varphi^{(c)}(Y') + \varphi^{(a)}(X)\varphi^{(b)}(X')\varphi^{(c)}(YY')\big)\\
&\qquad - \sum\nolimits_4 \varphi^{(a)}(X)\varphi^{(b)}(X')\varphi^{(c)}(Y)\varphi^{(d)}(Y'),
\end{aligned}$$

where $\sum_1$ is over

$$(a, b, c) = (e, e, e), (g, g, e), (g, e, g), (e, g, g),$$

$\sum_2$ is over

$$\begin{aligned}
(a, b, c, d) = {}& (e, e, e, e), (g, g, e, e), (g, e, g, e), (g, e, e, g),\\
& (e, g, g, e), (e, g, e, g), (e, e, g, g), (g, g, g, g),
\end{aligned}$$

$\sum_3$ is over

$$(a, b, c) = (g, e, e), (e, g, e), (e, e, g), (g, g, g),$$

and $\sum_4$ is over

$$\begin{aligned}
(a, b, c, d) = {}& (g, e, e, e), (e, g, e, e), (e, e, g, e), (e, e, e, g),\\
& (g, g, g, e), (g, g, e, g), (g, e, g, g), (e, g, g, g).
\end{aligned}$$

So far we have considered the setting of linear functionals without positivity. A natural question is: when does the associative product preserve the positivity of states? However, we do not treat this problem in this paper.

One can expect that the c-free and c-monotone products can be understood as commutative algebra-valued free and monotone products, respectively. This is, however, impossible for the following reason. For instance, consider random variables X, X', Y such that $\{X, X'\}$ and Y are free independent for a linear map φ with values in a two-dimensional commutative algebra. Let φ be identified with $(\varphi^{(0)}, \varphi^{(1)})$. Then the calculation rules for $\varphi^{(1)}(XY)$ and $\varphi^{(1)}(XYX')$ must be the same. If we try to realize the c-free independence with respect to the two linear functionals $(\varphi^{(0)}, \varphi^{(1)})$, then we need to choose the commutative algebra and its basis such that $\varphi^{(1)}(XY)$ coincides with $\varphi^{(1)}(X)\varphi^{(1)}(Y)$. This automatically implies that $\varphi^{(1)}(XYX') = \varphi^{(1)}(XX')\varphi^{(1)}(Y)$ by definition of free independence. Therefore, we cannot realize the c-free independence.

There are however connections to multi-state cases in a different context. There are natural products for more than one state: a c-free product [6], a c-monotone product [11], an ordered free product and an indented product [12]. Differential products can be defined for these products of states. For instance, let $*$ be the c-free product: $(\varphi, \psi) = (\varphi_1, \psi_1) * (\varphi_2, \psi_2)$. We can define the c-free product of pairs of $\mathbb{C}[\![t]\!]$-valued linear maps, similarly to the free case. A pair of $\mathbb{C}[\![t]\!]$-valued linear maps (φ^t, ψ^t) can be identified with a sequence of pairs $(\varphi^{(n)}, \psi^{(n)})_{n\geq 0}$. Then we can introduce an associative product for such sequences of pairs.

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